A study on standard n-ideal of a lattice

By

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Declaration

This is to certify that the thesis work entitled "**A study on standard n-ideal of a lattice**" has been carried out by **MD IMRAN HOSSEN** in the Department of **Mathematics**, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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Dedication

To

My parents

Md. Abdul Kader Biswas & Fatema Begum

Approval

This is to certify that the thesis work submitted by ***Md. Imran Hossen,*** Roll no. 1651501, entitled "***A study on standard n-ideal of a lattice***" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of ***Master of Science*** in the Department of ***Mathematics***, Khulna University of Engineering & Technology, Khulna, Bangladesh on 25 April 2019.

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Summary

Lattice theory is an important part of mathematics . Ideal lattice and n-ideal of a lattice have played many roles in development of lattice theory. Historically, lattice theory started with Boolean distributive lattices: as a result, the theory of ideal lattice and n-ideal of a lattice is the most extensive and most satisfying chapter in the history of lattice theory. Ideal lattice have provided the motivation for many results, in general lattice theory. Many conditions on lattices and on element and ideals of lattices are weakened forms of distributivity is imposed on lattices arising in various areas of mathematics, especially algebra.

In lattice theory there are different classes of lattices known as variety of lattices. Class of Boolean lattice is of course most powerful variety. Throughout this thesis we will be concerned with another large variety known as the class of ideal lattice and n-ideal of a lattice have been studied by several authors.

The realization of special role of ideal lattices moved to break with the traditional approach to lattice theory, which proceeds from partially ordered sets to general lattices, semi modular lattices, modular lattices and finally ideal lattices.

In this thesis we give several result on ideal and n-ideal which will certainly extend and generalize many results in lattice theory. In order to review, we include definations, examples, solved problems and proof of some theorems. The thesis contains four chapter.

Chapter 1 we have discussed the basic defination of relation, poset, lattice, complete lattice, convex sub lattice, complemented and relatively complemented lattice. We also proved that, Dual of a complete lattice is complete .

Chapter 2 have discussed basic concept of ideal and n-ideal of lattice. Here we study the defination and examples of ideal and n-ideal. Some imprtant theorem like "If n is a neutral element of a lattice, then $I_n(L)$ is modular if and only if L is modular".

Chapter 3 we have discussed Standard element and n-ideals. We also discussed in this chapter Congruence relation.

Chapter 4 deals with standard n-ideal and Principal n-ideal. This is the main part of this thesis work. In this chapter we have discussed some defination and some important theorems like "For a neutral element n and a standard n-ideal S and an n-ideal I, $S \cap I$ is also a standard n-ideal" .

Contents

CHAPTER 1

Preliminaries

1.1 Some defination of lattices:

1.1.1 Relation : Let A and B be two non-empty set, any subset of $A \times B$ (Cartesian Product) is called relation from A to B. The elements a, b $(a \in A, b \in B)$ are in relation with respect to R if $(a, b) \in R$.

For $(a, b) \in R$, we will also write "$a\ R\ b$" or "$a \equiv b\ (R)$" and read as "a is related to b by R".

Example : Let A={1, 2, 3}; B={4, 5}

Then A×B={(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)}

R_1={(1, 4), (1, 5)}, R_2={(2, 5)}, R_3={(3,4), (1, 5)} are all relations from A to B.

1.1.2 Inverse relation: Every relation R from A to B has an inverse relation R^{-1} from B to A which is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

In other words, the inverse relation R^{-1} consists of those ordered pairs which when reversed, i.e. permutated, belongs to R.

Example : Let A={1, 2, 3} and B={a, b}.Then

R={(1, a), (1, b), (3, a)} is a relation from A to B. Then the inverse relation of R is

$R^{-1} = \{(a, 1), (b, 1), (a, 3)\}$.

1.1.3 Reflexive relation: A relation R in a set A is called a reflexive relation if, for every $a \in A$, $(a, a) \in R$.

In other words, R is reflexive if every element in A is related to itself.

Example : Let A={1, 2, 3}. Then

R={(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)}

Here R is reflexive since (1, 1) , (2, 2) and (3, 3) belongs to the relation .

1.1.4 Symmetric relation: Let R be a subset of $A \times A$, i.e. let R be a relation in A . Then R is called a symmetric relation if $(a, b) \in R$ implies $(b, a) \in R$

that is, if a is related to b then b is also related to a.

Example : Let A={1, 2, 3}. Then

R={(1, 1), (3, 2), (2, 3)} is symmetric relation.

1.1.5 Anti-symmetric relation: Let R be a subset of $A\times A$, i.e. let R be a relation in A. Then R is called a anti- symmetric relation if $(a,\ b)\in R$ and $(b,\ a)\in R$ implies $a=b$

In other words, if $a \neq b$ then possibly a is related to b or possibly b is related to a, but never both.

Remark: Let D denoted the diagonal line of $A\times A$, i.e. the set of all ordered pairs $(a\,, a)\in A\times A$

Then a relation R in A is anti-symmetric if and only if $R\cap R^{-1}\subset D$.

Example 1.1.5: Let A={1, 2, 3} . Then

R_1 ={(1, 1)} R_2 ={(1, 2)} both are anti-symmetric relation.

1.1.6 Transitive relation: A relation R in a set A is called a transitive relation if

(a, b) $\in R$ and *(b, c)* $\in R$ implies *(a, c)* $\in R$

In other words, if a is related to b and b is related to c then a is related to c.

Example : Let A={1, 2, 3}. Then

R_1={(1, 2), (2, 2)} R_2={(1, 3), (3,3)} both are transitive relation .

1.1.7 Equivalence relation: A relation R in a set A is an equivalence relation if

(1) R is reflexive , that is for every $a\in A$, $(a\ ,a)\in R$

(2) R is symmetric, that is $(a,\ b)\in R \Rightarrow (b,\ a)\in R$

(3) R is transitive ,that is $(a,\ b)\in R$ and $(b,\ c)\in R \Rightarrow (a,\ c)\in R$.

Example : Let A={1, 2, 3} be a set and

R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3,1), (2, 3)} be a relation of $A\times A$ then the relation is an equivalence relation, since

(1) R is reflexive , {(1, 1), (2, 2), (3, 3)} $\in R$,

(2) R is symmetric, {(1, 2), (2, 1), (1, 3), (3, 1)} $\in R$ and

(3) R is transitive, {(2, 1), (1,3), (2,3)} $\in R$.

1.1.8 Partial order relation : A relation R is called a partial order relation if

1. R is reflexive i.e. $aRa\,,\forall a\in A$

2. R is anti symmetric if aRb and $bRa \Rightarrow a=b \ \ \forall\, a,b\in A$

3. R is Transitive if aRb and $bRc \Rightarrow aRc \ \ \forall\, a,\ b,\ c\in A$

Example : Let $A=\{1, 2, 3\}$

Then, R={(1, 1), (1, 2)} is Quasi order relation on A.

1.1.9 Quasi order relation: A relation R on a set A is called Quasi order relation if,

1. R is reflexive i.e. $aRa, \forall a \in A$

2. R is Transitive i.e. aRb and $bRc \Rightarrow aRc \ \forall a,b,c \in A$.

Example: Let $A=\{1, 2, 3\}$

Then, R={(1, 1), (2, 2), (3, 3), (1, 2)} is partial order relation.

1.1.10 Supremum: Suppose $(P,\leq)$ is a poset and $H \subseteq P, a \in P.$ Then a is an upper bound of H, if $h \leq a$ for all $h \in H$. An upper bound a of H is the least upper bound of H or supremum of H . If for any upper bound b of H such that $a \leq b$. Written as $a =$ **S**up H

for $\{a, b\}$, $\text{Sup}\{a, b\} = a \vee b$.

1.1.11 Infimum : Let $(P, \leq)$ is a poset and $H \subseteq P, a \in P.$ Then a is an lower bound of H, if $h \geq a$ for all $h \in H$. A lower bound a of H is the greatest lower bound of H or infimum of H . If for any lower bound b of H such that $b \leq a$ .We shall write $a =$ **Inf H**

for $\{a, b\}$, Inf $\{a, b\} = a \wedge b$.

1.1.12 Partially order set (Poset) : A non-empty set P, together with a binary relation R is said to be a Partially Orderd set or a Poset if

(P1) aRa for every $a \in P$, i.e., R is reflexive.

(P2) aRb and bRa implies $a = b$, i.e., R is anti-symmetric, for a, $b \in P$

(P3) aRb and bRc implies aRc, i.e. R is transitive, for $a, b, c \in P$.

Remark: For our convenience, we use the symbol "$\leq$" in place of R. We read $\leq$ as "less than or equal to". Thus if P is a poset then we automatically assume that "$\leq$" is the partial ordered relation in P, unless other symbol is mentioned.

1.1.13 Totally order set (Toset or Chain): If P is a poset in which every two members of P are comparable it is called a totally ordered set or a toset or a chain.

Thus if P is a chain and $x, y \in P$ then either $x \leq y$ or $y \leq x$.

Clearly also if x, y are distinct elements of a chain then either $x < y$ or $y < x$.

1.1.14 Well order set: A poset $(P, \leq)$ is called well ordered set if it is a total ordering and every non-empty subset of P has a least element.

1.1.15 Greatest element: Let P be a poset. If $\exists$ an element $a \in P$ s.t. $x \leq a$ for all $x \in P$ then a is called greatest or unit element of P. Greatest element if exists, will be unique.

1.1.16 Least element: Let P be a poset. If $\exists$ an element $b \in P$ s.t. $b \leq x$ for all $x \in P$ then b is called least or zero element of P. Least element if exists, will be unique.

Example: Let $X = \{1, 2, 3\}$. Then $(P(X), \subseteq)$ is a poset.

Let $A = \{\phi, \{1,2\}, \{2\}, \{3\}\}$ then $(A, \subseteq)$ is a poset with ϕ as least element. A has no greatest element. Let $B = \{\{1,2\}, \{2\}, \{3\}, \{1,2,3\}\}$ then B greatest element $\{1, 2, 3\}$ but no least elements. If $C = \{\phi, \{1\}, \{2\}, \{1,2\}\}$ then C has both least and greatest elements namely, ϕ and $\{1, 2\}$.

1.1.17 Maximal element: An element a in a poset P is called maximal element of P if $a < x$ for no $x \in p$.

Example : In the poset {2, 3, 4, 6, 7, 21} under divisibility 4, 6 and 21 are three maximal elements (none being the greatest).

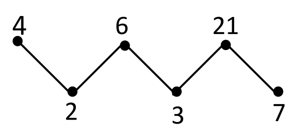

Fig-1

1.1.18 Minimal element: An element b in a poset P is called a minimal element of P if $x < b$ for no $x \in P$.

1.1.19 Upper bound of a set: Let S be a non empty subset of a poset P. An element $a \in P$ is called an upper bound of S if $x \leq a \ \forall x \in S$.

1.1.20 Lower bound of a set: An element $a \in P$ will be called lower bound of S if $a \leq x \ \forall x \in S$.

1.1.21 Lattice (Set theorical Defination): A poset $(L, \leq)$ is said to form a lattice if for every $a, b \in L, \mathrm{Sup}\{a, b)$ and $\mathrm{Inf}\{a, b\}$ exist in L.

In that case, we write

Sup $\{a, b\} = a \vee b$ (read a join b)

Inf $\{a, b\} = a \wedge b$ (read a meet b)

Other notations like $a+b$ and $a \cdot b$ or $a \cup b$ and $a \cap b$ are also used for Sup $\{a,b\}$ and $\text{Inf}\{a,b\}$.

1.1.22 Lattice (Algebrical Definition): A set L together with two binary operation $'\wedge'$ (meet) and $'\vee'$(join) is called a lattice if it satisfies the following identities

(i) idempotent law $\forall a \in L,\ a \wedge a = a,\ a \vee a = a$

(ii) commutative law $\forall a, b \in L,\ a \wedge b = b \wedge a,\ a \vee b = b \vee a$

(iii) associative law $\forall a, b, c \in L,\ a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$

(iv) absorption identities $\forall a, b \in L,\ a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

Example : Let X be a non empty set, then the poset $(P(X), \subseteq)$ of all subset is a lattice. Here for $A, B \in P(X)$

$A \wedge B = A \cap B$ and $A \vee B = A \cup B$

As particular case, when $X = \{1, 2, 3\}$

$P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

It represented by the following figure.

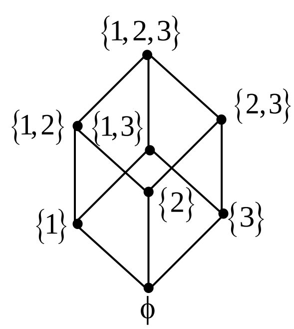

Fig-2

1.1.22 Bounded lattice: A lattice with smallest and largest element is called a bounded lattice. Smallest element is denoted by "0" and largest element denoted by "1" or "u".

Example 1.1.23: The bounded subset of all real number under usual relation $\leq$ is a bounded lattice.

1.1.24 Complete lattice : A lattice L is called a complete Lattice if every non empty subset of L has its Sup and Inf in L.

Example 1.1.24: Set of all sub space of a vector space V is a complete Lattice under set inclusion.

1.1.25 Sub lattice : Let $(L, \wedge, \vee)$ be a Lattice, A non empty subset S of L is called a sublattice of L if S itself is a lattice under same operation $\wedge$ and $\vee$ in L.

Example : Let $L = \{o, a, b, 1\}$ be a lattice.

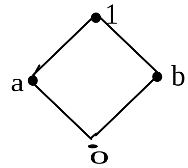

Fig-3

Sublattice of L are: $\{o, a, b, 1\}, \{0\}, \{a\}, \{b\}, \{1\}, \{0, a\}, \{0, b\}, \{0,1\}, \{a, 1\}, \{b,1\}$.

1.1.25 Convex sublattice: A sublattice S of a lattice L is called a convex sublattice of L. If for all $a, b \in S$, $[a \wedge b, a \vee b] \subseteq S$.

Example: Let $L = \{o, a, b, c, 1\}$ be a lattice.

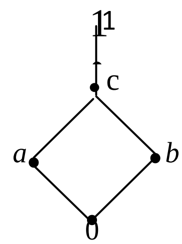

Fig- 4

Here $\{o, a, b, c\}$ is convex sublattice.

1.1.26 Semi lattice (Set theorical defination) : A poset is called a meet-semi lattice if for all $a, b \in P$.

Inf $\{a,b\}$ exist.

And a poset $(P, \leq)$ is called a join-semi lattice if for all $a, b \in P$. sup $\{a,b\}$ exists.

Both the meet and join semi lattice are called semi lattice.

1.1.27 Semi lattice (Algebrical Defination) : A non-empty set P together with a binary composition '$\wedge$' is called a meet-semi lattice and '$\vee$' called a join semi lattice, if for all $a, b, c \in P$.

(i) $a \wedge a = a, a \vee a = a$

(ii) $a \wedge b = b \wedge a, a \vee b = b \vee a$

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$

Both meet and join semi lattice are called semi lattice.

1.1.28 Complemented Lattice: A bounded lattice in which every element has a complement is called complemented lattice.

1.1.29 Sectionally Complemented Lattice: If L has $a, 0$ and all intervals $[0, a]$ are complemented , then L is said Sectionally Complemented Lattice.

1.1.30 Relatively Complemented Lattice: A relatively complemented lattice is a lattice in which every element has a relative complement in any interval containing it.

1.2. Some related thorem of lattices

Theorem 1.2.1: Dual of a complete lattice is a complete lattice.

Proof: Let (L, ρ) be a complete lattice and let $(\overline{L}, \overline{\rho})$ be its dual. Then $(\overline{L}, \overline{\rho})$ is a lattice.

We have to show that $(\overline{L}, \overline{\rho})$ is complete lattice.

Let $\varphi \neq S \subseteq \overline{L}$ be any subset of $\overline{L}$.

Since L is complete, $\sup S$ and $\inf S$ exist in L.

Let, $a = \inf S$ in L.

Then $a \rho x, \forall x \in L$

$\Rightarrow x \overline{\rho} a, \forall x \in \overline{L}$

$\Rightarrow a$ is an upper bound of S in $\overline{L}$.

Let b be any other upper bound of S in $\overline{L}$

Then $x \overline{\rho} a, \forall x \in \overline{L}$

$\Rightarrow b \rho x, \forall x \in \overline{L}$

$\Rightarrow b \rho a$ as $a = \inf S$ in $\overline{L}$.

$\Rightarrow a \overline{\rho} b$ or that 'a' is $l.u.b$ of S in $\overline{L}$

Similarly, we can show that $\sup S$ in L will be S in, $\overline{L}$. Hence $(\overline{L}, \overline{\rho})$ is complete. ■

Theorem 1.2.2: Product of two lattice is a lattice.

Proof: Let A and B be two lattices then we have already proved that

$A \times B = \{(a, b) : a \in A, b \in B\}$ is a poset under the relation $\leq$ defined by

$(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow a_1 \leq a_2$ in A $b_1 \leq b_2$ in B.

We show $A \times B$ forms a lattice.

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ be any elements.

Then $\{a_1, a_2\} \in A$ and $\{b_1, b_2\} \in B$.

Since A and B are lattices, $\{a_1, a_2\}$ and $\{b_1, b_2\}$ have Sup and Inf in A and B respectively.

Let $a_1 \wedge a_2 = \text{Inf}\{a_1, a_2\}$, $b_1 \wedge b_2 = \text{Inf}\{b_1, b_2\}$

then $a_1 \wedge a_2 \leq a_1$, $a_1 \wedge a_2 \leq a_2$

$b_1 \wedge b_2 \leq b_1$, $b_1 \wedge b_2 \leq b_2$

$\Rightarrow (a_1 \wedge a_2, b_1 \wedge b_2) \leq (a_1, b_1)$

$(a_1 \wedge a_2, b_1 \wedge b_2) \leq (a_2, b_2)$

$\Rightarrow (a_1 \wedge a_2, b_1 \wedge b_2)$ is a lower bound of $\{(a_1, b_1), (a_2, b_2)\}$

Suppose (c, d) is any lower bound of $\{(a_1, b_1), (a_2, b_2)\}$

Then $(c, d) \leq (a_1, b_1)$

$(c, d) \leq (a_2, b_2)$

$\Rightarrow c \leq a_1$, $c \leq a_2$, $d \leq b_1$, $d \leq b_2$

$\Rightarrow c$ is a lower bound of $\{a_1, a_2\}$ in A.

d is a lower bound of $\{b_1, b_2\}$ in B.

$\Rightarrow \quad c \leq a_1 \wedge a_2 = \text{Inf}\{a_1, a_2\}$

$d \leq b_1 \wedge b_2 = \text{Inf}\{a_1, a_2\}$

$\Rightarrow (c, d) \leq (a_1 \wedge a_2, b_1 \wedge b_2)$

So that $(a_1 \wedge a_2, b_1 \wedge b_2)$ is g.l.b. $\{(a_1, b_1), (a_2, b_2)\}$

Similarly(by duality) we can say that $(a_1 \vee a_2, b_1 \vee b_2)$ is l.u.b$\{(a_1, b_1), (a_2, b_2)\}$

Hence $A \times B$ is a lattice.

Also $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$

$(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$. ■

Theorem 1.2.3: Show that a poset is a lattice iff it is algebraically a lattice.

Proof: Clearly L is a non empty set.

So set $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$

Then $a \wedge a = \inf\{a, a\} = a; a \vee a = \sup\{a, a\} = a$

So $\wedge$ and $\vee$ are idempotent

$a \wedge b = \inf\{a,b\} = \inf\{b,a\} = b \wedge a$

$a \vee b = \sup\{a,b\} = \sup\{b,a\} = b \vee a$

$\therefore$ $\wedge$ and $\vee$ are commutative.

Next, $a \wedge (b \wedge c) = \inf\{a, b \wedge c\} = \inf\{a, \inf\{b,c\}\}$

$= \inf\{\inf\{a,b\}, c\} = \inf\{a \wedge b, c\}$

$= (a \wedge b) \wedge c$

$a \vee (b \vee c) = \sup\{a, b \vee c\} = \sup\{a, \sup\{b,c\}\}$

$= \sup\{\sup\{a,b\}, c\} = \sup\{a \vee b, c\}$

$= (a \vee b) \vee c$

so $\wedge$ and $\vee$ are associative.

Finally, $a \wedge (a \vee b) = a \wedge \sup\{a,b\} = \inf\{a, \sup\{a,b\}\} = a$

$a \vee (a \wedge b) = a \vee \inf\{a,b\} = \sup\{a, \inf\{a,b\}\} = a$

Hence $\wedge$ and $\vee$ satisfy two Absorption identity

So $L = (L; , \wedge, \vee)$ is a lattice.

(ii) Since $\wedge$ is idempotent i.e. $a \wedge a = a \quad \forall a \in L$

So $a \leq a$

$\therefore \leq$ is reflexive.

Since $\wedge$ is commutative

$\therefore a \wedge b = b \wedge a$

$\Rightarrow a = b \; [\therefore a \wedge b = a \;\; and \;\; a \vee b = b]$

So, $\leq$ is anti symmetric.

Let $a \leq b$ and $b \leq c$

Then $a = a \wedge b, \;\; b = b \wedge c$

$= a \wedge (b \wedge c)$

$= (a \wedge b) \wedge c$

$= a \wedge c$

$\Rightarrow a = a \wedge c$

$\Rightarrow a \leq c$

So, $\leq$ is transitive

$\therefore (L, \leq)$ is a poset. ■

Theorem 1.2.4 : A sub lattice S of a lattice L is a convex sublatice iff $\forall a, b \in S$ with $a \leq b$; $[a, b] \subseteq S$.

Proof : First suppose, S is a convex sublattice in L.

Then we have to show that $\forall\, a, b \in S, (a \leq b), [a, b] \subseteq S$.

Let $\forall a, b \in S$ be any elements, then by definition of a convex sublattice, we have,

$$[a \wedge b,\ a \vee b] \subseteq S \ldots\ldots\ldots(1)$$

But given that $a \leq b$

$a \wedge b = a,\ a \vee b = b$

Therefore (i) becomes $[a,\ b] \subseteq S$

Conversely suppose $\forall a, b \in S$ with $a \leq b$

$$[a,\ b] \subseteq S \ldots..(2)$$

we have to show that S is convex sublattice in L.

Since S is a sublattice of L

So, by definition of a sublattice.

$a \wedge b \in S$ and $a \vee b \in S, \forall a, b \in S$

Again, $\forall\ a, b \in S$ we know.

$a \wedge b \leq a \vee b$

So by given condition. [i.e. (2) become]

$[a \wedge b,\ a \vee b] \subseteq S.$

Therefore S is convex sublattice. ■

Theorem 1.2.5 : Two bounded lattice A and B are complemented if and only if $A \times B$ is complemented.

Proof: Let A and B be complemented and suppose o, u and o', u' are universal bounded of A and B respectively.

Then (o, o') and (u, u') will be least and greatest elements of $A \times B$

Let $(a, b) \in A \times B$ be any element.

Then $a \in A$, $b \in B$ and as A, B are complemented, $\exists\ a' \in A,\ b' \in B$ s.t,

$a \wedge a' = 0,\ a \vee a' = u,\ b \wedge b' = 0',\ b \vee b' = u'$

Now $\quad (a, b) \wedge (a', b') = (a \wedge a', b \wedge b') = (0, 0')$

$\quad (a, b) \vee (a', b') = (a \vee a', b \vee b') = (u, u')$

Shows that (a', b') is complement of (a,b) in $A \times B$.

Hence $A \times B$ is complemented.

Conversely, let $A \times B$ be complemented.

Let $a \in A, b \in B$ be any elements.

Then $(a,b) \in A \times B$ and thus has a complement, say (a',b')

Then $(a, b) \wedge (a', b') = (0, 0'), (a, b) \vee (a', b') = (u, u')$

$\Rightarrow (a \wedge a', b \wedge b') = (0,0'), (a \vee a', b \vee b') = (u,u')$

$\Rightarrow a \wedge a' = 0,\ a \vee a' = u$

$\Rightarrow b \wedge b' = 0', b \vee b' = u'$

i.e, a' and b' are complements of a and b respectively. Hence A and B are complemented. ■

Theorem 1.2.6 : Two lattice A and B are relatively complemented if and only if $A \times B$ is relatively complemented.

Proof: Let A, B be relatively complemented lattice.

Let $[(a_1,b_1), (a_2,b_2)]$ be any interval of $A \times B$ and suppose (x, y) is any element of this interval.

Then $(a_1,b_1) \leq (x, y) \leq (a_2, b_2)$ $a_1, a_2, x \in A;\ b_1, b_2, y \in B$

$\Rightarrow a_1 \leq x \leq a_2,\ b_1 \leq y \leq b_2$

$\Rightarrow x \in [a_1, a_2]$ an interval in A, $y \in [b_1, b_2]$ an interval in B.

Since A,B are relatively complemented, x,y have complements relative to $[a_1, a_2]$ and $[b_1, b_2]$ respectively.

Let x' and y' be these complements. Then

$$x \wedge x' = a_1,\ y \wedge y' = b_1$$

$$x \vee x' = a_2,\ y \vee y' = b_2$$

Now $(x, y) \wedge (x', y') = (x \wedge x', y \wedge y') = (a_1, b_1)$

$(x, y) \vee (x', y') = (x \vee x', y \vee y') = (a_2, b_2)$

$\Rightarrow (x', y')$ is complement of (x, y) related to $[(a_1, b_1), (a_2, b_2)]$

Thus any interval in $A \times B$ is complemented.

Hence $A \times B$ is relative complemented.

Conversely, let $A \times B$ be relatively complemented.

Let $[a_1, a_2]$ and $[b_1, b_2]$ be any intervals in *A* and *B*.

Let $x \in [a_1, a_2]$, $y \in [b_1, b_2]$ be any elements.

Then $a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$

$\Rightarrow (a_1, b_1) \leq (x, y) \leq (a_2, b_2)$

$\Rightarrow (x, y) \in [(a_1, b_1), (a_2, b_2)]$, an interval in $A \times B$

$\Rightarrow (x, y)$ has a complement, say (x', y') relative to this interval

Thus

$$(x, y) \wedge (x', \; y') = (a_1, b_1)$$
$$(x, y) \vee (x', \; y') = (a_2, b_2)$$
$$\Rightarrow (x \wedge x', \; y \wedge y') = (a_1, b_1)$$
$$(x \vee x', \; y \vee y') = (a_2, b_2)$$

$$\Rightarrow x \wedge x' = a_1, \;\; x \vee x' = a_2$$

$$y \wedge y' = b_1, \;\; y \vee y' = b_2$$

$\Rightarrow x'$ is complement of x relative to $[a_1, a_2]$

y' is complement of y relative to $[b_1, b_2]$

Hence *A*, *B* are relatively complemented. ■

Theorem 1.2.7: Dual of a complemented lattice is complemented.

Proof: Let (L, ρ) be a complemented lattice with o, u as least and greatest elements. Let $(\bar{L}, \bar{\rho})$ be the dual of (L, ρ). Then u, o are least and greatest elements of $\bar{L}$.

Let $a \in L = L$ be any element

Since $a \in L$, a' is complemented, $\exists \; a \in L$ s.t.,

$$a \wedge a' = 0, \; a \vee a' = u \text{ in } L$$

i.e., $o = \inf\{a, a'\}$ in *L*

$\Rightarrow o\rho a,\ o\rho a'$

$\Rightarrow a\bar{\rho}o,\ a'\bar{\rho}o$ in $\bar{L}$

$\Rightarrow$ o is an upper bound of $\{a, a'\}$ in $\bar{L}$

If *k* is any upper bound of $\{a, a'\}$ in $\bar{L}$ then $a\bar{\rho}k,\ a'\bar{\rho}k$

$\Rightarrow k\rho a, k\rho a' \Rightarrow k\rho o$ as o is Inf.

$\Rightarrow o\bar{\rho}k$

i.e., o is 1.u.b. $\{a, a'\}$ in $\bar{L}$

i.e., $a \vee a' = o$ in $\bar{L}$

Similarly, $a \wedge a' = u$ in $\bar{L}$

or that a' is complement of a in $\bar{L}$.Hence $\bar{L}$ is complemented. ■

CHAPTER-2

Basic concept of ideal and n-ideal of a lattice

Introduction: The idea of n-ideals in a lattice was first introduced by Cornish and Noor in several papers [1], [2], [3]. Let L be a lattice and $n \in L$ is a fixed element, a convex sublattice containing n is called an n-ideal. It is denoted by $I_n(L)$. If L has a "0", then replacing n by "0" an n-ideal becomes an ideal. Moreover if L has 1, an n-ideal becomes a filter by replacing n by 1. Thus, the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. So any result involving n-ideals will give a generalization of the results on ideals and filters with 0 and 1 respectively in a lattice.

Clearly $\langle a_1, a_2, a_3 \dots \dots \dots a_m \rangle_n = \langle a_1 \rangle_n \vee \dots \dots \dots \vee \langle a_m \rangle_n$.

The n-ideal generated by a finite number of elements is called a finitely generated n-ideal. The set of all finitely generated n-ideals is denoted by $F_n(L)$. Of course $F_n(L)$ is a lattice. The n-ideal generated by a single element is called a principal n-ideal. The set of all principal n-ideals of L is denoted by $P_n(L)$. We have

$$\langle a \rangle_n = \{x \in L : a \wedge n \leq x \leq a \vee n\}$$

The median operation

$$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$$

is very well known in lattice theory. This has been used by several authors including Birkhoff and Kiss [4] for bounded distributive lattices, Jakubik and Kalibiar [5] for distributive lattices and Sholander [6] for median algebra.

A lattice L with 0 is called sectionally complemented for all $x \in L$.

A distributive lattice with 0, which is sectionally complemented is called a generalized boolean lattice. For the background material we refer the reader to the texts of G. Grätzer [7], Birkhoff [8] and Rutherford [9].

In section 1, we have given some fundamental results on finitely generated n-ideals. We have shown that for a neutral element n of a lattice L, $P_n(L)$ is a lattice if and only if n is central. We have also shown that for a neutral element n, a lattice L is modular (distributive) if and only if $I_n(L)$ is modular (distributive). We proved that, in a distributive lattice L, if both supremum and infimum of two n-ideals are principal, then each of them is principal.

In section 2, we have studied the prime n-ideals of a lattice. Here we have generalized the separation property for distributive lattices given by M. H. Stone in terms of prime n-ideals. Then we showed that in a distributive lattice, every n-ideal is the intersection of prime n-ideals containing it.

2. 1. Finitely generated n-ideals.

We start this section with the following proposition which gives some descriptions of $F_n(L)$.

2.1.1 Proposition: Let L be a lattice and $n \in L$. For $a_1, a_2, a_3 \ldots \ldots \ldots a_m \in L$

(i) $\langle a_1, a_2, a_3 \ldots \ldots \ldots a_m \rangle_n \subseteq$

$\{y \in L : (a_1] \wedge (a_2] \wedge \ldots \ldots \ldots \wedge (a_m] \wedge (n] \subseteq (y] \subseteq (a_1] \vee (a_2] \ldots \ldots \vee (a_m] \vee (n]\}$

(ii) $\langle a_1, a_2, a_3 \ldots \ldots \ldots a_m \rangle_n = \{y \in L : a_1 \wedge a_2 \wedge a_3 \ldots \ldots \wedge a_m \wedge n \leq y \leq a_1 \vee a_2 \vee \ldots \ldots a_m \vee n\}$.

(iii) $\langle a_1, a_2, a_3 \ldots \ldots \ldots a_m \rangle_n =$

$\{y \in L : a_1 \wedge a_2 \wedge a_3 \ldots \ldots \wedge a_m \wedge n \leq y = (y \wedge a_1) \vee (y \wedge a_2) \vee \ldots \ldots (y \wedge a_m) \vee (y \wedge n)\}$.

when L is distributive.

(iv) For any $a \in L$

$\langle a \rangle_n = \{y \in L : a \wedge n \leq y = (y \wedge a) \vee (y \wedge n)\}$

$= \{y \in L : y = (y \wedge a) \vee (y \wedge n) \vee (a \wedge n)\}$

whenever n is standard.

(v) Each finitely generated n-ideal is generated two n-ideal

.

Indeed $\langle a_1, a_2, a_3 \ldots \ldots \ldots a_m \rangle_n = \langle \mathrm{a}_1 \wedge \mathrm{a}_2 \wedge \ldots \ldots \wedge \mathrm{a}_m \wedge \mathrm{n}, \mathrm{a}_1 \vee \ldots \ldots \vee \mathrm{a}_m \vee \mathrm{n} \rangle_n$.

(vi) $F_n(L)$ is a lattice and its members are simply the intervals $[a, b]$ such that $\mathrm{a} \leq \mathrm{n} \leq \mathrm{b}$ and for each intervals

$$[\mathrm{a}, \mathrm{b}] \vee [\mathrm{a}_1, \mathrm{b}_2] = [\mathrm{a} \wedge \mathrm{a}_1, \mathrm{b} \vee \mathrm{b}_1]$$

and $$[\mathrm{a}, \mathrm{b}] \wedge [\mathrm{a}_1, \mathrm{b}_2] = [\mathrm{a} \vee \mathrm{a}_1, \mathrm{b} \wedge \mathrm{b}_1] \quad .$$

Proof: (i) Right hand side is clearly an n-ideal containing $a_1, a_2, a_3 \ldots \ldots \ldots a_m$.

(ii) This clearly follows from (i) and by the convexity of n-ideals.

(iii) When L is distributive, then by (ii) $y \leq a_1 \vee a_2 \vee \ldots\ldots \vee a_m \vee n$ implice that $y = y \wedge [a_1 \vee a_2 \vee \ldots\ldots \vee a_m \vee n] = (y \wedge a_1) \vee (y \wedge a_2) \vee \ldots\ldots \vee (y \wedge a_m) \vee (y \wedge n)$, and (iii) follows.

(iv) By (ii) $\langle a \rangle_n = \{y \in L : a \wedge n \leq y \leq a \vee n\}$.

Then $y = y \wedge (a \vee n) = (y \wedge a) \vee (y \wedge n)$, when n is standard. This proves (iv)

(v) This clearly follows from (ii)

vi) First part is readily verifiable. For the second part, consider the intervals $[a,b]$ and $[a_1, b_1]$ where $a \leq n \leq b$, and $a_1 \leq n \leq b_2$

Then using (ii)we have, ,$[a,b] \vee [a_1, b_1] = \langle a, a_1, b, b_1 \rangle_n$

$= [a \wedge a_1 \wedge b \wedge b_1 \wedge n, a \vee a_1 \vee b \vee b_1 \vee n]$

$= [a \wedge a_1, b \vee b_1]$, while

$[a,b] \wedge [a_1, b_1] = [a \vee a_1, b \wedge b_1]$ is trivial.

In general, the set of principal n-ideals $P_n(L)$ is not necessarily a lattice. The case is different when n is a central element. The following theorem also gives a characterization of central element of a lattice L. ■

Theorem 2.1.2: If $P_n(L)$ is a lattice if and only if n is central, where n be a neutral element of a lattice.

Proof: Suppose $P_n(L)$ is a lattice and $x \leq n \leq y$.Then $[x, y] = \langle x \rangle_n \vee \langle y \rangle_n$.

Since $P_n(L)$ is a lattice , $\langle x \rangle_n \vee \langle y \rangle_n = \langle c \rangle_n$ for some $c \in L$.

This implies that c is the relative complement of n in $[x, y]$. Therefore n is central.

Conversely, suppose n is central.

Let $\langle x \rangle_n$, $\langle y \rangle_n \in P_n(L)$

Then using neutrality of n and proposition–2.1.1.

$\langle x \rangle_n \wedge \langle y \rangle_n = [x \wedge n, x \vee n] \wedge [y \wedge n, y \vee n]$

$= [(x \vee y) \wedge n, (x \wedge y) \vee n]$

and $\langle x \rangle_n \vee \langle y \rangle_n = [x \wedge y \wedge n, x \vee y \vee n]$

Since n is central, there exist c and d such that

$c \wedge n = (x \vee y) \wedge n, \quad c \vee n = (x \wedge y) \vee n$

And $\quad d \wedge n = x \wedge y \wedge n, \; d \vee n = x \vee y \vee n$

Which implies that $\langle x \rangle_n \wedge \langle y \rangle_n = \langle c \rangle_n$ and $\langle x \rangle_n \vee \langle y \rangle_n = \langle d \rangle_n$

and so $P_n(L)$ is a lattice. ■

Theorem 2.1.3: If $a, b \in L$ with $a \leq n \leq b$, the intervals $[a, n]$ and $[n, b]$ are complemented , then $F_n(L)$ is sectionally complemented when L be a lattice.

Proof: Let the interval $[a, n]$ and $[n, b]$ are complemented for all $a, b \in L$ with $a \leq n \leq b$.

Consider $\quad \langle a \rangle_n \subseteq [c, d] \subseteq [a, b]$

Then $a \leq c \leq n \leq d \leq b$ .Since $[a, n]$ and $[n, b]$ are complemented so there exist c' and d'

Such that $c \vee c' = n, \; c \wedge c' = a$

and $d \wedge d' = n, \; d \vee d' = b$

Thus $[c,d] \wedge [c',d'] = [c \vee c', d \wedge d'] = [n,n] = \langle n \rangle$

and $[c,d] \vee [c',d'] = [c \wedge c', \; d \vee d'] = [a,b]$

which implies that $[c, d]$ has a relative complement $[c', d']$. Hence $F_n(L)$ is sectionally complemented.

Conversely, suppose $F_n(L)$ is sectionally complemented.

Consider

$a \leq c \leq n \quad and \qquad n \leq d \leq b$

Then $\langle n \rangle \subseteq [c, \, d] \subseteq [a, \, b]$

Since $F_n(L)$ is sectionally complemented, so there exists $[c', d']$ such that $[c, d] \wedge [c', d'] = \langle n \rangle$

$[c, \; d] \vee [c', \, d'] = [a, b]$ This implies

$c \vee c' = n, \; c \wedge c' = a$

And $\quad d \wedge d' = n, \; d \vee d' = b$ That is c' is the relative complement of c in $[a, n]$ and d' is the relative complement of d in $[n, \, b]$. Hence $[a, n]$ and $[n, b]$ are complemented for all $a, b \in L$ with $a \leq n \leq b$. ■

Lemma 2.1.4: An element n of a lattice L is neutral if and only if

$$m(x,n,y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$$

$$= (x \vee y) \wedge (x \vee n) \wedge (y \vee n). \quad \blacksquare$$

Theorem 2.1.5: If n is a neutral element of a lattice L. Then $I_n(L)$ is modular if and only if L is modular.

Proof: Let $I, J, K \in I_n(L)$ where L is modular. with $K \subseteq I$.

By modularity

$$(I \wedge J) \vee K \subseteq I \wedge (J \vee K).$$

To prove the reverse inequality, let $x \in I \wedge (J \vee K)$. Then $x \in I$ *and* $x \in (J \vee K)$. Then $j_1 \wedge k_1 \leq x \leq j_2 \vee k_2$ for some $j_1, j_2 \in J, k_1, k_2 \in K$. Since $I \supseteq K$ so $x \wedge k_1 \in I$ and $x \vee k_2 \in I$. Then

$$m(x \wedge k_1, n, j_1) \wedge k_1 = k_1 \wedge [((x \wedge k_1) \vee n) \wedge (n \vee j_1) \wedge ((x \wedge k_1) \vee j_1)]$$

$$= [(x \wedge k_1) \vee n] \wedge (n \vee j_1) \wedge [(x \wedge k_1) \vee (k_1 \wedge j_1)], \quad \text{as } L \text{ is modular.}$$

$$\leq x, \text{as } j_1 \wedge k_1 \leq x$$

On the other hand

$$m(x \vee k_2, n, j_2) \vee k_2 =$$

$$\{[(x \vee k_2) \wedge n] \vee (n \wedge j_2) \vee [(x \vee k_2) \wedge j_2]\} \vee k_2,$$

$$= [(x \vee k_2) \wedge n] \vee (n \wedge j_2) \vee [(x \vee k_2) \wedge (k_2 \vee j_2)],$$

$\geq x$ *as* $j_2 \vee k_2 \geq x$, as L is modular.

So we have

$$m(x \wedge k_1, n, j_1) \wedge k_1 \leq x \leq m(x \vee k_2, n, j_2) \vee k_2$$

Hence $x \in (I \wedge J) \vee k$.

Therefore, $I \wedge (J \vee K) = (I \wedge J) \vee K$ with $k \subseteq I$ and so $I_n(L)$ is modular.

Conversely, suppose that $I_n(L)$ is modular.

Then for any $a, b, c \in L$ with $c \leq a$, consider the n-ideals $\langle a \vee n \rangle_n$, $\langle b \vee n \rangle_n$ and $\langle c \vee n \rangle_n$. Then of course

$\langle c \vee n \rangle_n \subseteq \langle a \vee n \rangle_n$. Since $I_n(L)$ is modular, So $\langle a \vee n \rangle_n \wedge [\langle b \vee n \rangle_n \vee \langle c \vee n \rangle_n]$

$= [\langle a \vee n\rangle_n \wedge \langle b \vee n\rangle_n] \vee \langle c \vee n\rangle_n$

Then by proposition 2.1.1 (vi) and by neutrality of n, it is easy to show that

$[a \wedge (b \vee c)] \vee n = [(a \wedge b) \vee c] \vee n$ … … (A)

Again, consider the n-ideals $\langle a \vee n\rangle_n$, $\langle b \vee n\rangle_n$ and $\langle c \vee n\rangle_n$ $c \leq a$ implies

$\langle a \vee n\rangle_n \subseteq \langle b \vee n\rangle_n$. Then using modularity of $I_n(L)$, we have

$\langle a \vee n\rangle_n \vee (\langle b \vee n\rangle_n \vee \langle c \vee n\rangle_n)$

$= \langle a \vee n\rangle_n \vee \langle b \vee n\rangle_n) \vee \langle c \vee n\rangle_n$ Then using proposition 2.1.1 (vi) again and the neutrality of n, it is easy to see that

$[a \wedge (b \vee c)] \wedge n = [(a \wedge b) \vee c] \wedge n$ … … (B)

From (A) & (B) we have $a \wedge (b \vee c) = (a \wedge b) \vee c$, with $c \leq a$, as n is neutral. Therefore L is modular. ■

From the proof of above theorem, it can be easily seen that the following corollary holds which is an improvement of the theorem.

Corollary 2.1.6: For a neutral element *n* of a lattice *L*, the following conditions are equivalent:-

(i) L is modular,

(ii) $I_n(L)$ is modular ,

(iii) $F_n(L)$ is modular. ■

Theorem 2.1.7: If L is distributive if and only if $I_n(L)$ is distributive, where L be a lattice with neutral element n .

Proof: Suppose L is distributive, Let $I, J, K \in I_n(L)$. Then obviously,

$(I \wedge J) \vee (I \wedge K) \subseteq I \wedge (J \vee K)$ .To prove the reverse inequality, let $x \in I \wedge (J \vee K)$ which implies $x \in I$ and $x \in (J \vee K)$. Then $j_1 \wedge k_1 \leq x \leq j_2 \vee k_2$ for some $j_1, j_2 \in J$, $k_1, k_2 \in K$. Since L is distributive,

$m(x,n,j_1) \wedge m(x,n,k_1) = [(x \wedge n) \vee (x \wedge j_1) \vee (n \wedge j_1)] \wedge [(x \wedge n) \vee (x \wedge k_1) \vee (n \wedge k_1)]$

$= (x \wedge n) \vee (n \wedge j_1 \wedge k_1) \vee (x \wedge j_1 \wedge k_1)$

$\leq x \vee (j_1 \wedge k_1) = x$

$$m(x,n,j_2)\wedge m(x,n,k_2)=[(x\wedge n)\vee(x\wedge j_2)\vee(n\wedge j_2)]\wedge[(x\wedge n)\vee(x\wedge k_2)\vee(n\wedge k_2)]$$

$$=(n\wedge(x\vee j_2\vee k_2)\vee(x\wedge(j_2\wedge k_2)),$$

$$=[n\wedge(j_2\vee k_2)]\vee x\geq x$$

Then we have

$$m(x,n,j_1)\wedge m(x,n,k_1)\leq x\leq m(x,n,j_2)\vee m(x,n,k_2)$$

and so $x\in(I\wedge J)\vee(I\wedge K)$.

Therefore $I\wedge(J\vee K)=(I\wedge J)\vee(I\wedge K)$, and so $I_n(L)$ is distributive. ■

Following corollary immediately follows from the above proof which is also an improvement of the above theorem.

Corollary 2.1.8: Let L be a lattice with a neutral element *n*. Then the following conditions are equivalent:

(i) L is distributive,

(ii) $I_n(L)$ is distributive,

(iii) $F_n(L)$ is distributive. ■

To prove this we need the following lemma:

Lemma 2.1.9: Any finitely generated n-ideal $F_n(L)$ which is contained in a principal n-ideal $P_n(L)$ is principal, where L be a distributive lattice .

Proof: Let $[b,\ c]$ be a finitely generated n-ideal such that $b\leq n\leq c$. Let $\langle a\rangle_n$ be a principal n-ideal such that $[b,\ c]\subseteq\langle a\rangle_n=[a\wedge n,a\vee n]$. Then $a\wedge n\leq b\leq n\leq c\leq a\vee n$.

Suppose $t=(a\wedge c)\vee b$.

Then

$t\wedge n=[(a\wedge c)\vee b]\wedge n=(n\wedge a\wedge c)\vee(n\wedge b)$, as L is distributive.

$=b\wedge n=b$

$$t\vee n=[(a\wedge c)\vee b]\vee n=(a\wedge c)\vee n$$
$$=(a\vee n)\wedge(c\vee n)$$

as L is distributive $c\vee n=c$

Hence $[b,c]=[t\wedge n,\ t\vee n]=\langle t\rangle_n$

Therefore, $[b,c]$ is a principal n-ideal. ■

Theorem 2.1.10: If $I \vee J$ and $I \wedge J$ are principal n-ideals, then I and J are also principal, where I and J be n-ideals of a distributive lattice .

Proof: Thus $I \vee J$ and $I \wedge J$ are principal n-ideals

So Let, $I \vee J = \langle a \rangle_n$ and $I \wedge J = \langle b \rangle_n$.

Then for all $\mathrm{i} \in \mathrm{I}$, $\mathrm{j} \in \mathrm{J}$, $\mathrm{j} \leq \mathrm{a} \wedge \mathrm{n}$ and $i,\ j \geq a \wedge n.$

So there exist $\mathrm{i}_1, \mathrm{i}_2 \in \mathrm{I}$, and $\mathrm{j}_1, \mathrm{j}_2 \in \mathrm{J}$ such that $a \wedge n = i_1 \wedge j_1$ and $a \vee n = i_2 \wedge j_2$.

Consider the n-ideal $[b \wedge i_1 \wedge n,\ b \vee i_2 \vee n]$. Since $[b \wedge i_1 \wedge n,\ b \vee i_2 \vee n] \subseteq I \subseteq \langle a \rangle_n$, $[b \wedge i_1 \wedge n,\ b \vee i_2 \vee n] = \langle t \rangle_n$, by lemma 2.1.9 for some $t \in L$. Then

$$< \mathrm{a} >_{\mathrm{n}} = \mathrm{J} \vee \mathrm{I} \supseteq \mathrm{J} \vee [\mathrm{b} \wedge \mathrm{i}_1 \wedge \mathrm{n},\ \mathrm{b} \vee \mathrm{i}_2 \vee \mathrm{n}]$$

$$= [\mathrm{j}_1 \wedge \mathrm{n} \wedge \mathrm{b} \wedge \mathrm{i}_1,\ \mathrm{j}_2 \vee n \vee \mathrm{b} \vee \mathrm{i}_2]$$

$$\supseteq [\mathrm{a} \wedge \mathrm{n},\ \mathrm{a} \vee \mathrm{n}] = < \mathrm{a} >_{\mathrm{n}} .$$

This implies that

$$\mathrm{I} \vee \mathrm{J} = \mathrm{J} \vee [\mathrm{b} \wedge \mathrm{i}_1 \wedge \mathrm{n},\ \mathrm{b} \vee \mathrm{i}_2 \vee \mathrm{n}] = \mathrm{J} \vee \langle \mathrm{t} \rangle_{\mathrm{n}}$$

Further, $\langle b \rangle_n = J \wedge I \supseteq J \wedge [b \wedge i_1 \wedge n,\ \ b \vee i_2 \vee n]$

$$\supseteq J \wedge [b \wedge n,\ b \vee n] = \langle b \rangle_n$$

Which implies that

$$J \wedge I = J \wedge [b \wedge i_1 \wedge n, b \vee i_2 \vee n]$$

$$= J \wedge \langle t \rangle_n$$

Since L is distributive, $I_n(L)$ is also distributive .

we obtain that $I = \langle t \rangle_n$. Similarly we can show that J is also principal. ■

2.2 Prime n-ideals.

Theorem 2.2.1: If I is an n-ideal and D is a convex sublattice of L with $I \cap D = \Phi$. Then L contain a prime n-ideal P such that $P \supseteq I$ and $P \cap D = \Phi$. Where L is a distributive lattice.

Proof: Suppose X be the set of all n-ideals of L that contains I and that are disjoint from D. Since $I \in X$, X is non-empty. Let C be a chain in X and let $T = \cup\{X : X \in C\}$. If $a, b \in T$, then $a \in X$, $b \in Y$ for some $X, Y \in C$. Since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. Then $a, b \in Y$, and so $a \wedge b$, $a \vee b \in Y \subseteq T$, as Y is an n-ideal. Thus, T is a sublattice.

If $a, b \in T$ and $a \le r \le b, r \in L$ then $a, b \in Y$ for some $Y \in C$ and so $r \in Y \subseteq T$ as Y is convex. Moreover $n \in T$. Therefore T is an n-ideal. Obviously $T \supseteq I$ and $T \cap D = \Phi$, which verifies that T is the maximum element of C. Hence by Zorn's lemma, X has a maximal element, say P. We claim that P is a prime n-ideal.

Indeed, if P is not prime, then there exist $a, b \in L$ such that $a, b \notin P$ but $m(a, n, b) \in P$. Then by the maximality of P, $(P \vee \langle a \rangle_n) \cap D \neq \Phi$. Then there exist $x, y \in D$ such that $p_1 \wedge a \wedge n \le x \le p_2 \vee a \vee n$ and $p_3 \wedge b \wedge n \le y \le p_4 \vee b \vee n$ for some $p_1, p_2, p_3, p_4 \in P$. Since $m(a, n, b) = (a \wedge n) \vee (b \wedge n) \vee (a \wedge b) \in P$, taking infimum with $p_1 \wedge p_3 \wedge n$, we have $(p_1 \wedge p_3 \wedge a \wedge n) \vee (p_1 \wedge p_3 \wedge b \wedge n) \in P$.

Choosing $r = ((p_1 \wedge p_3 \wedge a \wedge n) \vee (p_1 \wedge p_3 \wedge b \wedge n)$, we have $r \le x \vee y$ with $r \in P$.

Since $x \le r \vee x \le x \vee y, y \le r \vee y \le x \vee y$ and D is a convex sublattice, so $r \vee x, r \vee y \in D$. Therefore $(r \vee x) \wedge (r \vee y) \in D$.

Again, $r \vee x \le p_2 \vee a \vee n \le p_2 \vee p_4 \vee a \vee n$ and $r \vee y \le p_4 \vee b \vee n \le p_2 \vee p_4 \vee b \vee n$ implies $(r \vee x) \wedge (r \vee y) \le (p_2 \vee p_4 \vee a \vee n) \wedge (p_2 \vee p_4 \vee b \vee n) = s$ (say).

Since $m(a, n, b) = (a \vee n) \wedge (b \vee n) \wedge (a \vee b) \in P$, taking supremum with $p_2 \vee p_4 \vee n$, we have $s \in P$. Also, $r \le (r \vee x) \wedge (r \vee y) \le s$. Thus, again by convexity of P, $(r \vee x) \wedge (r \vee y) \in P$. This implies $P \cap D \neq \Phi$, which leads to a contradiction. Therefore, P is a prime n-ideal . ■

We conclude this section with the following corollaries.

Corollary 2.2.2: Let I be an n-ideal of a distributive lattice L and let $a \notin I, a \in L$. Then there exists a prime n-ideal P of L such that $P \supseteq I$ and $a \notin P$. ■

Corollary 2.2.3: Every $I_n(L)$ of a distributive lattice L is the intersection of all prime n-ideals containing it.

Proof: Suppose P is a prime n-ideal.

Let $\mathrm{I}_1 = \cap\{\mathrm{P} : \mathrm{P} \supseteq \mathrm{I},\ \mathrm{P}$ is a prime n-ideal of $L\}$.

If $I \neq I_1$, then there is an $a \in I_1 - I$. Then by above corollary, there is a prime n-ideal P with $P \supseteq I, a \notin P$. But $a \notin P \supseteq I_1$ gives a contradiction.

So every $I_n(L)$ of a distributive lattice L is the intersection of all prime n-ideals containing it. ■

CHAPTER-3

STANDARD ELEMENT AND n-IDEALS.

3.1. Some notion and notations

The partial ordering relation will be denoted by $<$, in case of theoretical set lattice (that is the elements of which are certain subsets of a given set) by $\subset$. In lattices the meet and the join will be designated by $\cap$ and $\cup$. And the complete meet and complete join by $\wedge$ and $\vee$. The least and greatest element of a partially ordered set (or of a lattice) we denote by 0 and 1. If a covers b (i.e. $a>$b, but $a>$x$>$b for no x), then we write $a>$b.

If a(x) is a property defined on the set *H*, then we define $\{x:a(x)\}$as the set of all $x \in H$ for which a(x) is true. Hence in partially ordered sets $< a >_n = \{x : x \wedge a \le x \le x \vee a\}$is the principal n-ideal generated by a, while $\{x : a \le x \le b\}$ is the interval $[a,b]$ provided that $a \le b$. If b covers a, then the interval $[a,b]$ is a prime interval. The dual principal n-ideal is denoted by $< a >_n^d$.

If any two elements a ,b of L, satisfying $a<b$, may be connected by a finite maximal chains of the lattice L are finite and bounded, then L is called finite length. If all intervals of the lattice L are of finite length, then L is of locally finite length. If L has a "n` and is of locally finite length, furthermore for all $a \in L$, in [*n*] any two maximal chains are of the same length, then we say that in L' the Jordan-Dedekind chain condition is satisfied. In this case the length of any maximal chain of the interval [*n*] will be denoted by $L(a)$, and *d(x)* is called the dimension function.

Let *P* and *Q* be partially ordered sets. The ordinal sum of *P* and *Q* is defined as the partially ordered set, which is the set union of *P* and *Q*, and the partial ordering remains unaltered in *P* and *Q*, while $x<y$ holds for all $x \in P$ and $y \in Q$; this partially ordered set will be denoted by $P \oplus Q$. The set of all n-ideals of a lattice *L*, partially ordered under set inclusion, form a lattice, which will be denoted by $I_n(L)$.

Lemma 3.1.1: $I_n(L)$ is a conditionally complete lattice. The meet of a set of n-ideals (if it exists) is the set-theoretical meet. The join of the n-ideals $I_a (a \in A)$ is the set of all *x* such that

$$i_{a_1} \wedge i_{a_2} \wedge ....\wedge i_{a_n} \le x \le i_{a_1} \vee .....i_{a_n} (i_{a_j} \in I_{a_j})$$ for some elements a_j of *A*. ■

If A is a general algebra and Θ is a congruence relation of *A*, then the congruence classes of *A* modulo Θ form a general algebra $A(\Theta)$. This is a homomorphic image of *A*. According to [10], we have the following two general isomorphism theorems.

3.2 The first general isomorphism theorem :

Let A be a general algebra and A' a subalgebra of A, further let Θ be an equivalence relation of A such that every equivalence class of A may be represented by an element of A'. Let Θ' denote the equivalence relation of A' induced by Θ. If Θ is a congruence relation, then so is Θ' and $A(\Theta) \cong A'(\Theta')$.

The natural isomorphism makes a congruence class of A correspond to the contained congruence class of A'.

3.3 The second general isomorphism theorem :

Let A' be a homomorphic image of the general algebra A, let Θ be an equivalence relation of A, and denote Θ' the equivalence relation of A' under which the equivalence classes are the homomorphic images of those of A modulo Θ and suppose that no two different equivalence classes of A modulo Θ have the same homomorphic image. Then Θ is a congruence relation if and only if Θ' is one and in this case $A(\Theta) \cong A'(\Theta')$

The natural isomorphism makes an equivalence class of A correspond to its homomorphic image.

3.4 Congruence relations in lattices :

Let Θ be a congruence relation of the lattice L and denote by L/Θ be homomorphic image of L induced by the congruence relation Θ that is the lattice of all congruence classes. If L/Θ has a $[n]$, then the complete inverse image of the $[n]$ is an n-ideal of L, called the kernel of the homomorphism $L \to L/\Theta$.

A simple criterion for a binary relation η to be a congruence relation is formulated in the following Lemma.

Lemma 3.4.1: (G. Gratzer and Schmidt [18])

Let η be a binary relation defined on the lattice L. η is a congruence relation if and only if the following conditions hold for all $x, y, z \in L$

(a) $x \equiv x(\eta)$;

(b) $\mathrm{x} \vee \mathrm{y} \equiv \mathrm{x} \wedge \mathrm{y}(\eta)$ if and only if $x \equiv y(\eta)$;

(c) $x \geq y \geq z,\ x \equiv y(\eta), y \equiv z(\eta)$ imply $x \equiv z(\eta)$;

(d) $x \geq y$ and $x \equiv y(\eta)$, then $\mathrm{x} \vee \mathrm{z} \equiv \mathrm{y} \vee \mathrm{z}(\eta)$ and $\mathrm{x} \wedge \mathrm{z} \equiv \mathrm{y} \wedge \mathrm{z}(\eta)$.

The congruence relations of L will be denoted by Θ ,Φ,… … The set of all congruence relations of L, partially ordered by $\Theta \leq \Phi$ if and only if $x \equiv y(\Theta)$ implies $x \equiv y(\Phi)$, will be denoted by C(L). ■

Lemma 3.4.2: (Birikhoff [19] and Krisganan [20])

$C(L)$ is a complete lattice $x \equiv y(\wedge\Theta_\alpha)(a \in A)$ if and only if $x \equiv y(\Theta_\alpha)$ for all $a \in A$; $x \equiv y(\vee\Theta_\alpha)(a \in A)$ if and only if there exists a sequence of elements in $L, \imath$.

$x \vee y = z_0 \geq z_1 \geq .....\geq z_n = x \wedge y$ such that

$z_i \equiv z_{i-1}(\Theta_{ai})$(i=1, 2,……, n) for suitable $a_1,.............a_n \in A$. ■

Lemma 3.4.3 : $\Theta(L)$ is a complete lattice, $x \equiv y(\wedge\Theta_\alpha)$ if and only if $x \equiv y(\Theta_\alpha)$ for all $\alpha \in A$; $x \equiv y(\vee\Theta_\alpha)$ if and only if there exist in L a sequence of element $x \cup y = z_0 \geq z_1 \geq ....z_n = x \cap y$ such that $z_i \equiv z_{i-1}(\Theta_{\alpha_i})$ $(i = 1, 2.........n)$ for sublattice $a_1, a_2.........a_n \in A$.

The least and greatest elements of the lattice $C(L)$ will be denoted by $\boldsymbol{\omega}$ and $\imath$ respectively.

Proof: Let H be a subset of L, $[H]$ denote the least congruence relation under which any pair of elements of H is congruent. This we call the congruence relation induced by H. If H has just two elements, $H = \{a,b\}$ then $\Theta[H]$ will be written as Θ_{ab}. The congruence relation Θ_{ab} is called minimal. First we describe the following minimal congruence relation Θ_{ab}. To do this, we have to make some preparations. Given two pairs of elements a,b and c, d of L, suppose that either $\mathrm{c} \wedge \mathrm{d} \geq \mathrm{a} \wedge \mathrm{b}$.

And $(\mathrm{c} \cap \mathrm{d}) \cup (\mathrm{a} \cup \mathrm{b}) = \mathrm{c} \cup \mathrm{d}$, or $c \cup d \leq a \cup b$ and $(c \cup d) \cap (a \cap b) = c \cap d$

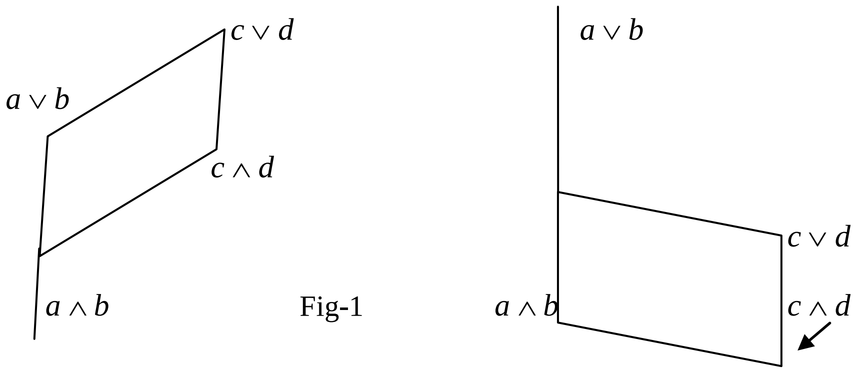

Fig-1

Then we say that a, b is weakly projective in one step to c, d and write $a, b \to c, d$. The situation is given in Fig.1. In other words $a, b \to c, d$ if and only if the intervals $[(a \vee b) \wedge c \wedge d, a \vee b]$, $[c \wedge d, c \vee d]$ or $[a \wedge b, (a \wedge b) \vee c \vee d], [c \wedge d, c \vee d]$ are transposes. If there exist two finite sequences of elements $a = x_0, x_1, \ldots\ldots\ldots x_n = c$ and $b = y_0, y_1, \ldots\ldots\ldots y_n = d$ in L such that

$a, b = x_0, y_0 \to x_1, y_1 \to \ldots\ldots\ldots \to x_n, y_n = c, d$ ……………………….(1)

Then we say that a ,b is weakly projective to c,d in notation: $a,b \to c,d$ or if we are also interested in the number n, then we write $a,b \to c,d$.

If $a,b \to c,d$ and $c,d, \to a,b$ then a,b and c, d are transposes, and we write $a,b \to c,d$. If the sequence (1) may be chosen in such a way that the neighbouring members are transpose, then a,b and c, d are called projective and we write $a,b \to c,d$.

The importance of this notion is shown by the fact that $a,b \to c,d$ and $a \equiv b(\Theta)$ imply $c \equiv d(\Theta)$ (applying this to $\Theta = \omega$, we get that a=b implies c=d, a fact which will be used several times).

Now we are able to describe Θ_{ab}:

According to [18], we have the following describtion: Let a, b, c, d be elements of the lattice L . $c \equiv d(\Theta_{ab})$ holds if and only if there exist $y_i \in L$ with

$c \vee d = y_0 \geq y_1 \geq \ldots\ldots \geq y_k = c \wedge d$ and $a,b \to y_{i-1}, y_i$ (i=1, 2,……, k)………………………..(2)

It is easy to describe $\Theta[H]$, using Lemma 3.4.2 and above. We have the following trivial identity:

$[H] = \vee\Theta_{ab}(a,b \in H)$ …………………………..(3)

The symbol [H] will be used mostly in case H is an n-ideal. Then one can prove the following important identity.

$[\vee I_\alpha] = \vee[I_\alpha]\ (I_\alpha \in I(L))$ ………………………(4)

The following definition is more importance in this chapter. Let L be a lattice and I an ideal of L. By the quofactor lattice L/I of the lattice L modulo the ideal I is meant the homomorphic image of L induced by (I), i.e.

$L/I \cong L(\Theta[I])$.

Finally, we mention the definition of permutability: the congruence relations Θ and Φ are called permutable if $a \equiv x(\Theta)$ and $x \equiv b(\Phi)$ imply the existence of a, y such that $a \equiv y(\Phi)$ and $y \equiv b(\Theta)$. ■

We recall the definition of standard elements:

The element s of the lattice L is standard if the equality

$\mathrm{x} \wedge (\mathrm{s} \vee \mathrm{y}) = (\mathrm{x} \wedge \mathrm{s}) \vee (\mathrm{x} \wedge \mathrm{y})$…………………(A) holds for all $x, y \in L$.

First of all, let us see some examples for standard elements, in the lattice L. p is a standard element. At the same time it is clear that P is not neutral. (Furthermore, in the same lattice $< r >_n$ is a homomorphism kernel but r is not standard.)

Obviously, any element of a distributive lattice L is standard. Furthermore, in any lattice the element n (if exist) are standard element. The simplest from for defining standard elements is the equality (A) however; it is not the most important property of a standard element. Some important characterizations of standard elements are given in the following theorem.

We conclude this chapter with the following results.

Theorem 3.4.4 : (The fundamental characterization theorem of standard elements) the following conditions upon an element s of the lattice L are equivalent:

(α) s is a standard element;

(β) the equality $\mathrm{u} = (\mathrm{u} \wedge \mathrm{s}) \vee (\mathrm{u} \wedge \mathrm{t})$ holds whenever $\mathrm{u} \le \mathrm{s} \vee \mathrm{t}$ $(\mathrm{u}, \mathrm{t} \in \mathrm{L})$;

(γ) the relation Θ_s, defined by $x \equiv y(\Theta_s)$ if and only if $(x \wedge y) \vee s_1 = x \vee y$ for some $s_1 \leq s$ is a congruence relation ;

(δ) for all $x, y \in L$

(i) $s \vee (x \wedge y) = (s \vee x) \wedge (s \vee y)$

(ii) $s \wedge x = s \wedge y$ and $s \vee x = s \vee y$ imply $x = y$.

Proof: We have proved the equivalence of the four conditions cicely

(α) implies (β). Indeed if (α) holds and $u \leq s \vee t$, then $u = u \wedge (s \vee t)$ Owing to (A) we get $u = (u \wedge s) \vee (u \wedge t)$, which was to be proved.

(β) implies (γ). Using condition (β) and Lemma 3.4.1 we will prove that Θ_s is a congruence relation.

(a) $x \equiv x(\Theta_s)$. Indeed for any $x \in L$, the equality $(x \wedge x) \vee (x \wedge s) = x$ trivially holds, so if we put $s_1 = x \wedge s$, we get the assertion.

(b) $x \wedge y \equiv x \vee y$ (Θ_s). This is trivial from the definition of Θ_s.

(c) $x \geq y \geq z$, $x \equiv y$ (Θ_s) and $y \equiv z(\Theta_s)$. By hypothesis $x = y \vee s_1$ and $y = z \vee s_2$ for suitable elements $s_1, s_2 \leq s$ Consequently $x = y \vee s_1 = (z \vee s_2) \vee s_1 = z \vee (s_1 \vee s_2)$ for $s_1 \vee s_2 \leq s$, that means $x \equiv z(\Theta_s)$.

(d) In case $x \geq y$ and $x \equiv y$ (Θ_s) holds, $x \vee z \equiv y \vee z$ (Θ_s) and $x \wedge z \equiv y \wedge z$ (Θ_s). In fact, by assumption $x = y \vee s_1$ ($s_1 \leq s$), and hence we get $x \vee z = (y \vee z) \vee s_1$, that is $x \vee z \equiv y \vee z$ (Θ_s). To prove the second assertion we start from the relations $x = y \vee s_1$ and $x \wedge z \leq y \vee s_1 \leq y \vee s$. Applying condition (β) to $u = x \wedge z, t = y$ and using $x \wedge y = y$, we get

$x \wedge z = (x \wedge z \wedge s) \vee (x \wedge z \wedge y) = (y \wedge z) \vee s_2$, where $s_2 = x \wedge z \wedge s \leq s$, which means $x \wedge z \equiv y \wedge z$ (Θ_s)

(γ) implies (δ). First we prove that (γ) implies (δ)(i). According to the definition of Θ_s, the congruence $x \equiv s \vee x(\Theta_s)$ and $y \equiv s \vee y$ (Θ_s) hold for arbitrary $x, y \in L$. We get $x \wedge y \equiv (s \vee x) \wedge (s \vee y)$ (Θ_s). By monotonicity. $x \wedge y \leq (s \vee x) \wedge (s \vee y)$, hence again by the definition of Θ_s. it follows that $(s \vee x) \wedge (s \vee y) = (x \wedge y) \vee s_1$ with suitable $s_1 \leq s$. Joining with s and keeping the inequalities $s_1 \leq s$ and $s \leq (s \vee x) \wedge (s \vee y)$ in view, we derive $s \vee (x \wedge y) = (s \vee x) \wedge (s \vee y)$, which is nothing else than (δ)(i).

Secondly, we prove that (γ) implies (δ)(ii). Let the elements x and y be chosen as in (δ)(ii). We know that $s \vee y = y$ (Θ_s), so meeting with x and using $x \vee s = y \vee s$ we get $x = (x \vee s) \wedge y = (y \vee s) \wedge x \equiv y \wedge x$ (Θ_s), consequently, using (γ), $(x \wedge y) \vee s_1 = x$ with suitable $s_1 \leq s$. From the last equality $s_1 \leq x$, accordingly $s_1 \leq s \wedge x = s \wedge y \leq y$ (in the meantime we have used the sub-position $s \wedge x = s \wedge y$ of (δ)(ii) thus $x = (x \wedge y) \vee s_1 \leq (x \wedge y) \vee y = y$. We may conclude similarly that $y \leq x$, and thus x=y, which was to be proved.

(δ) implies (α). Let x and y be arbitrary elements of L and define $a = x \wedge (s \vee y)$ and $b = (x \wedge s) \vee (x \wedge y)$. By (δ)(ii), it suffices to prove that $s \wedge a = s \wedge b$ and $s \vee a = s \vee b$.

To prove the equality we start from $s \wedge a$:

$s \wedge a = s \wedge [x \wedge (s \vee y)] = x \wedge [s \wedge (s \vee y)] = x \wedge s.$

It follows from the monotonicity that $x \wedge s \leq b = (x \wedge s) \vee (x \wedge y) \leq [x \wedge (s \vee y)] \vee [x \wedge (s \vee y)] = a$.

Meeting with s, we get $s \wedge x \leq s \wedge b \leq s \wedge a$. But we have already proved that $s \wedge x = s \wedge a$, and so $s \wedge a = s \wedge b$. To prove $s \vee a = s \vee b$ we start from $s \vee a$ and use (δ)(i) in several times:

$s \vee a = s \vee [x \wedge (s \vee y)] = (s \vee x) \wedge [s \vee (s \vee y)] = (s \vee x) \wedge (s \vee y) = s \vee (x \wedge y) = s \vee (x \wedge s) \vee (x \wedge y) = s \vee b,$

Hence proved. ■

Lemma 3.4.5 : Let S be a standard element of the lattice L. Then $< s >_n$ is a homomorphism kernel, namely $[< s >_n] = \Theta_s$. Conversely, if $x \equiv y[< s >_n]$ hold when and only when $(x \wedge y) \vee s_1 = x \vee y$ with a suitable $s_1 \leq s$, then s is a standard element.

Proof: The congruence relation Θ_s obviously satisfies $\Theta \to [<s>_n]$. Consequently $<s>_n$ is in the kernel of the homomorphism induced by Θ_s. We have to prove that $<s>_n$ is just the kernel. Otherwise there exists an $x > s$ with $x \equiv s$ (Θ_s). By definition, it follows $x = s \vee s_1$ $(s_1 \leq s)$ which is obviously a contradiction. Conversely, if $\Theta\ [<s>_n] = \Theta_s$, then Θ_s is a congruence relation, since $\Theta\ [<s>_n]$ is one and then from condition (γ) of Theorem 3.4.4 it follows that s is a standard element. ■

We have formulated Lemma 3.4.5 separately despite the fact that it is an almost trivial variant of condition (γ) of Theorem 3.4.4 because it points out that property of the standard elements which we think to be the most important one. It may be reformulated as follows: if (s] is a principal ideal of L, then $x \equiv y \quad \Theta \quad [\langle s \rangle_n]$ if and only if there exist a sequence of elements $x \vee y = z_0 \geq z_1 \geq z_2 \geq \ldots\ldots \geq z_m = x \wedge y$ of L, an $s_1 \leq s$, and a sequence of integers $n_1, n_2, \ldots \ldots n_m$ such that $s_1, s \rightarrow z_{i-1}, z_i$ (i=1, 2, 3……,m). Now the definition of standardness is as follows: s is standard if and only if $n_i=1$ may be chosen for all i. It follows then we may suppose m=1 as well.

CHAPTER-4

Standard n-ideal and Principal n-ideal

Introduction: We discuss some fundamental properties of n-ideals, which are basic to this thesis. Here we give an explicit description of $F_n(L)$ and $P_n(L)$ which are essential for the development of this thesis. Though $F_n(L)$ is always a lattice , $P_n(L)$ is not even a semilattice. But when n is a neutral element, $P_n(L)$ becomes a meet semilattice. Moreover , we show that $P_n(L)$ is a lattice if and only if *n* is a central element, and then in fact, $P_n(L) = F_n(L)$. Standard elements and ideals in a lattice were introduced by Gratzer and Schmidt [11]. Some additional work has done by Janowitz [12] while Fried and Schmidt [13] have extended the idea of standard ideals to convex sublattices.

According to Gratzer and Schmidt [11], if *a* is an element of a lattice *L*, then

(i) If it is distributive then

$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$, for all $x, y \in L$.

(ii) If it is standard then

$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$, for all $x, y \in L$.

(iii) If it is neutral if for all $x, y \in L$ then

$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$

Gratzer [10] has shown that an element *n* in a lattice *L* is neutral if and only if

$(n \wedge x) \vee (n \wedge y) \vee (x \wedge y) = (n \vee x) \wedge (n \vee y) \wedge (x \vee y)$ for all $x, y \in L$.

An ideal *S* of a lattice *L* is called standard if it is a standard element of the lattice of ideals $I(L)$.

Fried and Schmidt [13] have extended the idea of standard ideals to convex sublattices. Moreover, Nieminen (convex) sublattices. On the other hand, in a more recent paper Dixit and Paliwal [15], [16] have established some results on standard, neutral and distributive (convex) sublattices. But there technique is quite different from those of above authors. We denote the set of all convex sublattices of *L* by Csub (*L*). According to [13] and [17], we define two operation $\wedge$ and $\dot{\vee}$ (these notations have been used by Nieminen in [17] on Csub (L) by

$$A \dot{\wedge} B = < \{ a \dot{\wedge} b: a \in A, b \in B\} >$$

And $A \dot{\vee} B = <\{a \dot{\vee} b: a \in A, b \in B\} >$

For all $A, B \in Csub(L)$ where $< H >$ denotes the convex sublattice generated by a subset H of L.

If A and B are both ideals then $A \dot{\vee} B$ and $A \dot{\wedge} B$ are exactly the join and meet of A and B in the ideal lattice.

However, in general case neither $A \subseteq A \dot{\vee} B$ and $A \dot{\wedge} B \subseteq A$ are valid. For example if $A = \{a\}$ and $B = \{b\}$, then both inequalities imply $A = B$.

According to [15], a convex sublattics of a lattice L is called a standard convex sublattice (or simply a "standard sublattice") if

$$I \dot{\wedge} <S, K> = <I \dot{\wedge} S, I \dot{\wedge} K>$$

And $I \dot{\vee} <S, K> = <I \dot{\vee} S, I \dot{\vee} K>$ hold for any pair $\{I, K\}$ of $Csub(L)$ whenever either $S \cap K$ nor $I \cap <S, K>$ are empty , where $\cap$ denotes the set theoretical intersection.

In this chapter, we have given a characterization of standard n-ideals using the concept of standard sublattice when n is a neutral element. For a neutral element n of a lattice L, we prove the following:

(i) An n-ideal is standard if and only if it is a standard sublattice.
(ii) The intersection of a standard n-ideal and n-ideal I of a lattice L is a standard n-ideal in I.
(iii) The principal n-ideal $< a >_n$ of a lattice L is a standard n-ideal if and only if $a \vee n$ is standard and is dual standard.
(iv) For an arbitrary n-ideal I and a standard n-ideal S of a lattice L, if $I \vee S$ and $I \wedge S$ are principal n-ideals, then I itself is a principal n-ideal.

4.1. Standard n-ideal

According to Fried and Schmidt [13, Th.-1], we have a fundamental characterization theorem for standard convex sublattices:

Theorem 4.1.1: The following conditions are equivalent for each convex sublattice S of a lattice L:

(a) S is a standard sublattice,

(b) Let K be any convex sublattice of L such that $K \cap S \neq \Phi$. Then to each $x \in < S, K >$ there exist $s_1, s_2 \in S$, $a_1, a_2 \in K$ such that

$$x = (x \wedge s_1) \vee (x \wedge a_1) = (x \wedge s_2) \vee (x \wedge a_2)$$

(c) For any convex sublattice K of L and for each $s_2, s_1' \in S$, there are elements $s_1, s_2' \in S$, $a_1, a_2 \in K$ such that

$$x = (x \wedge s_1) \vee (x \wedge (a_1 \vee s_2))$$

$$= (x \wedge s_2') \wedge (x \wedge (a_2 \wedge s_1'))$$

(d) The relation $\Theta[S]$ on L defined by

$x \equiv y\ (\Theta\ [S])$ if and only if

$$x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$$

and $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$ with suitable $t, s \in S$ is a congruence relation. ■

Defination (**standard n-ideal**)**:** An n-ideal S of a lattice L is called a standard n-ideal if it is a standard element of the lattice $I_n(L)$. Where S is called standard if for all $I, J \in I_n(L)$, $I \cap (S \vee J) = (I \cap S) \vee (I \cap J)$.

Proposition 4.1.2: [13, Pro.2] An ideal S of a lattice L is Standard if and only if it is a standard sublattice. Recall that an n-ideal I of a lattice L is called a standard n-ideal if it is a standard element of $I_n(L)$ the lattice of n-ideals. ■

The following theorem gives an extension of proposition 4.1.1 above.

Theorem 4.1.3: If a neutral element n of a lattice L and an n-ideal is standard if and only if it is a standard sublattice.

Proof: First assume that an n-ideal S of a lattice L is a standard sublattice. That is, for all convex sublattice I & K of L with

$S \cap K \neq \Phi$ and $I \cap \langle S,\ K \rangle \neq \Phi$,

We have $I \dot{\wedge} \langle S, K\rangle = \langle I \dot{\wedge} S, I \dot{\wedge} K\rangle$

And $I \dot{\vee} \langle S, K\rangle = \langle I \dot{\vee} S, I \dot{\vee} K\rangle$

We are to show that S is a standard n-ideal in $I_n(L)$.

That is for all n-ideal $I, K \in I_n(L)$

$$\mathrm{I} \cap (\mathrm{S} \vee \mathrm{K}) = (\mathrm{I} \cap \mathrm{S}) \vee (\mathrm{I} \cap \mathrm{K}) .$$

Clearly, $(I \cap S) \vee (I \cap K) \subseteq I \cap (S \vee K)$.

So let $x \in \mathrm{I} \cap (\mathrm{S} \vee \mathrm{K})$. Then $x \in I$ and $x \in S \vee K$

so we have

$$x = (x \wedge s_1) \vee (x \wedge a_1) = (x \vee s_2) \wedge (x \vee a_2),$$

for some $\mathrm{s}_1, \mathrm{s}_2 \in \mathrm{S}$ and $\mathrm{a}_1, \mathrm{a}_2 \in \mathrm{K}$.

Now

$$\begin{aligned}
x &= (x \wedge s_1) \vee (x \wedge a_1) \\
&\leq [(x \wedge s_1) \vee (x \wedge n) \vee (s_1 \wedge n)] \vee [(x \wedge a_1) \vee (x \wedge n) \vee (a_1 \wedge n)] \\
&= m(x,\ n,\ s_1) \vee m(x,\ n,\ a_1)
\end{aligned}$$

that is $x \leq m(x,\ n,\ s_1) \vee m(x,\ n,\ a_1)$

again

$$\begin{aligned}
x &= (x \vee s_2) \wedge (x \vee a_2) \\
&\geq [(x \vee s_2) \wedge (x \vee n) \wedge (s_2 \vee n)] \wedge [(x \vee a_2) \wedge (x \vee n) \wedge (a_2 \wedge n)] \\
&= m^d(x,\ n,\ s_2) \wedge m^d(x,\ n,\ a_2) \\
&= m(x,\ n,\ s_2) \wedge m(x,\ n,\ a_2)
\end{aligned}$$

as n is neutral.

Hence $m(x,\ n,\ s_2) \wedge m(x,\ n,\ a_2) \leq x \leq m(x,\ n,\ s_1) \vee m(x,\ n,\ a_1)$

Which implies $x \in (I \cap S) \vee (I \cap K)$.

Thus, $I \cap (S \vee K) = (I \cap S) \vee (I \cap K)$. So S is a standard n-ideal.

Conversely, Suppose that n-ideal S of a Lattice L is standard. Consider any convex sublattice K of L such that $S \cap K \neq \Phi$. Since S is an n-ideal, clearly,

$< S, K > = < S, < K >_n >$

Let

$$x \in < x >_n \cap < (S, < K >_n)$$
$$= (< x >_n \cap S) \vee (< x >_n \cap < K >_n),$$

as S is a standard n-ideal. This implies

$< x >_n = (< x >_n \cap S) \vee (< x >_n \cap < K >_n)$……………..(1)

Since $x \vee n$ is the largest element of $< x >_n$.

So we have $x \vee n = m(x \vee n, n, s_1) \vee m(x \vee n, n, t)$

for some $s \in S, t \in < K >_n$.

$$= (x \vee n) \wedge s_1) \vee ((x \vee n) \wedge t) \vee n$$
$$= (x \wedge s_1) \vee ((x \wedge t) \vee n),$$

as n is neutral

Now $t \in < K >_n$ implies $t \leq t_1 \vee n$ for some $t_1 \in K$

Then

$$x \vee n \leq (x \wedge s_1) \vee (x \wedge (t_1 \vee n)) \vee n$$
$$= (x \wedge s_1) \vee (x \wedge t_1) \vee n$$
$$\leq (x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee n \leq x \vee n$$

Which implies that $x \vee n = (x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee n$

Then

$$x = x \wedge (x \vee n)$$
$$= x \wedge [(x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee n]$$
$$= [x \wedge \{(x \wedge (s_1 \vee n)) \vee (x \wedge t_1)\}] \vee (x \wedge n)$$

as n is neutral

$$= (x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee (x \wedge n)$$
$$= (x \wedge (s_1 \vee n)) \vee (x \wedge t_1),$$

where $s_1 \vee n \in S, t_1 \in K$.

Since $x \wedge n$ is the smallest element of $< X >_n$, using the relation (1) a dual proof of above shows that $x = (x \vee (s_2 \wedge n)) \wedge (x \vee t_2)$ for some $s_2 \in S$ and $t_2 \in K$

Hence from Th. 4.1.1 (b) we obtain that S is a standard sublattice.

Now, we give characterizations for standard n-ideals when n is a neutral element. We prefer to call it the "Fundamental characterization Theorem" for standard n-ideals. ■

Theorem 4.1.4: For a neutral element n of a lattice L, Then the following conditions are equivalent:

(a) S is a standard n-ideal;

(b) K be any n-ideal

$$S \vee K = (x \wedge s_1) \vee (x \wedge k_1)$$
$$= (x \wedge s_1') \vee (x \wedge k_1') \vee (x \wedge n)$$

and

$$x = (x \vee s_2) \wedge (x \vee k_2)$$
$$= (x \vee s_2') \wedge (x \vee k_2') \wedge (x \wedge n)$$

For some $s_1, s_2, s_1', s_2' \in S; k_1, k_2, k_1', k_2' \in K$

(c) The relation $\Theta(S)$ on L defined by $x \equiv y\, \Theta(S)$ if and only if $x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$ and $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$, for some $t,s \in S$, is a congruence relation.

Proof: (a)$\Rightarrow$(b) . Suppose S is a standard n-ideal and K be any n-ideal. Let $x \in S \vee K$. Since K is also a convex sublattice of L, we have from the proof of theorem 4.1.3,

$$x = (x \wedge (s_1 \vee n)) \vee (x \wedge t_1)$$
$$= (x \vee (s_2 \wedge n)) \wedge (x \wedge t_2)$$

for some $s_1, s_2 \in S$; $t_1, t_2 \in K$. Since n is neutral, from above we also have

$$x = (x \wedge s_1) \vee (x \wedge t_1) \vee (x \wedge n)$$
$$= (x \vee s_2) \wedge (x \vee t_2) \wedge (x \wedge n).$$

Thus (b) holds.

Now, suppose (b) $\Rightarrow$(c). Let (b) holds. Let $\Theta(S)$ be defined as

$$x \equiv y \ \Theta(S) \text{ if and only if } \quad x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$$

and $\quad x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y).$

for $x \geq y$

$y = (y \vee t) \wedge x$ and $x = (x \wedge s) \vee y$ for some $t, s \in S$ with $s \geq t$.

Obviously, $\Theta(S)$ is reflexive and symmetric.

Moreover, $\quad x \equiv y \ \Theta(S)$ if and only if $x \wedge y \equiv x \vee y \ \Theta(S)$

Now suppose $x \geq y \geq z$ with $x \equiv y \ \Theta(S)$ and $y \equiv z \ \Theta(S)$.

Then $x = (x \wedge s_1) \vee y,\ y = (y \vee t_1) \wedge x$ and $y = (y \wedge s_2) \vee z,\ z = (z \vee t_2) \wedge y$ for some $s_1, s_2, a_1, a_2 \in S$.

Then

$$x = (x \wedge s_1) \vee y = (x \wedge s_1) \vee (y \wedge s_2) \vee z$$
$$\leq (x \wedge s_1) \vee (x \wedge s_2) \vee z$$
$$\leq (x \wedge (s_1 \vee s_2) \vee z \leq x$$

which implies

$$x = (x \wedge (s_1 \vee s_2)) \vee z.$$

This shows that

$y \equiv z \ \Theta(S)$.

For the substitution property,suppose $x \geq y$ and $x \equiv y \ \Theta(S)$. Then $x = (x \wedge s) \vee y$ and $y = (y \wedge s_2) \vee z$, $z = (z \vee t_2) \wedge y$ for some s_1, s_2, t_1, $t_2 \in S$.

Then

$$\begin{aligned} x &= (x \wedge s_1) \vee y = (x \wedge s_1) \vee (y \wedge s_2) \vee z \\ &\leq (x \wedge s_1) \vee (x \wedge s_2) \vee z \\ &\leq (x \wedge (s_1 \vee s_2) \vee z \leq x. \end{aligned}$$

which implies

$$x = (x \wedge (s_1 \vee s_2)) \vee z.$$

Similarly, we can show that

$$x \equiv y \quad \Theta(S)$$

For the substitution property, suppose $x \geq y$ and $x \equiv y \ \Theta(S)$. Then $x = (x \wedge s) \vee y$ and $y = (y \vee t) \wedge x$ for some $s, t \in S$. From these relations it is easy to find $s, t \in S$ with $t \leq S$ satisfying the relations. Then for every $z \in L$, $y \wedge z \leq x \wedge z$ and $y \wedge z \leq t \vee (y \wedge z)$.

Therefore,

$$\begin{aligned} y \wedge z &\leq (t \vee (y \wedge z)) \wedge (x \wedge z) \\ &\leq (t \vee y) \wedge (x \wedge z) \\ &= ((t \vee y) \wedge x) \wedge z \\ &= y \wedge z. \end{aligned}$$

This implies that $y \wedge z = (t \vee (y \wedge z)) \wedge (x \wedge z)$.

Let K be the n-ideal , so

$$< t \wedge y \wedge z, \ y >_n .$$

Since $s, t \wedge y \wedge z \in S \vee K$, so by the convexity of

$S \vee K$, $t \wedge y \wedge z \leq t \wedge y \leq t \wedge x \leq s \wedge x \leq s$ as $t \leq s$.

This implies that $s \wedge x \in S \vee K$ Hence $x = (s \wedge x) \vee y \in S \vee K$ Also, by the convexity of $S \vee K$, $t \wedge y \wedge z \leq y \wedge z \leq x \wedge z \leq x$ implies $y \wedge z$, $x \wedge z \in S \vee K$. Then by (b) we have

$x \wedge z = (x \wedge z \wedge s_1) \vee (x \wedge z \wedge k_1) \vee (x \wedge z \wedge n)$ for some $s_1 \in S, k_1 \in K$.

$= (x \wedge z \wedge s_1) \vee (x \wedge z \wedge n) \vee (x \wedge z \wedge n)$ as $y \vee n$ is the largest element of K.

$= (x \wedge z \wedge s_1) \vee (y \wedge z) \vee (x \wedge z \wedge n)$ as n is neutral.

$= ((x \wedge z) \wedge (s_1 \vee n)) \vee (y \wedge z)$

where $s_1 \vee n \in S$. Therefore $x \wedge z \equiv y \wedge z$ $\Theta(S)$ dually we can prove $x \vee z \equiv y \vee z \, \Theta(S)$.

Hence holds. ■

Corollary 4.1.5: Let n is a neutral element in a lattice L. Then for a standard n-ideal S of L, $\Theta(S)$ will be the smallest congruence relation of L, which containing S as a class.

Proof: Here any two element of S are related with $\Theta(S)$.

Now , Let $x \equiv y$ $\Theta(S)$ with $x \geq y$.

Then by theorem we have $y = (y \vee t) \wedge x$ and $x = (x \wedge s) \vee y$ for some $s, t \in S$.

Suppose, $y \in S$ then

$y \leq x = (x \wedge s) \vee y \leq y \vee s$. Then by the convexity of S, $x \in S$.

On the other hand, if $x \in S$, then $x \geq y = (y \vee t) \wedge x \geq t \wedge x$ implies $y \in S$.

Hence $\Theta(S)$ contains S as a class.

Let Φ be a congruence relation containing S as a class. We have $x \equiv y$ $\Theta(S)$ with $x \geq y$,

$y = (y \vee t) \wedge x$ and $x = (x \wedge s) \vee y$ for some $s, t \in S$.

Now

$$x = (x \wedge s) \vee y \equiv (x \wedge n) \vee y \ \Phi$$
$$= (x \vee y) \wedge (n \vee y)$$

as n is neutral.

$= x \wedge (n \vee y) \equiv x \wedge (y \vee t) \Phi = y \ \Phi$.

This implies $\Theta(S) \subseteq \Phi$. Hence $\Theta(S)$ is the smallest congruence containing S as a class. ■

Corollay 4.1.6: If S and T are two standard n-ideals of a lattice L and n is a neutral element, then $S \bigcap T$ is standard n-ideal.

Proof: Here , S and T are two standard n-ideals and clearly $S \cap T$ is n-ideal.

Let $x \equiv y$ (Θ(S) ∩ Θ(T)) with $x \geq y$. Since $x \equiv y$ Θ(S), so

we have $x = (x \wedge s_1) \vee y$ and $y = (y \vee s_2) \wedge x$, for some $s_1, s_2 \in S$. Here we can consider $s_2 \leq n \leq s_1$. Now $x \equiv y$ Θ(S) implies $x \wedge s_1 \equiv y \wedge s_1$ Θ(T), and so there exists $t_1 \in T$, $t_1 \geq n$ such that $x \wedge s_1 = (x \wedge s_1) \wedge t_1) \vee (y \wedge s_1)$.

Then $x = (x \wedge s_1) \vee y = [((x \wedge s_1) \wedge t_1) \vee (y \wedge s_1)] \vee y$.

$= (x \wedge s_1 \wedge t_1) \vee y = (x \wedge (s_1 \wedge t_1)) \vee y.$

Again $x \equiv y\,\Theta(T)$, implies $x \vee s_2 \equiv y \vee s_2 \Theta(T)$. Then we can find $t_2 \in T$ with $t_2 \leq n$ such that

$y \vee s_2 = ((y \vee s_2) \vee t_2) \wedge (x \vee s_2)$. Then

$$\begin{aligned} y = (y \vee s_2) \wedge x &= [((y \vee s_2) \vee t_2) \wedge (x \vee s_2)] \wedge x. \\ &= (y \vee s_2 \vee t_2) \wedge (x \vee s_2) \wedge x \\ &= (y \vee (s_2 \vee t_2) \wedge x. \end{aligned}$$

Now, $n \leq s_1 \wedge t_1 \leq s_1$ and $n \leq s_1 \wedge t_1 \leq t_1$ implies

$s_1 \wedge t_1 \in S \cap T$. Also $s_2 \leq s_2 \vee t_2 \leq n$ and $t_2 \leq s_2 \vee t_2 \leq n$ implies $s_2 \vee t_2 \in S \cap T$.

Hence $x \equiv y$ Θ(S ∩ T). Therefore

Θ(S ∩ T)= Θ(S) ∩ Θ(T).

Hence $S \cap T$ is also a standard n-ideal. ■

Lemma 4.1.7: For a neutral element n and a standard n-ideal S and an n-ideal I , $S \cap I$ is also a standard n-ideal .

Proof: Suppose S be a standard n-ideal and I be an n-ideal of L. We are to show that $S \cap I$ is a standard n-ideal in I. Consider an n-ideal K of I, which is also an n-ideal of L. Now,

$x \in (S \cap I) \vee K \subseteq S \vee K$, since S is standard so we have $x=(x\wedge s)\vee(x\wedge k)$, for some $s\in S,\ k\in K$. By the monotionity, we can choose both $s\geq n,\ k\geq n$. put $s'=(x\vee n)\wedge s$. Then $s'\leq s$

and $n=(x\vee n)\wedge n\leq(x\vee n)\wedge s=s'\leq x\vee n$.

Since $x\vee n\in I$, so by convexity of S and I,

$s'\in S\cap I$. Also $x\wedge s'=x\wedge s$. Thus

$x=(x\wedge s')\vee(x\vee k)$ for some $s'\in S\cap I,\ k\in K$

Also by duality we get $x=(x\wedge s')\vee(x\vee k')$

for some $s'\in S\cap I,\ k'\in K$.
So $S\cap I$ is standard in I. ■

Lemma 4.1.8: If n is a neutral element of a lattice L and Φ is a homomorphism of L onto a lattice L', where $\Phi(n)=n',\ n'\in L'$.Then for any standard n-ideal I for L, $\Phi(I)$ is a standard n'-ideal in L'.

Proof: Clearly $\Phi(I)$ is a sublattice of L'. Let $p\leq t\leq q$, where $p,q\in\Phi(I),\ t\in L'$. Then $p=\Phi(x)$ and $q=\Phi(y)$ for some x, y$\in$I. Since Φ is onto, $t=\Phi(r)$ for some $r\in L$.

Then $\Phi(r)=\Phi(r)\wedge\Phi(y)=\Phi(r\wedge y)$

And

$$\begin{aligned}\Phi(r)&=\Phi(r)\vee\Phi(x)\\&=\Phi(x)\vee\Phi(r\wedge y)\\&=\Phi(x\vee(r\wedge y)\end{aligned}$$

Now, $x\leq x\vee(r\wedge y)\leq x\vee y$ and so by convexity we have

$x\vee(r\wedge y)\in I$. Thus $t=\Phi(x\vee(r\wedge y))\in\Phi(I)$.

Hence $\Phi(I)$ is a convex sublattice of L'.

Moreover $\Phi(n)=n'$ implies $\Phi(I)$ is an n'-ideal in L'.

For standardness, we shall prove (b) of theorem 4.1.4 for $\Phi(I)$. Let K' be any n'-ideal in L' .Then $K=\Phi(K)$ for some n-ideal K of L.

Let $y \in \Phi(I) \vee \Phi(K) \subseteq \Phi(I \vee K)$.

Then $y = \Phi(x)$ for some $x \in I \vee K$. Since I is a standard n-ideal of L, using (b) of Theorem 4.1.4

we have $x = (x \wedge i_1) \vee (x \wedge k_1) \vee (x \wedge n)$, for some $i_1 \in I$, $k_1 \in K$

$= (x \vee i_2) \wedge (x \vee k_2) \wedge (x \vee n)$, $(x \vee i_2) \wedge (x \vee k_2) \wedge (x \vee n)$, For some $i_2 \in I$, $k_2 \in K$.

Then

$$\begin{aligned} y &= \Phi(x) \\ &= \Phi(x \wedge i_1) \vee \Phi(x \wedge n) \\ &= [\Phi(x) \wedge \Phi(i_1)] \vee [\Phi(x) \wedge \Phi(n)] \\ &= [y \wedge \Phi(i_1)] \vee [y \wedge \Phi(k_1)] \vee [y \wedge n']. \end{aligned}$$

Also,

$$\begin{aligned} y &= \Phi(x) \\ &= [y \vee \Phi(i_2)] \wedge [y \vee \Phi(k_2)] \wedge [y \vee n'.] \end{aligned}$$

From Grätzer and Schmidt [11], we know that ideal $(s]$ is standard if and only if s is standard in L. This is true for principal n-ideal when n is a neutral element. In fact this not even true when L is a complemented lattice, where n is neutral. There $< a >_n$ is standard in $I_n(L)$ but a is not standard in L. Moreover b is standard in L but $< b >_n$ is not standard.

So for any standard n-ideal I for L, $\Phi(I)$ is a standard n'-ideal in L'. ■

Theorem 4.1 .9: Let n be neutral element of a lattice L. Let S and T be two standard n-ideals of L. Then

(i) $\Theta(S \cap T) = \Theta(S) \cap \Theta(T)$

(ii) $\Theta(S \cup T) = \Theta(S) \cup \Theta(T)$

Proof: (i) This has already been proved in corollary 4.1.6,

(ii) Clearly, $\Theta(S) \vee \Theta(T) \subseteq \Theta(S \vee T)$ Θ(S) . To prove the reverse inequality,

let $x \equiv \Theta(S \vee T)$ with $x \geq y$.

Then $y = (y \vee p) \wedge x$ and $x = (x \wedge p) \vee y$, for some $p, q \in S \vee T$.

Then

$P = (p \wedge s_1) \vee (p \wedge t_1)$ and $P = (p \wedge s_2) \vee (p \wedge t_2)$

$q = (q \wedge s_3) \vee (p \wedge t_3)$ and $q = (q \wedge s_4) \vee (p \wedge t_4)$

for some $s_1, s_2, s_3, s_4 \in S$ and $t_1, t_2, t_3, t_4 \in T$

Now,

$$\begin{aligned} P &= (p \wedge s_1) \vee (p \wedge t_1) \\ &= (p \wedge n) \vee (p \wedge t_1)\Theta(S) \\ &= (p \wedge n) \vee (p \wedge n)\Theta(T) \\ &= p \wedge n. \end{aligned}$$

Thus, $p = p \wedge n(\Theta(S) \vee \Theta(T))$

Again

$$\begin{aligned} p &= (p \vee s_2) \wedge (p \vee t_2) \\ &= (p \vee n) \wedge (p \vee t_2)\Theta(S) \\ &= (p \vee n) \wedge (p \vee n)\Theta(T) \\ &= p \vee n. \end{aligned}$$

Thus, $p \equiv p \vee n(\Theta(S) \vee \Theta(T))$. This implies

$p \wedge n \equiv p \vee n(\Theta(S) \vee \Theta(T))$

and so $p \equiv n(\Theta(S) \vee \Theta(T))$.

Similarly, we have $q \equiv n(\Theta(S) \vee \Theta(T))$.

Now,

$$\begin{aligned} y &= (y \vee p) \wedge x \\ &\equiv (y \vee n) \wedge x(\Theta(S) \vee \Theta(T) \\ &= (y \wedge x) \vee (n \wedge x), \end{aligned}$$

as n is neutral.

$$\begin{aligned} &= y \vee (x \wedge n) \\ &= y \vee (x \wedge q)\ (\Theta(S) \vee \Theta(T)) \\ &= x \end{aligned}$$

This implies $x \equiv y(\Theta(S) \vee \Theta(T))$.

Therefore, $\Theta(S \cup T) = \Theta(S) \cup \Theta(T)$

which proves (ii). ■

4.2. Principal n-ideal

Recall that a distributive lattice L with 0 is called a normal lattice if its every prime ideal contains a unique minimal prime ideal. Following result gives a characterization of normal lattices.

Theorem 4.2.1: For a distributive lattice L with 0, the following conditions are equivalent :

i) Any two distnict minimal prime ideals are co maximal.

ii) L is normal.

iii) For any $x, y \in L$, $(x \wedge y]^* = (x]^* \vee (y]^*$.

iv) For any $x, y \in L$ with $x \wedge y = 0$ implies $(x]^* \vee (y]^* = L$.

Moreover , when L has a largest element 1, then each of the above conditions is equivalent to for any $x, y \in L$, $x \wedge y = 0$ implies $x_1, y_1 \in L$, such that $x \wedge x_1 = 0 = y \wedge y_1$ and $x_1 \vee y_1 = 1$. ■

Theorem 4.2.2: For a distributive lattice L, $n \in L$ with $F_n(L)$ is normal if and only if $(n]^d$ and $[n)$ are normal. ■

Generalized Stone lattice : A distributive lattice L with 0 is called generalized Stone lattice if for each $x \in L$, $(x]^* \vee (x]^{**} = L$.

we know that L is generalized Stone if and only if $[0, x]$ is a Stone sub lattice for each $x \in L$.

Moreover, a distributive lattice L with 0 is generalized stone if and only if it is normal and pseudo complemented.

Complement of a lattice: For an element $a \in L$, a' is called complement of a lattice if $a \wedge a' = 0$ and $a \vee a' = 1$.

Complemented lattice: A Bounded lattice in which every element has a complement is called Complemented lattice.

Corollary 4.2.3: Suppose $F_n(L)$ is a sectionally pseudo complemented distributive lattice, then $F_n(L)$ is generalized stone if and only if $(n]$ is dual generalized stone and $[n)$ is generalized Stone. ■

Lemma 4.2.4 : Suppose n be a neutral element of a lattice L. Then any finitely generated n-ideal $F_n(L)$ which is contained in a principal n-ideal $P_n(L)$ is principal .

Proof : Let $[b, c]$ be a finitely generated n-ideal such that $b \le n \le c$. Let $< a >_n$ be a principal n-ideals which contains $[b, c]$. Then $a \wedge n \le b \le n \le c \le a \vee n$. Suppose $t = (a \vee b) \wedge c$. Since n is neutral, we have

$$\begin{aligned} n \wedge t &= n \wedge [(a \vee b) \wedge c] = n \wedge (a \vee b) \\ &= (n \wedge a) \vee (n \wedge b) = n \wedge b = b \end{aligned}$$

and

$$\begin{aligned} n \vee t &= n \vee [(a \vee b) \wedge c] \\ &= (n \vee a \vee b) \wedge (n \vee c) \\ &= (n \vee a) \wedge c = c. \end{aligned}$$

Hence $[b, c] = [n \wedge t, n \vee t] = < t >_n$.

Therefore $[b, c]$ is a principal n-ideal. ■

Theorem 4.2.5: Let I be an arbitrary n-ideal and S be a standard n-ideal of a lattice L, where n is neutral. If $I \vee S$ and $I \cap S$ are principal n-ideals, then I itself is a principal n-ideal.

Proof: Let $I \vee S = < a >_n = [a \wedge n, a \vee n]$ and $I \cap S = < b >_n = [b \wedge n, b \vee n]$. Since S is a standard n-ideal, then

$a \vee n = [(a \vee n) \wedge s] \vee ((a \vee n) \wedge x)$ for some $s \in S$, $x \in I$

$= s \vee x$.

Again, $a \wedge n \in S \vee I$

If there exist $s_1 \in S$ and $x_1 \in I$ such that $a \wedge n = ((a \wedge n) \vee s_1) \wedge ((a \wedge n) \vee x_1) = s_1 \wedge x_1$.

Now, consider the n-ideal $[b \wedge x_1 \wedge n, b \vee x \vee n]$. Obviously, $[b \wedge x_1 \wedge n, b \vee x \vee n] \subseteq I \subseteq < a >_n$. So by above lemma, $[b \wedge x_1 \wedge n, b \vee x \vee n]$ is a principal n-ideal say $< t >_n$ for some $t \in L$.

Then

$$
\begin{aligned}
<a>_n = I \vee S &\supseteq S \vee [b \wedge x_1 \wedge n, b \vee x \vee n] \\
&\supseteq [s_1 \wedge n, s \vee n] \vee [b \wedge x_1 \wedge n, b \vee x \vee n] \\
&= [s_1 \wedge n \wedge b \wedge x_1 \wedge n, s \vee n \vee b \vee x \vee n] \\
&= [a \wedge n, a \vee n] = <a>_n .
\end{aligned}
$$

This implies

$$
\begin{aligned}
S \vee I &= S \vee [b \wedge x_1 \wedge n, b \vee x \vee n] \\
&= S \vee <t>_n \ldots\ldots\ldots\ldots\ldots (A)
\end{aligned}
$$

Further,

$$
\begin{aligned}
<b>_n = S \cap I &\supseteq S \cap [b \wedge x_1 \wedge n, b \vee x \vee n] \\
&\supseteq S \cap [b \wedge n, b \vee n] = <b>_n
\end{aligned}
$$

As $\quad b \wedge x_1 \wedge n \le b \wedge n \le b \vee n \le b \vee x \vee n.$

This implies

$$
S \cap I = S \cap [b \wedge x_1 \wedge n, b \vee x \vee n] = S \cap <t>_n \ldots\ldots\ldots\ldots\ldots (B)
$$

Since S is standard so we have from (A) & (B),

$I = <t>_n$. Therefore I is a principal n-ideal. ■

In this section we shall deduce some important properties of standard elements and n-ideals from the fundamental characterization theorem. If S is a standard n-ideal, then we call the congruence relation $\Theta(S)$, generated by S, a standard n-congruence relation. If $S = <s>_n$, then $\Theta(S) = \Theta(<S>_n)$ and so $\Theta(<s>_n)$ is a standard n-congruence relation which we call principal standard n-congruence . Firstly, we prove some results on the connection between standard n-ideals and standard n-congruence relations.

Recommendation and Application

Conclusion and Future recommendation: From the discussion of all previous chapter it can be concluded and recommended that the concept of standard n-ideals can be introduce in join semilattice. Then using these results, we can study those $F_n(L)$ and $P_n(L)$ which are normal, relatively normal, where L is a join-semilattice with 0. In other words all the works of this thesis can be extended for join semilattice.

Application: Lattice theory has a lot of application in different fields.Boolean lattice has applications in the field of hardware and software development of computer science.Also it has wide applications in networking.it can be applied to develop theories in other branches of algebra,such as group theory,Ring etc.

One of the major application of Boolean lattices is the switching system, which are network of switches that involve two state device 0 and 1 for off and on respectively.

REFERENCES

1. W.H. Cornish and A.S.A. Noor, *Around a neutral element of a lattice*. Comment. Math. Univ. Coroline, 28.2.(1987).

2. A.S.A. Noor, *Isotopes of nearlattices*, Ganit (J. Bangladesh Math. Soc) Vol.1, No. 1 (1981), 63-72.

3. A.S.A. Noor, *A ternary operation in a medial nearlattice*, The Rajshahi University studies, (part B) xiii : (1985), 89-96

4. G. Birkhoff and S.A. Kiss, *A ternary operation in a distributive lattices*, Bull. Amer. Math. Soc.53; 749-752(1947).

5. J. Jakubik and M.Kolibiar, *On some properties of a pair of lattices (Russian),* Czechosl. Math. J. 4(1954), 1-27.

6. M. Sholander, *Medians, lattices and trees, Proc. Amer. Math*. Soc. 5(1954), 808-812.

7. G. Grätzer, *Lattice Theory. First concepts and distributive lattices*, Freeman, san Francisco, 1971.

8. G. Birkhoff, *Lattice theory, Amer. Math. Soc. Colloq. Publ*.,25, 3rd Edition, (1967).

9. D.E. Rutherford- *Introduction to lattice Theory*, Oliver and Boyd, 1965.

10. L. REDEI, *Algebra. I* (Leipzig, 1959).

11. Grätzer and E.T. Schmidt *Standard ideals in lattices,* Acta Math. Acad. Sci. Hung. 12(1961),17-86.

12. M.F. janowitz, *A characterization of standard ideals*, Acta Math. Acad. Sci. Hung. 16(1965),289-301.

13. E. Fried and E.T. Schmidt, *Standard sublattices,* Alg. Univ. 5(1975), 203-211.

14. G. Grätzer , *A characterlization of neutral elements in lattice* (Note on lattice theory 1) , Magyar Tud. Akad.Math. Kutato Int. Kozl, 7 (1962),191-192.

15. V.N. Dixit and D.K. Paliwal, *On convex sublattices, Indian J.Pure appl. Math.* 17(6)(1986), 769-777.

16. V.N. Dixit, *perspectivity and projectivity in NTC and W-modular lattices,* Ind. J. Pure appl. Math.

17. G. Grätzer , *General lattice theory*, Birkhauser verlag Basel (1978).

18. G. Grätzer and E.T. Schmidt, *Ideals and congruence relations in lattices*, Acta Math Acad. Sci. Hung., 9 (1958), pp. 137-175

19. G. BIRKHOFF, *On the structure of abstract algebras*, Proc. Cambr. Phil. Soc., 31 (1935), pp. 433-454.

20. V.S. KRISHNAN, *Binary relations, congruences and homomorphisms*, Journ. Madras Univ., Ser. B, 16.

21. R.P. DILWORTH, *The structure of relatively complemented lattices*, Annals of Math., 51 (1950), pp. 348-359

22. W.H. Cornish , *Normal lattices*, J.Austral. Math.Soc. 14(1972),200-215.

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