STANDARD IDEAL AND FILTER OF A LATTICE

By

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Dedicated

To My Parents

Declaration

I hereby declare that this thesis entitled "Standard Ideal and Filter of a Lattice" submitted for the partial fulfillment of the degree of Master of Philosophy is done by myself under the supervision of **Dr. Md. Abul Kalam Azad** and is not submitted else where for any other degree or diploma.

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Approval

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Finally, I would like to shoulder upon all the errors and shortcoming in the study if there be any, I am extremely sorry for that.

(Md. Mizanur Rahman)

CONTENTS

CHAPTER ONE

Introduction

This thesis studies the nature of Standard ideal of a lattice. The idea of standard ideal in lattice was first introduced by G. Gratzer and E.T. Schmidt. The characterization of standard ideal was first introduced by M. F. Jamowitz. It had extended the ideal to convex sub lattices and proved many result of homomorphism by E. Fried and E.T. Schmidt.

First we can define infimum of two ideals of a lattice in their set theoretic intersection but supremum of two ideals I and J. In a lattice L is given by

$I \vee J = \{x \in L : x \leq i \vee j, \mathrm{I} \vee \mathrm{J}$ for some $i \in I, j \in J\}$. In a distributive lattice, two ideals I and J, the supremum i.e., $I \vee J = \{i \vee j : i \in I, j \in J,$ where i, j exists$\}$.

But in a general lattice the formula for the supremum of two ideals is not easy. We start in chapter one the lemmas which gives the formula for the supremum of two ideals.

An ideal I of a lattice L is called standard if and only if I is standard as an element of $I(L)$ the lattice of all ideals of L.

That is of any ideals $I, L \in I(L), I \wedge (J \vee S) = (I \wedge L) \vee (I \wedge S)$

Any element of a lattice is standard if and only if it is distributive and modular. Thus, in a modular lattice every distributive element is standard. Not only that in a modular lattice every standard element is also neutral. Therefore, an ideal is standard if and only if it is both distributive and modular. Since a neutral element n of L is modular if and only if $I(L)$ is modular. So every distributive ideal of L is standard when L is modular and n is neutral.

A congruence φ of a lattice L is called standard if for some standard ideal S of L.

A meet semi lattice together with the properly that any two elements possessing a common upper bound have a suprimum. For any two lattice L_1 and L_2,

a map $\varphi : L_1 \rightarrow L_2$ is called an isotone if for $x, y \in L$ any with $x \leq y$ implies $\varphi(x) \leq \varphi(y)$, also the above mapping is called a meet homomorphism if for all $x, y \in L$, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$. Therefore, meet homomorphism is an isotone and $\varphi(x) \vee \varphi(y) \leq \varphi(x \vee y)$. Therefore, $\varphi(x) \vee \varphi(y)$ exist by upper bound property of L_2, Chinthayamma Malliah and Parameshwara Bhatta have characterize those lattices, whose all congruence are standard and neutral. Here we generalize characterization of those lattice whose all congruence are standard.

In this thesis, we have studied several properties of Standard ideal of a lattice. Moreover, we give several results on Standard ideal of a lattice which certainly extend and generalize many results in lattice theory.

In Chapter two, we have discussed ideals, congruence, length and covering conditions, For any subset K of a lattice L, $(K]$ denotes the ideal generated by K.

Infimum of two ideals of a lattice is their set theoretic intersection. supremum of two ideals I and J in a lattice L is given by

I $\vee$ J= $I \vee J = \{X \in L / X \leq i \vee j$ for some $i \in I, j \in J\}$. Cornish and Hickman in [3] showed that in a distributive lattice L for two ideals I and J,

$I \vee J = \{i \vee j : i \in I, j \in J,$ where $i \vee j$ exists$\}$. But in a general lattice the formula for the supremum of two ideals is not very easy. Which are explain with some examples and generalized many theorems of them.

In Chapter three, Standard and Neutral elements of a lattice and Traces have been discussed. Standard elements in lattices were first studied in depth by Gratzer and schmid [15]. Since then little attenton has been paid to these notions. A lower Semi lattice is said to have the upper bound property if the supremum of any two elements automatically exists when they share a common upper bound. According to Gratzer and Schmidt [15] if a is an element of a lattice L then,

(i) a is called distributive if

$(a \vee (r \wedge s) = (a \vee r) \wedge (a \vee s)$ *for all* $r, s \in L;$

(ii) a is called standard if

$r \wedge (s \vee a) = (r \wedge s) \vee (r \wedge a)$ *for all* $r, s \in L;$

(iii) a is called neutral if the sub lattice generated by r, s and a is distributive for all $i, j \in L$

i.e., $(a \wedge r) \vee (r \wedge s) \vee (s \wedge a) = (a \vee r) \wedge (r \wedge s) \wedge (s \vee a)$ for all $r, s \in L$.

Standard and Neutral elements are essential for the further development of standard ideals.

In chapter four we give a description of Prime ideals, minimal prime ideals and normal. We have also studied Minimal prime n- ideals of a lattice. We give some characterizations on minimal prime n-ideals which are essential for the further development of this chapter. Here we provide a number of results which are generalizations of the results on Normal and generalized Normal lattices.

In chapter five we studied relatively pseudocomplemented of a lattice. We have also studied Multiplier extentions of pseudocomplemented lattices. These have been studied by Cornish and Hicman [3] and many other authors. Here we extend several results of Cornish and Hicman to lattices.

Pseudocomplemented distributive lattices satisfying Lee's identities form educational subclasses denoted by B_n, $-1 \le n \le \omega$. Cornish and Mandelker have studied distributive lattices analogues to B_1 -lattices and relatively B_1-lattices. Moreover, Cornish, Beazer and Davey have idependently obtained several characterizations of sectionally B_m lattices and relatively B_m lattices.

These have been studied by Cornish and Hicman and many other authors. Here we extend several results of Cornish and Hicman to lattices. .

Chapter six introduces the concept of standard ideals, homomorphism, kernels,which have been studied by Gratzer, Schmidt and many other authors. We have given a characterization of standard ideals also characterise in a lattice every standard ideal in a homomorphism kernel of at least one congruence relation. Noor [32] has introduced the concept of standard n- ideals of a lattice. We conclude this thesis with some more properties of standard and neutral ideals, which are the basic concept of this thesis.

CHAPTER TWO

IDEALS AND CONGRUENCES OF A LATTICE

Introduction: The intention of this Section is to outline and fix the notation for some the concepts of lattices which are basic to this thesis. We also formulate some results on arbitrary lattices for later use. For the background material in lattice theory, we refer the render to the text of Brikhoff [11], Gratzer [12], Rutherford [34],Talukder and Noor [39] and Khanna [22].

By a lattice L, we will always mean a lower semi lattice which has the property that any two elements possessing a common upper bound, have a supremun. Cornish and Hickman [3] referred this property in their analysis as the upper bound property and a semilattice of this nature as a semilattice with the upper bound property. We shall see later, the behavior of such a semilattice is closer to that of a lattice than an ordinary semilattice. For the sake of brevity, we prefer to use the term lattice in place of semilattice with the upper bound property.

The upper bound property appears in Gratzer and Lakser [13], While Rozen [35] shows that it is the result of placing certain associativity conditions on the partial join operation . Moreover, more recently Evans [9] referred nearlattices as conditional lattices. By conditional lattice he means a lower semilattice L with the condition that for each $x \in L$, $\{y \in L / y \leq x\}$ is a lattice ; and it is very easy to cheek that this condition is equivalent to the upper bound property of L.

Whenever a lattice has a least element we will denote it by 0. If $x_1, x_2, \cdots\cdots, x_n$ are elements of a lattice then by $x_1 \vee x_2 \vee \cdots\cdots x_n$, we mean that the supremum of $x_1, x_2 \cdots\cdots x_n$ exists and $x_1, x_2 \cdots\cdots x_n$, is the symbol denoting this supremum .

2.1 LATTICE

A non empty subset k of a lattice L is called a sub lattice of L if for any $a,b\in k$ both $a\wedge b$ and $a\vee b$ (whenever it exists in L) belong to k ($\wedge$ and $\vee$ are taken in L) and the $\wedge$ and $\vee$ of k are the restrictions of the $\wedge$ and $\vee$ of L to k. Moreover, a sub lattice k of a lattice L is called a sublattice of L if $a\vee b\in K$ for all $a,b\in K$. A lattice L is called modular if for any $a,b,c\in L$ with $c\leq a, a\wedge(b\vee c)=(a\wedge b)\vee c$ whenever $b\vee c$ exists.

A Lattice L is called distributive for any $x, x_1, x_2 \cdots\cdots x_n$,

$x\wedge(x_1\vee x_2\vee x_3)\vee\cdots\cdots\vee x_n\equiv(x\wedge x_1)\vee(x\wedge x_2)\vee\cdots\cdots(x\vee x_n)$,

whenever $x\vee x_1\vee x_2\vee\cdots\cdots x_n$ exists. Notice that the right hand expression always exist by the upper bound property of L.

Lemma 2.1.1: A lattice L is modular if and only if $(x]=\{y\in L \,/\, y\leq x\}$ is a modular lattice for each $x\in L$.

Consider the following lattices:

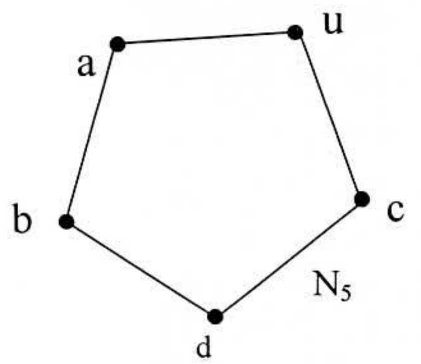


Figure-2.1

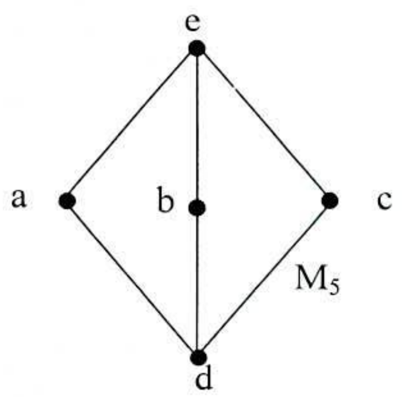


Figure- 2.2

Hickman in [19], [20] has given the following extension of a very fundamental result of lattice theory.

Theorem 2.1.2: A lattice L is distributive if and only if L does not contain a sublattice isomorphic to N_5 or M_5.

Now we give another extension of a fundamental result of lattice theory.

Theorem 2.1.3: A lattice L is modular if and only if L does not contain a sub lattice isomorphic to N_5.

Proof: Suppose L does not contain any sub lattice isomorphic to N_5, then (x] does not contain any sub lattice isomorphic to N_5 for each $x \in L$. Thus, a fundamental result of lattice theory says that (x] is modular for each $x \in S$ as (x] is a sublattice of L. Hence L is modular by Lemma 2.1.1.

Conversely, let L be modular. If L contains a sub lattice isomorphic to N_5, then letting e as the largest element of the sub lattice. We see that (e] is not modular [by lattice theory]. Thus, by Lemma 2.1.1 is not modular and this gives a contradiction. This completes the proof.

In this context it should be mentioned that many lattice theorists' e.g Balbes [2] Varlet [39], Hickman [20] and Shum [34] have worked with a class of semi lattices L which has the property that for each $x, a_1, a_2, --------, a_r \in L$,

if $a_1 \vee a_2 \vee ----------- \vee a_r$, exists

then $(x \wedge a_1) \vee (x \wedge a_2) \vee --------- \vee (x \wedge a_r)$ exists

and equals $x \wedge (a_1 \vee a_2 \vee ---------- \vee a_r)$.

R. Balbes [2] called them as prime semi lattices while D.E. Rutherford [34] referred them as weakly distributive semi lattices □

Theorem 2.1.4: Let $< R+ >$ be a ring and L be the set of all ideals of R. Then $(L \geq)$ forms a lattice, where for any $A, B \in L, A \wedge B = A \cap B$

and $A \vee B = A + B < A \cup B <$ then L is modular.

Proof: Let $A, B, C \in L$ be any three members with $A \supseteq B$.

We claim $A \cap (B + C) = B = (A \cap C)$,

Let $x \in A \cap (B + C)$ be any element . Then $x \in A$ and $x \in B + C \Rightarrow x \in A$

and $x = b + c, b \in B, c \in C$. Now $b \in B \subseteq A, b + c = x \in A$.

Thus, $(b + c) - b \in A \Rightarrow (c + b) - b \in A$ or that $c \in A \Rightarrow x \in A \cap C$,

i.e., $x = b + c, b \in B, c \in A \cap C$. Thus, $x \in B + (A \cap C)$ i.e $A \cap (B + C) \subseteq B + (A \cap C)$

Again by modular inequality (which holds in every lattice) $A \cap (B + C) \supseteq B + (A \cap C)$.

Hence $A \cap (B + C) = B + (A \cap C)$.

L is a modular lattice $\square$

Theorem 2.1.5: The normal subgroups of a group ordered by set inclusion form a modular lattice.

Proof: Let G be any group and L be the set of all normal subgroup of G. Then $L \neq Q$ as $G \in L(L \subseteq)$ is then a poset. For any $A, B \in L$, let $A \wedge B = A \cap B$ which is well defined as intersection of two normal subgroups is a normal subgroup and of course, $A \cap B$ is the largest subset of A and B.

Again, define $A \vee B = AB$. Which is also well defined as AB is a normal Subgroup. Whenever A and B are normal. Also $A \subseteq AB, B \subseteq AB$ (as $a \in A \Rightarrow aa = ac \in AB$ etc).

That AB is the smallest normal subgroup containing A and B is also trivially seen to be true. Indeed if C is any normal subgroup containing A and B, then $AB \subseteq C$. ($x \in AB \Rightarrow x = ab \in C$ as $a \in A \subseteq C, b \in B \subseteq C$).

Finally to check the modularity condition .

Let $A, B, C \in L$ with $A \supseteq B$ be any members we show $A \wedge (B \vee C) = B \vee (A \wedge C)$

i.e $A \cap BC = B(A \cap C)$. Let $x \in A \cap BC$ be any element.

Then $x \in A$ and $x \in BC$

$\Rightarrow \exists\ b \in B, c \in C$ s.t $x = bc$

$x \in A \Rightarrow bc \in A$ also $b \in B \subseteq A \Rightarrow b^{-1} \in A$.

Thus $b^{-1}bc \in A \Rightarrow c \in A \Rightarrow c \in A \cap C$

So $b \in B, c \in A \cap C \Rightarrow bc \in B(A \cap C) \Rightarrow x \in B(A \cap C)$.

Again if $y \in B(A \cap C)$, then $y = bk$ where $b \in B, k \in A \cap C$.

Now $b \in B \subseteq A, k \in A \Rightarrow bk \in A$. Also $b \in B, k \in C \Rightarrow bk \cup BC$.

Thus, $bk \in A \cap BC \Rightarrow B(A \cap C) \subseteq A \cap BC$

Hence $A \cap BC = B(A \cap C)$ □

Theorem 2.1.6: Any non modular lattice L contains a sub lattice isomorphic with the pentagonal lattice.

Proof: Since L is non modular $\exists$ at least three elements a, b, c with $a \geq b$,

s,t $a \wedge (b \wedge c) \neq b \vee (a \wedge c)$.

We must have $a > b$, and as in any lattice the modular inequality

$a \geq b, a \wedge (b \wedge c) \geq (b \vee a \vee c)$ holds,

we get $a \wedge (b \vee c) \succ b \vee (a \wedge c)$

Consider the chain

$a \wedge c \leq b \vee (a \wedge c) \prec a \wedge (b \vee c) \leq b \vee c$ .... (i)

We show at all places strict inequality holds.

Suppose $a \wedge c = b \vee (a \wedge c)$. Then $b \leq a \wedge c \Rightarrow b \vee c \leq (a \wedge c) \vee c$.

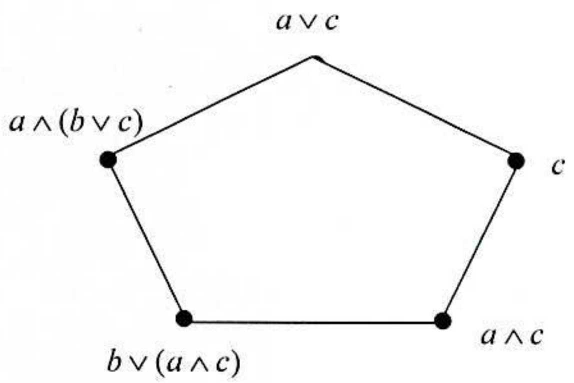


Fig-2.3

$\Rightarrow b \vee c \leq c \vee c \Rightarrow b \vee c = c$

$\Rightarrow a \wedge (b \vee c) = a \wedge c$, a contradiction to (i) .

Thus, $a \wedge c < b \vee (a \wedge c)$.

Similarly, $a \wedge (b \vee c) < b \vee c$.

Hence chain (I) becomes

$a \wedge c < b \vee (a \wedge c) < a \wedge (b \vee c) < b \vee c$-------------(2)

Consider now the chain $a \wedge c \leq c \leq b \vee c$.

As seen above $b \vee c = c$ leads to contradiction and

similarly, $a \wedge c = c$ would give a contradiction.

Hence $a \wedge c < c < b \vee c$

we show c does not lie in chain (2). For this it is sufficient to proved that c is not comparable with $a \wedge (b \vee c)$.

Suppose $a \wedge (b \vee c) \leq c$.

$a \wedge (a \wedge (b \vee c)) \leq a \wedge c$
$\Rightarrow a \wedge (b \vee c \leq a \wedge c$

a contradiction to (2).

Again, if $a \wedge (b \vee c) > c$, then as $a \geq a \wedge (b \vee c)$. We find $a > c$

which gives $a \wedge c = c$ a contradiction to (3).

Hence the chain (2) and (3) form a pentagonal subset

$S = \{a \wedge c, b \vee (a \wedge c), a \wedge (b \vee c), b \vee c, c\}$ of L.

We show now this pentagonal subset is a sublattice for that meet and join of any two elements of S should lie inside S. Meet and join of any two comparable elements being one of them is clearly in S.

So we need to check it for only non comparable elements.

Now $[a \wedge (b \vee c)] \wedge c = a \wedge [(b \vee c) \wedge c] = a \wedge c \in S$.

Also $[a \wedge (b \vee c) \vee c] \geq [b \vee (a \wedge c)] \vee c$ by (2)

$= b \vee [(a \wedge c)c] = b \vee c$

and $a \wedge (b \vee c) \leq b \vee c$ gives $a \wedge (b \vee c) \vee c \leq (b \vee c) \vee c = b \vee c$.

Thus, $[a \wedge (b \vee c)] \vee c = b \vee c \in S$.

Similarly,

we can show $[b \vee (a \wedge c)] \vee c = b \vee c \in S$

$[b \vee (a \wedge c)] \wedge c = a \wedge c \in S$.

Hence S forms a sub lattice of L. Proving our assertion (1) □

Hiekman in [20] has defined a ternary operation j by $j(x, y, z) = (x \wedge y) \vee (y \wedge z)$ on a lattice L (which exists by the upper bound property of L). In fact he has shown that (also see Lyndon [24], theorem 4]) the resulting algebras of the type (L ; j) form a variety, which is referred to as the variety of join algebras and following are its defining identities.

$(i) j(x,x,x)=x$
$(ii) j(x,y,x)=j(y,x,y)$
$(iii) j(j(x,y,x),z,j(x,y,x)=j(x,j(y,z,y),x0$
$(iv) j(x,y,z)=j(z,y,x)$
$(v) j(j(x,y,z),j(x,y,x),j(x,y,z)=j(x,y,x)$
$(vi) j(j(x,y,x),y,z)=j(x,y,x)$
$(vii) j(x,y,j(x,z,x))=j(x,y,x)$
$(viii) j(j(x,y,j(w,y,z),j(x,y,j(x,y,z))=j(x,y,z)$

We do not want to elaborate it further as it is beyond the scope to this thesis.

We call a lattice L medial lattice it for all $x,y,z \in L$,

$m(x,y,z)=(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists. For a (lower) semitattice S, if $m(x,y,z)$ exists for all $x,y,z \in S$, then it is not hard to see that S has the upper bound property and hence is a lattice. Distributive medial lattice were first studied by Sholander in [36] and recently by Evans in [9]. Sholander preferaed to call these as median semi lattices. There he showed that every medial lattice L can be characterized by means of algebra (s;m) of type $< 3 >$ known as median algebra, satisfying the following two identities:

(i) $m(a,a,b)=a$.

(ii) $m(m(a,b,c),m(a,b,d),e)=m(m(c,d,e),a,b)$.

A lattice L is said to have the three properties if for any $a,b,c \in L, a \vee b \vee c$ exists, whenever $a \vee b, b \vee c$ and $c \vee a$ exist. Lattice with the three properties were discussed by Evans in [9], where he referred. It is strong conditional lattices.

The equivalence of (i) and (iii) of the following lemma is trivial, while the proof of (i) $\Leftrightarrow$ (ii) is inductive □

Lemma 2.1.7: For a lattice L the following conditions are equivalent.

(i) L has the three properly.

(ii) Every pair of a finite number n≥3) of elements of L possess a supremum ensures the existence of the supremum of all the n elements.

2.2. IDEALS OF LATTICES

A non empty subset I of a lattice L is called an ideal if it is hereditary and closed under existent finite suprema. We denote the set of all ideals of L by $I(L)$. If L has a smallest element 0 then $I(L)$ is an algebraic closure system on L, and is consequently an algebraic lattice. However, if L dose not possess smallest element then we can only assert that $I(L) \cup \{\phi\}$ is an algebraic closure system.

For any subset K of a lattice L, $(K]$ denotes the ideal generated by K.

Infimum of two ideals of a lattice is their set theoretic intersection.Supermum of two ideals I and J in a lattice L is given by

$I \vee J = \langle x \in L / x \leq i \vee j$ for some $i \in I, j \in J\}$. Cornish and Hickman in [3] showed that in a distributive lattice L for two ideals I and J.

$I \vee J = \{i \vee j / i \in I, j \in J,$ where $i \vee j$ exists$\}$. But in a general lattice the formula for the supremum of two ideals is not very easy. We start this section with the following lemma which gives the formula for the supremum of two ideals. It is in fact Gragter [11, p-54] for partial lattice.

Definition (Ideal): A sub lattice I of a lattice L is called an ideal of L if $i \in I$ i and $a \in L$ implies that $a \wedge i \in I$.

Equivalently, a non empty subset I of a lattice L is an ideal if

(i) $a, b \in I, a \vee b \in I$
(ii) $a \in I$ and $i \in L$ implies that $a \wedge i \in I$.

Let $L = \{1,2,3,5,6,10,15,30\}$ be a lattice of factors of 30 under divisibility.

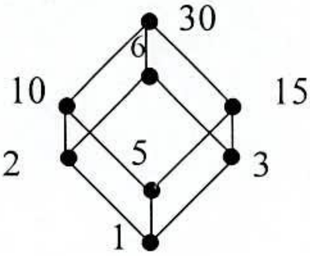


Figure-2.5

Then {1}, {1,2}, {1,3},{1,5}, {1,2,5,10}, {1,3,5,15} ,{1,2,3,6}, {1,2.3,5,6,10,15} are all the ideals of L.

Lemma 2.2.1: Let I and J be ideals of a lattice L. Let $B_0 = I \cup J$,

$B_n = \{x \in L / x \le y \vee Z; \vee$ exists and $y, z \in B_{n-1}\}$ for n= 1,2,3 .... and K= $\overset{a}{\underset{n=0}{\cup}} B_n$.

Then $K = I \vee J$.

Proof: Since $B_0 \subseteq B_1 \subseteq B_2 \subseteq --------- \subseteq B_n \subseteq ---------$, K is an ideal containing I and J. Suppose H is any ideal containing I and J. Of course, $B_0 \subseteq H$. We proceed by induction. Suppose $B_{n-1} \subseteq H$ for some $n \ge 1$ and let $x \in B_n$. Then $x \le y \vee z$ with $y, z \in B_{n-1}$ sinec $B_{n-1} \subseteq H$ and H is an ideal, $y \vee z \in H$ and $x \in H$. That is $B_{n-1} \subseteq H$ for ever n. Thus, $K = I \vee J$ □

Lemma 2.2.2: Let K be a non empty subset of lattice L. Then $(K] = \overset{a}{\underset{n=0}{\cup}} \{B_n / n \ge 0\}$,

where $B_0 = \{t \in s / t = i(k_1, t, k_2\}$, for some $k_1, k_2 \in K\}$ and $B_n = \{t \in L / t = j(a_1, t, a_2)$ for some $a_1, a_2 \in B_{n-1}\}$ for $n \ge 1$.

Proof: For any $k \in K$, clearly $K = J(k,k,k)$ and so $K \subseteq B_0$ similarly, for any $a \in B_{n-1}$

$a = j(a,a,a)$ a implies that $B_{n-1} \subseteq B_n$, Thus,

$K \subseteq B_0 \subseteq B_1 \subseteq ------ \subseteq B_{n-1} \subseteq B_n$ ------------.

Let t$\in \bigcup_{n=0}^{a} A_n; n = 0,1,2,3,$---------, and $t_1 \in S$ such that $t_1 \geq t$. Then $t \in B_m$

for some $m \geq 0$ clearly, $t_1 = j(t,t_1,t)$ t₁ and so $t_1 \in B_{m+1}$. Thus $\bigcup_{n=0}^{a} B_n$ is hereditary.

Now suppose, $t_1, t_2 \in \bigcup_{n=0}^{a} B_n$ and $t_1 \vee t_2$ exist. Let $t_1 \in B_r$ and $t_2 \in B_s$ for some $r,s \geq 0$

with $r \leq s$ (say).Then $t_1, t_2 \in B_s$ and $t_1 \vee t_2 = j(t_1, t_1 \vee t_2, t)$ says $t_1 \vee t_2 \in B_{s+1}$.

Finally, suppose H is an ideal containing K. If $x \in B_0$.

Then $x = j(k_1,x,k_2) = (k_1 \wedge x) \vee (k_2 \vee x)$ for some $k_1,k_2 \in K$. As $K \subseteq H$ and H is an ideal, $K_1 \wedge x, K_2 \wedge x, \in H$ and so $x \in H$. Again we use the induction.Suppose $B_{n-1} \subseteq H$ for some $n \geq 1$. Let $x \in B_n$ so that $x = j(a_1,x,a_2)$ for some $a_1,a_2 \in B_{n-1}$.

Then $x \in H$ as $a_1,a_2 \in H$ and $x = (a_1 \wedge \chi) \vee (a_2 \wedge \chi)$ □

Lemma 2.2.3: A non empty subset K of a lattice L is an ideal if only if $x \in k$ whenever x is an element of L such that $x = j(k_1,x,k_2)$ for same $k_1,k_2, \in K$.

Proof: Since the only if part is of obvious, suppose $x \in k$ whenever x is an element of S and $x = j(k_1,x,k_2)$ for some $k_1,k_2 \in K$. Then clearly B_0(of Lemma 2.2.2) $\subseteq K$. Now for any $x \in B_1$, $x = (a_1,x,a_2)$ for some $a_1,a_2 \in B_0 \in K$ .Thus $x \in K$ and so $B_1 \subseteq K$.

Hence using induction .

We obtain that $(K] = \bigcup_{n=0}^{a} B_n \subseteq K$, i.e $K = (K]$. Therefore K is an ideal □

We now give an alternative formula for the supremum of two ideals in an arbitrary lattice.

Lemma 2.2.4: For any two ideals K_1 and K_2, $K_1 \vee K_2 = \overset{a}{\underset{n=0}{\cup}} B_n$,where

$B_0 = \{x \in L / x = j(k_1, k_l k_2), k_i \in K_1\}$ and $B_n = \{x \in L / x = j(b_1, x, b_2), b_1 b_2 \in B_{n-1}\}$ and

n=0, 1, 2................

Proof: $K_1, K_2 \subseteq B_0 \subseteq B_1 \subseteq ....\subseteq B_{n-1} \subseteq B_n \subseteq ....$

Suppose $b \in \overset{a}{\underset{n=0}{\cup}} B_n$ and $b_1 \leq b; b \in L$. Then $b \in B_n$ for some $m \geq o$. Also $b_1 = j(b, b_1, b)$

and so $b_1 \in B_{m+1,}$ Thus $\overset{a}{\underset{n=0}{\cup}} B_n$ is hereditary. Now suppose exists $t_1, t_2 \in \overset{a}{\underset{n=0}{\cup}} B_n$, such that

$t_1 \vee t_2$ exists. Then there exist $r^1 \geq 0$ Such that $t_1 \in B_1$ and $t_2 \in B_1$ If $r \leq \mathrm{I}$. Then $t_1, t_2 \in B_1$

and $t_1 \vee t_2 = j(t_, t_1 \vee t_2, t_2)$ implies that $t_1 \vee t_2 \in B_{l+1}$. Hence $\overset{a}{\underset{n=0}{\cup}} B_n$ is an ideal.

Finally, suppose H is an ideal containing K_1 and K_2. If $x \in B_o$ then

$x = j(k_1\, x, k_2) = (k_1 \wedge x) \vee (k_2 \wedge x)$ for some $k_1 \in K_1$ and $k_2 \in K_2$ since H is an ideal and

$K_1, K_2 \subseteq H$. Clearly $x \in H$. Then using the induction on n it is very easy to see that

$H \supseteq B_n$ for each n □

Theorem 2.2.5: Cornish and Hickman [3,Theorem 1.1].

The following conditions on a lattice L are equivalent:

(i) L is distributive.

(ii) For any $H \in H(L)$,

$(H] = \{t / h_1 \vee -------- \vee h_n / h_1 ----- h_n \in H$.

(iii) For any $I, J \in J(L)$, $I \vee J = \{a_1 \vee .... \vee a_n / a_1, ..., a_n \in I \vee J\}$

(iv) $J(L)$ is a distributive lattice.

(v) The map $f : H \to (H]$ is a lattice homomorphism of $H(L)$ onto $J(L)$ (which preserves arbitrary suprema).

Observe here that (iii) of above could easily be improved by 2.2.4 to (iii).

For any $I, J \in J(L), I \vee J = \{i \vee j / i \in I, j \in J\}$.

Let $J_f(L)$ form hence forth denotes the set of all finitely generated ideals of a lattice L. Of course $J_f(L)$ is an upper subsemilattice of $J(s)$.

Also for any $x_1, x_2, \ldots\ldots\ldots x_n \in L(x_1, x_2, \ldots\ldots\ldots x_m)$ is clearly the supremum of

$(x_1] \vee (x_2] \vee \ldots\ldots\ldots\ldots \vee (x_m]$.

When L is distributive,

$$(x_1, x_2, \ldots\ldots, x_m)] \cap (y_1, y_2, \ldots, y_n)$$
$$= ((x_1] \vee (x_2] \vee \ldots\ldots\ldots \vee (x_m]) \cap ((y_1] \vee (y_2] \ldots\ldots\ldots\ldots \vee (y_n]) = \cup_{i,j} (x_i \wedge y_j)$$ for

any $x_1, x_2, \ldots\ldots\ldots x_m, y_1, y_2, \ldots\ldots\ldots y_n \in L$ (by 1.2.5) and so $J_f(L)$ is a distributive sub lattice of $J(L)$. c.f Cornish and Hickman [3].

A Lattice L is said to be finitely smooth if the intersection of two finitely generated ideals is itself finitely generated. For example, (i) distributive lattice, (ii) finite lattices, (iii) lattices, which are finitely smooth. Hickman in [20] exhibited a lattice which is not finitely smooth.

By Cornish and Hickman [3], we know that a lattice L is distributive if and only if $I(L)$ is so. Our next result shows that the case is not the some with the modularity.

Theorem 2.2.6: Let L be a lattice. If $I(L)$ is modular then L is also modular but the converse is not necessarily true.

Proof: Suppose $I(L)$ is modular. Let $a, b, c \in L$ with $c \leq a$ and $b \vee c$ exists.

Then $(c] \subseteq (a]$. Since $I(L)$ is modular, so $[a \wedge (b \vee c) = (a] \wedge ((b] \vee (c]$

$$= ((a] \wedge (b]) \vee (c] = (a \wedge b) \vee c].$$

Thus implies that $a \wedge (b \vee c) = (a \wedge b) \vee c$, and so L is modular.

Lattice L of figure 2.5 shows that the converse of this result is not true.

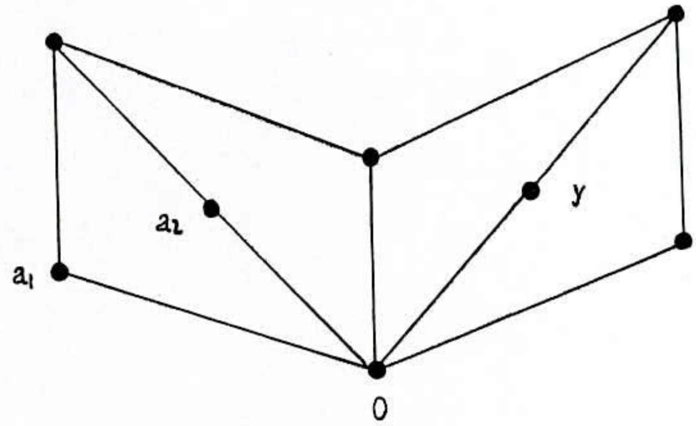


Fig 2.6

Notice that $(r]$ is modular for each $r \in L$. But in $I(L)$ clearly $\{(0],(a_1],(a_1,y],(a_2,b],L\}$ is a pentagonal sublattice.

A filter F of a lattice L is a non empty subset of L such that if $f_1 f_2 \in F$ and $x \in L$ with $f_1 \leq x$, then both $f_1 \wedge f_2$ and x are in F. A filter G is called a prime filter if $G \neq L$ and at least one of $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$ is in G whenever $x_1 \vee x_2 \ldots\ldots\ldots\ldots \vee x_n$ exists and is in G An ideal p is a lattice L is called a prime ideal it $P \neq L$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$. It is not hard to see that a filter F of a lattice L is prime if and only if $L - F$ is a prime ideal.

The set of filters of a lattice is an upper semilattice; yet it is not a lattice in general, as there is no guarantee that the intersection of two filters is non empty.

The join $F_1 \vee F_2$ of two filters is given by

$F_1 \vee F_2 = \{t \in L / t \geq f_1 \wedge f_2$ for some $f_1 \in F_1, f_2 \in F_2\}$.

The smallest filter containing a sub semi lattice H of L is $\{t \in L / t \geq h$ for some $h \in H$ $\}$ and is denoted by $[H)$.

Moreover, the description of the join of filters shows that for all

$a, b \in L, [a) \vee [b) = [a \wedge b)$ □

Following theorem and corollary is due to Noor and Rahman [24] which is an extension of a well known theorem of lattice theory .

Theorem 2.2.7: Let L be a lattice. The following conditions are equivalent.

(i) L is distributive.

(ii) For any ideal I and any filter F of L such that $I \wedge F = \phi$, there exists a prime ideal $P \supseteq I$ P$\supseteq$I and disjoint from F.

Corollary 2.2.8: A lattice L is distributives if and only if every ideal is the intersection of all prime ideals containing it.

Theorem 2.2.9: A lattice L is modular if and only if the ideal lattice of L is modular.

Proof: Let the lattice L be modular.

Also let $A, B, C \in I(L)$ be three members s.t $B \supseteq A$.

We show $A \cap (B \vee C) = B \vee (A \cap C)$.

Let $x \in A \wedge (B \vee C)$ be any element.

Then $x \in A$ and $x \in B \vee C$.

$\Rightarrow$ $x \in A$ and $x \leq b \vee c$ for some $b \in B$, $c \in C$.

Since $b \in B \subseteq A, x \vee b \in A$. Let $x \vee b = a$

Now $x \leq b \vee c, x \leq a \Rightarrow x \leq a \wedge (b \vee c)$

$\Rightarrow$ $x \leq b \vee (a \wedge c)$ as $a \geq b^1$ and L is modular

Again $a \wedge c \leq a, a \in A \Rightarrow a \wedge c \in a$.

$a \wedge c \leq c, c \in C \Rightarrow a \wedge c \in C$.

Thus $a \wedge c \in A \cap C$ and as $b \in B$, we find $x \in B \vee (A \cap C)$

i.e $A \cap (B \vee C) \subseteq B \vee (A \cap C)$

$B \vee (A \cap C) \subseteq A \cap (B \vee C)$ follows by modular inequality or to prove it independently.

Let $y \in B \vee (A \cap C)$.

Then $y \le b \vee k$ where $b \in B$, $k \in (A \cap C)$

Thus $y \le b \vee k(b \in B \subseteq A, k \in A \Rightarrow b \vee k \in A \Rightarrow b \vee k \in A \Rightarrow y \in A$.

Also $y \le b \vee k, b \in B, k \in C \Rightarrow y \in B \vee C$

i.e., $y \in A \cap (B \vee C)$

Showing that $B \vee (A \cap C) \subseteq A \cap (B \vee C)$.

Hence $A \wedge (B \cup C) = B \wedge (A \cap C)$ of that I(L) is modular.

Conversely, let $I(L)$ be modular, since L can be imbedded in to $I(L)$, it is isomorphic to a sub lattice of $I(L)$. This sub lattice must be modular as $I(L)$ is modular. Hence L is modular □

Lemma-2.2.10: Union of two ideal may not be an ideal.

Proof : Let us suppose two ideals $A = \{1,2\}$ $B = \{1,3\}$ of a lattice $L = \{1,2,3,4,6,12\}$ under divisibility. But $A \cup B$ is not an ideal, because $2,3 \in A \cup B$ but $2 \vee 3 = 6 \notin A \cup B$ □

Theorem 2.2.11: Every convex sub lattice of a lattice L is the intersection of an ideal and a dual ideal.

Proof: Let S be a convex sub lattice of a latice L. Also let $A = \{x \in L \exists s \in S, x \le S\}$.

Then $A \ne \varphi$ as $S \subseteq A$. Notice $s \le S$ $\forall$ $s \in S$.

We show A is an ideal of L. Let $x, y \in A$ be any elements.

Then, there exist $S_1, S \in S$, such that $x \le S_1, y \le S_2$

$\Rightarrow x \vee y \le s_1 \vee s_2 \Rightarrow x \vee y \in A$ as $s_1 \vee s_2 \in S$.

Again let $x \in A$ and $i \in L$ any elements. Then $x \le S_s$ for some $s \in S$.

Now $x \wedge l \le x \le s$ $\Rightarrow$ $x \wedge l \in$ A .

Hence A is an ideal of L.

Let $A^1 = \{x \in L / \exists s \in S, s \le x\}$ then by duality it follows that A^1 is a dual ideal of L.

We show $S = A \cap A^1$ $S \subseteq A \cap A^1$ (by def of A and A^1). Let $t \in A \wedge A^1$. Then $t \in A$ and $t \in A^1$

$\Rightarrow \exists\ s_1, s_2 \in S$, such that $s_1 \le t, t \le s_2$ i.e., $s_1 \le t \le s_2$, $t \in [s_1, s_2]$. Since S is convex sublattice , $s_1, s_2 \in S$, $[s_1, s_2] \le S \Rightarrow t \in s \Rightarrow a \wedge A^1 \subseteq S$.

Hence $S = A \cap A^1$ □

Theorem 2.2.12: Dual of a modular lattice is modular.

Proof: Let L be a modular lattice. Let $a, b, c \in L$, since L is modular.

$\therefore a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\ \forall\ a, b, c, \in L$.

Now we have to show that dual of L is modular

i.e., $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\ \forall\ a, b, c, \in L^1$.

Here L^1 is the dual of L. Let $a, b, c \in L^1$ be any there element, then

$$(a \wedge b) \vee (a \wedge c) = [(a \wedge b) \vee a] \wedge [(a \wedge b) \vee c]$$

$$= a \wedge b \vee [(a \wedge b) \vee c]$$

$$= a \wedge [(c \vee a) \wedge (c \vee b)]$$

$$= a \wedge (c \vee a) \wedge (c \vee b)$$

$$= a \wedge (b \vee c).$$

Therefore, L^1 is modular.

Hence dual of a modular lattice is modular □

Theorem:2.2.13 L is distributive if the identity

$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ holds in L.

Proof: Let L be a distributive lattice. Then

$$(x\vee y)\wedge(y\vee z)\wedge(z\vee x)=\{x\wedge[(y\vee z)\wedge(z\vee x)]\}\vee\{y\wedge[(y\vee z)\wedge(z\vee x)]$$
$$=[\{x\wedge(z\vee x)\}\wedge(y\vee z)]\vee[\{y\wedge)y\vee z)\}\wedge(z\vee x)]$$
$$=[x\wedge(y\vee z)]\vee[y\wedge(z\vee x)]$$
$$=(x\wedge y)\vee(x\wedge z)\vee(y\wedge z)\vee(y\wedge x)$$
$$=(x\wedge y)\vee(y\wedge z)\vee(z\wedge x).$$

Conversely, we first show that L is modular.

Let a,b,c be any three elements of L with $a\geq b$.Then

$$a\wedge(b\vee c)=[a\wedge(a\wedge c)]\wedge(b\vee c)$$
$$=(a\vee b)\wedge(a\vee c)\wedge(b\vee c)$$
$$=(a\vee b)\wedge(b\vee c)\wedge(c\vee a)$$
$$=(a\wedge b)\vee(b\vee c)\vee(c\wedge a)$$
$$=(b\vee(b\wedge c))\vee(c\wedge a)$$
$$=b\vee(a\wedge c)$$

i.e., L is modular.

Now for any $x,y,z\in L$,

$$x\wedge(y\vee z)=[x\wedge(x\vee z)]\wedge(y\vee z)$$
$$=[x\wedge(x\vee y)\wedge(x\vee z)]\wedge(y\vee x)$$
$$=x\wedge[(x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$$
$$=x\wedge[(y\wedge z)\vee(x\wedge y)\vee(z\wedge x)]$$

Now using modularity, as $x\geq x\wedge y$, $x\geq z\wedge x$ gives $x\geq(x\wedge y)\vee(z\wedge x)$,

we get $x\wedge(y\vee z)=[(x\wedge y)\vee(z\vee x)\vee(y\wedge z\wedge x)$

$$=(x\wedge y)\vee[(z\wedge x)\vee[(z\wedge x)\wedge y]]$$

$$(x\wedge y)\vee(z\wedge x).$$

Hence L is distributive □

. **Theorem 2.2.14** : Every distributive lattice is modular, but not conversely.

Proof: Let us suppose that L is distributive and $x,y,z\in L$.

Therefore, $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ $\forall x,y,z,\in L$, Let $x\geq y$. Then

$$
\begin{aligned}
x \wedge (y \vee z) &= [x \wedge (x \vee z)] \wedge (y \vee z) \\
&= (x \vee y) \wedge (x \vee z) \wedge (y \vee z) \\
&= (x \vee y) \wedge (y \vee z) \wedge (z \vee x) \\
&= (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \\
&= (y \vee (y \wedge z)) \vee (z \wedge x) \\
&= y \vee (x \wedge z).
\end{aligned}
$$

Therefore, L is modular.

Conversely, from the Fig the lattice M_5 is net distributive, but it is modular.

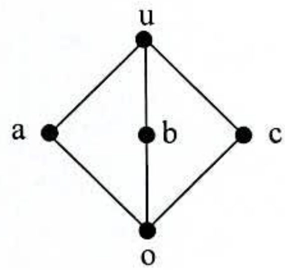


Fig-2.7

Notice $a \wedge (b \vee c) = a$, whereas $(a \wedge b) \vee (a \wedge c) = 0$

I.e., $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$.

Hence the theorem □

Theorem 2.2.15: Any chain is a distributive lattice.

Proof: Let x, y, z be any three members of a chain.

Then any two of these are comparable.

Suppose $x \leq y, \quad x \geq z, \quad y \leq z$.

Then $x \leq y \leq z \leq x \Rightarrow x = y = z$.

Thus $x \wedge (y \vee z)x = x = (x \wedge y) \vee (x \wedge z)$.

If $x \leq y, x \geq z, z \leq y$,

then $z \leq x, \; x \leq y, \; z \leq y$. Thus

$x \wedge (y \vee z) = x \wedge y = x$

$(x \wedge y) \vee (x \wedge z) = x \vee z = x$

One can check that under different cases $(x \leq y,\ x \leq z;\ x \leq z,\ x \geq y,\ x \geq y,\ x \geq z)$ the condition of destructivity holds and thus a chain is always a distributive lattice □

Cor: A non modular lattice contain at least five elements or a lattice with up to four elements is always modular.

Remark: It is possible that we may have a modular lattice which contains a pentagonal subset. Consider for instance, the lattice L of factors at 240. The lattice is given by the diagram.

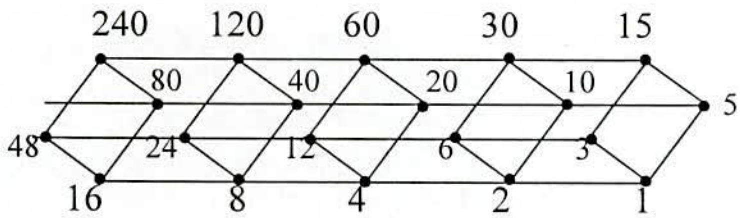


Fig-2.8

We notice $S = \{2,6,10,12,60\}$ is a pentagonal subset of L but not a Sub lattice For in L.

$10 \vee 6 = 30 \neq S$ $10 \vee 6 = 30 \neq S$. Again L is modular, as it is cardinal product of three chains,

$A = \{0 < 1 < 2 < 3\}$ $B = \{0 < 1\}$ $C = \{0 < 1\}$ and a chain being modular gives product of chains to be modular.

Theorem 2.2.16: A lattice L is modular iff it does not contain a pentagonal sublattice.

Theorem 2.2.17: A modular lattice is non distributive iff it contains a sublattice isomorphic with M_5.

Proof: Let L be a modular lattice which is not distributive. We know in any lattice.

$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a) \quad \forall\ a,b,c$.

Again a lattice is distributive iff the above is an equality.

Hence as L is not distributive $\exists$ at least three elements a,b,c in L, such that

$$(a \wedge b) \vee (b \vee c) \vee (c \vee a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$$

Let $p = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

$q = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Then $q < p$.

Consider now the tree elements

$r = p \wedge (q \vee a) = q \vee (p \wedge a)$
$s = p \wedge (q \vee b) = q \vee (p \wedge b)$
$t = p \wedge (q \vee c) = q \vee (p \wedge c).$

Then by definition of r, s, t we find $q \leq r \leq p,\ q \leq s \leq p,\ q \leq t \leq p$............(1)

we will show p, q, r, s, t form a sub lattice of L, isomorphic to M_5.

Now

$$\begin{aligned} r \wedge s &= \{p \wedge (q \vee a)\} \wedge \{p \wedge (q \vee b)] \\ &= p \wedge (q \vee a) \wedge (q \vee b) \\ &= p \wedge [a \vee (b \wedge c) \wedge \{b \vee (c \wedge a)\}] \end{aligned}$$

As $q = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ (using absorption)

$$\begin{aligned} q \vee a &= a \vee (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = a \vee (b \wedge c) \vee (c \wedge a) \\ &= a \vee (c \wedge a) \vee (b \wedge c) = a \vee (b \wedge c) \end{aligned}$$

Similarly $a \vee b = b \vee (c \wedge a)$.

Thus $r \wedge s = p \wedge [(c \wedge a) \vee (a \vee (b \wedge c) \wedge b]$

As $a \vee (b \wedge c) \geq a \geq a \wedge c$ and using modularity.

i.e., $r \wedge s = p \wedge [(c \wedge a) \vee \{b \wedge (b \wedge c) b \vee a\}]$

$$= p \wedge [(c \wedge a) \vee (b \wedge c) \vee (b \wedge a)\ ]$$

$$= p \wedge q = q.$$

By duality we can say that $r \vee s = p$.

Thus $r \wedge s = q < p = r \vee s$ and also then $r \neq s$

(indeed $r = s \Rightarrow r \wedge s = s = q, r \vee s = s = p$ or $p = q$

By symmetry, we can say

$$s \wedge t = q < p = s \vee t$$
$$t \wedge r = q < p = t \vee r$$

And $s \neq t, t \neq r$

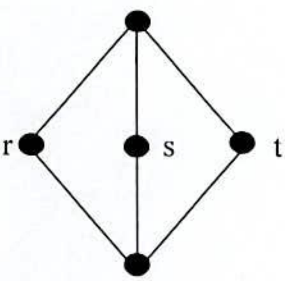


Fig-2.9

We now show equality does not hold in (1).

Suppose $q = r$ then as $q \leq s \leq p$, we get $r \leq s \Rightarrow r \wedge s = r$,

$r \vee s = s, p = s$.

Similarly, $q \leq t \leq p$ gives $r \leq t$

$r \wedge t = r, r \vee t = t, p = t$

or that $s = t$, which is not true.

Similarly, other equalities do not hold in (1),

Hence $q < r < p, q < s < p, q < t < p$.

Combining all the results prove above it is obvious that $\{p, q, r, s, t\}$ forms a sub lattice, which is isomorphic to M_5. Conversely, let L be a lattice which has a sub lattice isomorphic to M_5. Then L cannot be distributive as M_5 is not distributive. It is then that A lattice is distributive iff it does not contain a pentagonal sub lattice or M_5 sublattice □

Congruence

Definition (Congruence): An equivalence relation $\Theta_{\wedge}$ (that is a reflexive, symmetric and transitive binary relation) of a lattice L is called a congruence relation if $a_i \equiv b_i\,(\Theta)$ for $i=1,2\,(a_i\, b_i \in L$ then (i) $a_1 \wedge a_2 \equiv b_1 \wedge b_2\ (\Theta)$, and (ii) $a_1 \vee a_2 \equiv b_1 \vee b_2\ (\Theta)$ provided $a_1 \vee a_2$ and $b_1 \vee b_2$ exists.

It can be easily shown that for an equivalence relation Θ on L, the above conditions are equivalent to the conditions that for $a,b \in L$ if $a \equiv b\ (\Theta)$, then (i) $a \wedge t = b \wedge t(\Theta)$ for all $t \in L$ and (ii) $a \vee t = b \vee t\ (\Theta)$ for all $t \in L$, provided both $a \vee t$ and $b \vee t$ exist.

The set $C(L)$ of all congruence on L is an algebraic closure system on $L \times L$ and hence , when ordered by set inclusion, is an algebraic lattice.

Cornish and Hickman [3] showed that for an ideal I of a distributive lattice L, the relation $\Theta(I)$ defined by $a \equiv b\ (\Theta(1))$ if and only if $(a] \vee I = (b] \vee I$ is the smallest congruence having I as a congruence class. Moreover the equivalence relation R (I) defined $yRa \equiv b\,(R\ (1))$ if and only if for any $l \in L, a \wedge l \in \mathrm{I}$ is equivalent to $b \wedge l \in I$ is the largest congruence having I as a congruence class.

Suppose L is a distributive lattice and $a \in L$, we will use Θ_a as an abbreviation for $\Theta((a])$. Moreover Ψ_a denote the congruence, defined by $x \equiv y(\Psi_a)$if and only if $x \wedge a = y \wedge a$.

Cornish and Hickman [3] also showed that for any two elements a, b of a distributive lattice L with $x \leq y$, the smallest congruence identifying x and y is equal to $\Psi_x \cap \Theta$ and we denote if by $\Theta(x, y)$. Also in a distributive lattice L , they observed that if L

has a smallest element 0, then clearly $\Theta_a = \Theta(0, a)$ for any $a \in L$. Moreover, it is easy to see that (i) $\Theta_x \vee \Psi_x = l,$ the largest congruence of L.

(ii) $\Theta_x \cap \Psi_x = w$, the smallest congruence of L and

(iii) $\Theta(x,y)^1 = \Theta_x \vee \Psi_y$, where $x \le y$.

Now suppose L is an arbitrary lattice and $E(L)$ denotes its lattice of equivalence relations F or $\varphi_1, \varphi_2 \in E(L)$, $\varphi_1 \vee \varphi_2$ denotes their suprimum; $a \equiv b(\varphi_1 \vee \varphi_2)$ if and only it there exists $a = z_0, z_1, \ldots\ldots\ldots z_n = b$ such that $z_{i-1} \equiv z_i$ $(\varphi_1\ or\ \varphi_2)$ for $i = 1,2,,\ldots\ldots n$.

Theorem 2.3.1: A lattice L is algebraic iff it is isomorphic to the lattice of all ideals of a join-semi lattice with 0.

Proof: Let F be a join-semilattice with 0; we want to prove that $I(F)$ is algebraic. We know that $I(F)$ is complete. We claim that for $a \in F, (a]$ is a compact element of $I(F)$. Let $X \subseteq I(F)$ and let $(a] \subseteq \{I / I \in X\}$.

But we have $\vee(I / I \in x) = \{x : x \le t_0 \vee \ldots\ldots\ldots \vee t_{n-1}, t_1 \in I_1, I_1 \in x\}$.

Therefore $a \le t_0 \vee \ldots\ldots\ldots \vee t_{n-1}, t_1 \in I, I_1 \in x$. Thus with $x_1 = \{I_0 \ldots\ldots I_{n-1}\}$

$(a] \subseteq \vee(I / I \in x_1)$. Since for any $I \in I(F)$

We have $I = \vee((a] / a \in I)$. we see that $I(F)$ is algebraic.

Now let L be an algebraic lattice and let F be the set of compact elements of L. Obviously $0 \in F$, let $a, b \in F$, $a \vee b \le VX, X \subseteq L$. Then $a \le a \vee b \le VX$ and so $a \le Vx_0$ for some finite $X_0 \subseteq X$, similarly $b \le v_1$ for some finite $X_1 \in X$. Thus $a \vee b \le V(X_0 \cup X_1)$, and $X_0 \cup X_1$ is a finite subset of X. So $a \vee b \in F$.

Therefore, (F;V) is a join-semi lattice with 0. Consider the map $\varphi: a \to \{x / x \in F, x \le a\}$ $a \in L$. Obviously φ maps L into $I(F)$, by the definition of an algebraic lattice, $a = va\varphi$, and thus φ is one-to-one.

To prove that φ is onto, let $I \in I(F)$. $a = vI$ .Then $a\varphi \equiv l$, let $x \in a\varphi$. Then $x \le VI$, So that by the compactness of x, $x \leq VI_1$ for some finite $I_1 \subseteq I$. Therefore, $x \in I$ proving that $a\varphi \subseteq I$.

Consequently $a\varphi = I$ and so φ is on to. Thus φ is an isomorphism □

Now we connect the foregoing with congruence lattices.

Theorem 2.3.2: For any lattice L , $C(L)$ is a distributive sublattice of $E(L)$.

Proof: Suppose $\Theta, \varphi \in C(L)$. Define Ψ to be the supremum of Θ and φ in the lattice equivalence relations $E(L)$ on L .

Let $a \equiv b(\Psi)$. Then there exists $a = z_0, z_1 .... z_n = b$ such that $z_{i-1} \equiv z_i\ (\Theta or \varphi)$.

Thus for any $t \in L$ $z_{i-1} \wedge t \equiv z_i \wedge t(\Theta or \varphi)$ as $\Theta, \varphi \in c(L)$.

Hence $a \wedge t \equiv b \wedge t(\Psi)$and consequently Ψ is semi lattice congruence. Then in particular $a \wedge b = a(\Psi)$ and $a \wedge b \equiv b(\Psi)$. To show that Ψ is a congruence, let $a \equiv b(\Psi)$ with $a \leq b$, and choose any $t \in L$ such that both $a \vee t$ and $b \vee t$ exist. Then there exists $z_0, z_1, z_2, ... z_n$, such that $a = z_0, z_n = b$ and $z_{i-1} = z_i(\Theta or \varphi)$

Put $w_i = z_1 \wedge b$ for all $i = o,1,..........n,$ then $a = w_0, w_n = b$ $w_{i-1} \equiv w_i, (\Theta or \varphi)$.

Hence by the upper bound property $w_i \vee t$ exists for all $i = o,1 .... n(as\, w_i\ , t \leq b \vee t)$ and $w_{i-1} \vee t \equiv w_i \vee t(\Theta or \varphi)$ for all $i = 1,2,..........n(as \Theta, \varphi \in C(L))$. i.e., $a \vee t = b \vee t(\Psi)$.

Then Ψ is congruence on L . Therefore $C(L)$ is a sub lattice of the lattice $E(L)$.

To show the distributivity of $C(L)$, let $a \equiv b(\Theta \cap (\Theta_1 \vee \Theta_2)$.

Then $a \wedge b \equiv y(\Theta)$ and $(\Theta_1 v \Theta_2)$. Also, $a \wedge b \equiv a(\Theta)$ and $(\Theta_1 \vee \Theta_2)$.

Since $a \wedge b = b(\Theta_1 \vee \Theta_2)$, there exists $t_0, t_1 .... t_n$ such that (as we have seen in the proof of the first part) $a \wedge b = t_0, t_n = b, t_{i-1} \equiv t_1\ (\Theta_1\ or\, \Theta_2)$ and $a \wedge b = t_0 \leq t_i \leq b$ for each $i = 0,1 .... n.$ Hence $t_{i-1} = t_i\ (\Theta)$ for all $i = 1,2,............ n$ and so

$t_{i-1} \equiv t_i(\Theta \cap \Theta_1) or (\Theta \cap \Theta_2\ )$.

Thus $a \wedge b \equiv b((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$. By symmetry, $a \wedge b \equiv a((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2)$ and the proof completes by transitivity of the congruences.

In lattice theory it is well known that a lattice is distributive if and only if every ideal is a class of some congruence. Following theorem gives a generalization of this result in case of lattices □

Theorem:2.3.3: L is distributive if and only if every ideal is a class of some congruence.

Proof: Suppose L is distributive. Then for each ideal I of L, $\Theta(I)$ is the smallest congruence containing I as a class.

To prove the converse, let each ideal of L be a congruence class with respect to some congruence on L. Supposes L is not distributive. Then by theorem:2.1.2 we have either N_5 (Figure 2.2) or M_5 (Figure2.3) as a sublattice of L. In both cases consider $I = (a]$ and suppose I is a congruence class with respect ot Θ. Since $d \in I$, $d \equiv a(\Theta)$.

Now $d = b \wedge c = b \wedge (a \vee c) \equiv b \wedge (d \vee c) = b \wedge c = d(\Theta)$ I.e $b \equiv d(\Theta)$and this implies $b \in I$, I.e $b \leq a$ which is a contradiction. Thus L is distributive.

An equivalence relation C on a lattice L is called a congruence relation if $a_1 C b_1$ and $a_2 C b_2$ imply $(a_1 \wedge a_2) C (b_1 \wedge b_2)$ and $(a_1 \vee a_2) C (b_1 \vee b_2)$.

We know C would partition L in equivalence classes, where for any $a \in L$ equivalence class of 'a' is given by $C(a) = \{x \in L / xCa\}$ □

Theorem 2.3.4: Let L and M be lattices and suppose C_1 and C_2 are congruence relations on L and M respectively. Then a relation $C = C_1 \times C_2$ on $L \times M$ by $(a,b)C(x,y) \Rightarrow aC_1x, bC_2y, a, x \in l, b, y \in M$ then $C = C_1 \times C_2$ is a congruence relation on $L \times M$. Conversely any congruence relation on $L \times M$ is of this type.

Proof: Since $aC_1a, bC_2b, \quad \forall a \in L, b \in M$.

We get $(a,b)C(a,b) \quad \forall (a,b) \in L \times M$ or that C is reflexive.

Again let $(a,b)C(x,y) \Rightarrow aC_1x, bC_2y$

$$\Rightarrow xC_1a, yC_2b$$

$$\Rightarrow (x,y)C(a,b)$$

$\Rightarrow C$ is symmetric.

Similarly it is seen that C is transitive.

Let now $(a,b)C(x,y)$, $(p,q)C(r,s)$

$\Rightarrow \ aC_1x, bC_2y, pC_1r, qC_1s$

$\Rightarrow (a \wedge p)C_1(x \wedge r)$ and $(b \wedge q)C_2(y \wedge s)$

$\Rightarrow (a \wedge p, b \wedge q)C(x \wedge r, y \wedge s)$

$\Rightarrow (a,b) \wedge (p,q)C(x,y) \wedge (r,s)$.

Similarly, we can prove that $(a,b) \vee (p,q)C(x,y) \vee (r,s)$.

Hence C is a congruence relation on $L \times M$.

Conversely, let C be a congruence relation on $L \times M$.

Define a relation C_1 on L by $aC_1b \Leftrightarrow (a,x)C(b,x)$ for some $x \in M$, $a,b \in L$ Let $y \in M$ be any element, then since $(a \wedge b, y)$ $(a \wedge b, y) \in L \times M$ and C is a congruence relation on $L \times M$.

We get $(a \wedge b, y)C(a \wedge b, y)$, similarly $(a \vee b, y)C(a \vee b, y)$.

Now $(a,x)C(b,x), (a \wedge b, y) \vee (a,x)C(a \wedge b, y) \vee (b,x)$

$\Leftrightarrow (a, y \vee x)C(b, y \vee x)$

$\Leftrightarrow (a \vee b, y) \wedge (a, y \vee x)C(a \vee b, y)(b, y \vee x)$

$\Leftrightarrow (a,y)C(b,y)$.

We thus notice $(a,x)C(b,x)$ for some $x \in M$ is equivalent to saying that $(a,x)C(b,x)$ for all $x \in M$.

So we define $aC_1b \Leftrightarrow (a,x)C(b,x)$ aC₁b for all $x \in M$. It is easy to verify that C_1 is a congruence relation on L. Similarly, we define a relation C_2 on M by $aC_2b \Rightarrow ((x,a)C(x,b)$ aC₂b $\forall x \in L, (a,b \in M)$. We claim $C = C_1 \times C_2$

Let $(a,b)C(p,q), a,p \in L, b,q \in M$ then $(a \vee p, b \wedge q) \wedge (a,b)C(a \vee p, b \wedge q) \wedge (p,q)$

$\Rightarrow (a, b \wedge q)C(p, b \wedge q)$ for some $b \wedge q \in M$. $\Rightarrow aCp$

we can say bC_2q and thus get $(a,b)C_1 \times C_2(p,q)$. Again, if $(a,b)C_1 \times C_2(p,q)$, them aC_1p and bC_1q $\Rightarrow (a,x)C(p,x)$ and $(y,b)C(y,q)$ for all $x \in M, y \in L$.

In particular

$(a, b \wedge q)C\ (p, b \wedge q)$ *and* $(a \wedge p, b)C(a \wedge p, q)$ $\quad x = b \wedge q \in M$

$\quad y = a \wedge p \in L$

$\Rightarrow (a, b \wedge q) \vee (a \wedge p, b) C\ (p, b \wedge q) \vee (a \wedge p, q)$
$\Rightarrow (a,b)C\ (p,q)$.

Hence $(a,b)C(p,q) \Leftrightarrow (a,b)c_1 \times c_2(p,q)$

or that $C = C_1 \times C_2$ □

Definition(Convex sublattice): The subset K of the lattice L is called convex sub lattice if $a,b \in K, c \in L$ and $a \le c \le b$ imply that $c \in K$.

Theorem 2.3.5: Let Θ be a congruence relation of L. Then for every $a \in L[a]\ \Theta$ is a convex sublattice.

Proof: Let $x, y \in [a]\ \Theta$; then $x \equiv a\Theta$ and $y \equiv a\Theta$.

Therefore, $x \wedge y \equiv\ a \wedge a = a(\Theta)$, *and* $x \vee y = a \vee a = a\Theta$, *proving that* $[a]\ \Theta$ *is a* sublattice. If $x \le t \le y, x, x \in [a]\Theta$, then $x = a(\Theta)$ and $y = a(\Theta)$.

Therefore, $t = t \wedge y = t \wedge a(\Theta)$, and $t = t \vee x = (t \wedge a) \vee x \equiv (t \wedge a) \vee a = a\ (\Theta)$, proving that $[\,a\,]\ \Theta$ is convex □

Theorem 2.3.6: (The Homomorphism Theorem) : Every homomorphism image of a lattice L is isomorphic to a suitable quotient lattice of L. In fact, if $\phi : L \to L_1$ is a homomorphism of L onto L and if Θ is the congruence relation defined by $x \equiv y\Theta$ iff $\phi(x) = \phi(y)$, then L, $J\ \Theta\ \equiv\ L_1$ is an isomorphism and is given by

$\psi : [x]\Theta \to \phi(x), x\Theta L_1$

Proof: Since Φ is a homomorphism and (Θ) is obviously a congruence, to prove that Ψ is an isomorphism we need to check

i) To show that Θ is well defined: let $[x]\Theta - [y](\Theta)$.

Then $\phi(x) = P(y) \Rightarrow ([x]\Theta)\psi \equiv ([y]\Theta)\psi$ i.e., Ψ is well defined.

ii) To show that Ψ is one-one $\Psi[x] = \Psi[y], \Theta \Rightarrow \phi(x) = \phi(y)$ then $x \equiv y(\Theta)$ and so $[x](\Theta) \equiv [y](\Theta)$ i.e Ψ is one-one.

iii) To show that Ψ is onto: Let $x \in L$. Since Φ is onto. There is any $y \in L$ with $\phi(y) = x$. Thus $[y](\Theta)y\psi = x$ i.e., Ψ is onto .

iv) To show that Ψ is a homomorphism: Let $[x]\Theta, [y]\Theta \in L / \Theta$,

Therefore $\psi([x]\Theta \wedge [y]\Theta) = \psi([x \wedge y]\Theta) = \phi(x) \wedge \phi(y)$

$= \psi(x)\Theta \wedge \psi(y\Theta)$ and

$\psi([x]\Theta \vee [y]\Theta) = \psi([x \vee y]\Theta) = \phi(x \vee y) = \phi(x) \vee \phi(y)\psi([x]\Theta \vee \psi(y)\Theta$

i.e., Ψ is homomorphism then the theorem is proved □

2.4 Length and covering conditions

A finite chain with n elements is said to have length n-1. We say a covers b if $b \leq a$ and there exists no c such that $b \leq c \leq a$.

A chain $x_1 < x_2 < .... < x_n$ is called a minimal chain if each x_{i+1} covers x_i. Suppose now $[a,b]$ is an interval in a lattice and if amongst all chains from a to b, there is one of maximum length n. We say $[a,b]$ has tength n. Thus it is the sup of lengths of chains from a to b. We denote it by $l[a,b] = n$. In case some chains from a to b have infinite length here $[a,b]$ has infinite length.

Let L be a lattice with least element \`o' and greatest element u then as $L = [0,u]$, length of L is defined to be length of the interval $[0,u]$.

All finite lattices have finite length, infinite lattices can also have finite length as the lattice given by Fig has finite length 2 but it is infinite.

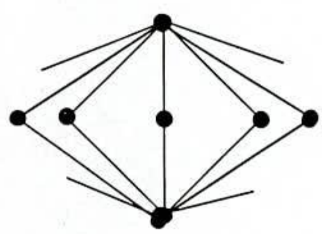


Fig-2.10

Theorem 2.4.1: Length of a pentagonal lattice is 3.

Proof: Consider the pentagonal lattice as shown in the Fig .

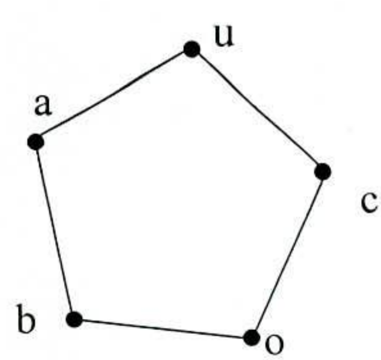


Fig-2.11

It has five chains. $0 < u$, $0 < a < u$, $0 < b < u$, $0 < c < u$, $0 < b < a < u$ from 0 to u. The last two being maximal chains. The chains have lengths 1, 2, 2, 2, 3. Therefore $l[0,u] = 3$ and hence length of the pentagonal lattice is 3 □

Jordan-Dedekind condition: Let L be a lattice of finite length. Then L satisfies the Jordan-Dedekind condition if all maximal chains between same end points have same length.

Remark: The pentagonal lattice does not satisfy the jordain- Dedekind condition. Because there are two maximal chains from o to u and have different lengths 2 and 3 □

Theorem 2.4.2: Let L be a lattice of finite length. Suppose in L whenever x, y cover $x \wedge y$ implies $x \vee y$ covers x and y. Then L Wsatisfies the Jordain- Dedekind condition.

Proof: Let a, b be any two comparable points $(a \leq b)$. we show all maximal chains from a to b have same length $l[a,b]$. Since all chains from a to b are finite, at least one maximal chain exists of finite length from a to b. We show all maximal chains are of the same length.

We prove the result by induction on n, the length $l[a,b]$. If $l[a,b] = 1$, then b covers a and thus there is only one maximal chain from a to b with length 1 and hence the result holds for n= 1.

Let the result be true for x =m-1

Let $a < x_1 < x_2 < \ldots\ldots\ldots\ldots < x_m = b$

$a < y_1 < y_2 < \ldots\ldots\ldots\ldots\ldots < y_k = b$ be two maximal chains from a to b of lengths m and k we show $k = m$.

Case (i) If $x_1 = y$ then $x_1 < x_2 < \ldots\ldots\ldots\ldots\ldots < x_m = b$ $y_1 < y_2 < \ldots\ldots\ldots\ldots\ldots < y_k = b$ .are two maximal chains from x_1 to b with length $m-1, k-1$ and as the result holds for $m-1, k-1 = m-1 \Rightarrow k = m$.

Case (ii) $x_1 \neq y_1$. Here x_1 and y_1 cover $a = x_1 \wedge y_1$.

Thus by given condition $x_1 \vee y_1$ cover x_1 and y_1.

Let $x_1 \vee y_1 = t$. Since $x < b_1, y < b_1 = x_1 \vee y_1 \leq b$ and we find t and b are comparable.

Let $t < z_1 < z_2 < \ldots\ldots\ldots\ldots z_i = b$

be a maximal chain from t to b with length i.

Now $x_1 < x_2 < \ldots\ldots\ldots\ldots < x_m = b \qquad x_1 < t < z_1, \ldots\ldots, z_i = b$

are two maximal chains from x_1 to b of lengths $m-1$ and $i+1$ (Note t covers x_1).

But the result holds for m-1 and thus $i+1 = m-1$.

Again, the chains $y_1 < y_2 < \ldots\ldots\ldots\ldots < y_k = b$, $y_1 < t < z_1 < \ldots\ldots\ldots\ldots z_i = b$

are maximal chains from y_1 to b with length $k-1$, and $i+1$ i.e are maximal chains from y_1 to b with lengths $k-1$ and $m-1$.

But result holds for $m-1$, and so $k-1 = m-1 \Rightarrow k = m$

i.e., the result holds for n=m.

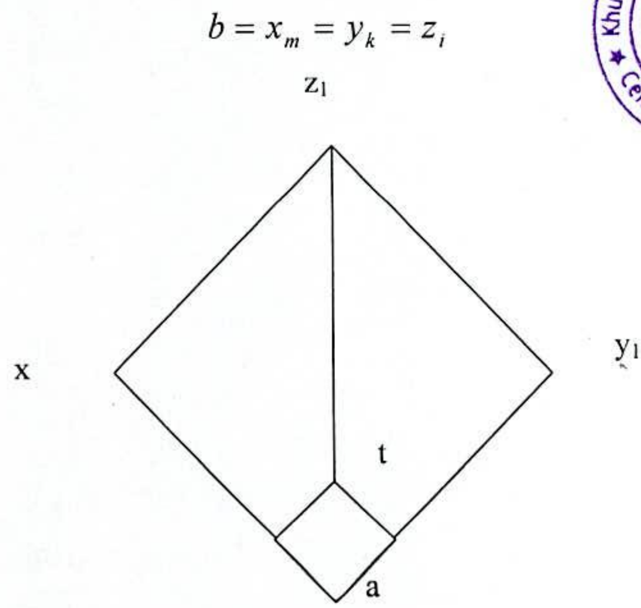


Fig-2.12

Hence by induction hypothesis, the result holds for all n and our assertion is proved □

Atom: An element a in lattice L called an atom if it covers o. i.e a is atom iff $a \neq 0$ and $x \wedge a = a$ or $x \wedge a = 0$, $\forall\ x \in L$.

Dual atom: An element b is called dual atom if the greatest element u of a lattice cover b.

Complements: Let $[a,b]$ be an interval in a lattice L. Let $x \in [a,b]]$ be any element if $\exists\ y \in L$, s.t, $x \wedge y = a, x \vee y = b$ we say y is a complement of x relative to $[a,b]$.

Theorem 2.4.3: No ideal of a complemented lattice which is a proper sub lattice can contain both an element and its complement.

Proof: Let L be a complemented lattice. Then $0, u \in L$. Let I be an ideal of L such that I is a proper sublattice of L. Suppose $\exists$ an element a in I such that its complement a^1 is also in I .Then $a \wedge a^1 = 0, a \vee a^1 = u$ Since I is a sub lattice, $a \wedge a^1, a \vee a^1$ are in I. I.e $0, u \in I$,

Now if $I \in L$ be any element then as $U \in I, I \wedge u \bigcap I \Rightarrow 1 \in I \Rightarrow L \subseteq i \Rightarrow I = L$,

a contradiction. Hence the theorem □

Theorem 2.4.4: Let L be a uniquely complemented lattice and let a be an atom in L. Then a^1 i.e the complement of a is a dual atom of L.

Proof: Since L is uniquely complemented lattice, every element has a unique complement.

Suppose a^1 is not a dual atom, then $\exists$ at least on x, s,t,

$$a^1 < x < u \Rightarrow a^1 \vee a \leq x \vee a$$

$$\Rightarrow u \leq x \leq u \quad \Rightarrow \quad u = x \vee a$$

Now if $a \leq x$ then $x \vee a = x \Rightarrow x = u$, not true.

Again if $a \leq x$, then $a \wedge x = 0$ (note a is an atom). Thus $a \wedge x = 0, a \vee x = u \Rightarrow x = a^1$

$a \wedge x$, again a contradiction.

Hence a^1 is a dual atom □

CHAPTER THREE

Standard Element of a lattice

Introduction: Standard elements in lattices were first studied in depth by Gratzer and schmid [15]. Since then little Attenton has been paid to these notions. A lower semi lattice is said to have the upper bound property if the supremum of any two elements automatically exists when they share a common upper bound according to Gratzer and Schmidt [15] if a is an element of a lattice L then,

(i) a is called distributive if

$$(a \vee (r \wedge s) = (a \vee r) \wedge (a \vee s) \ \textit{for all} \ \ r,s \in L;$$

(ii) a is called standard if

$$r \wedge (s \vee a) = (r \wedge s) \vee (r \wedge a) \ \textit{for all} \ r,s \in L;$$

(iii) a is called neutral if the sub lattice generated by r,s and a is distributive for all $r,s \in L$

i.e $(a \wedge r) \vee (r \wedge s) \vee (s \wedge a) = (a \vee r) \wedge (r \vee s) \wedge (s \vee a)$ for all $r,s \in L$.

It is easily seen that a standard elements is distributive and a neutral element is both standard and distributive. In a distributive lattice, the three notions coincide .It was shown by Gratzer and Schmidt [15] that an element n in a lattice L is neutral if and only it for all $r,s \in L$, $r \wedge (s \vee n) = (r \wedge s) \vee (r \wedge n)$ *and* $n \wedge (r \vee s) = (n \wedge r) \vee (n \wedge s)$. Also Gratzer [11] has shown that an element n in a lattice L is neutral if and only if

$(n \wedge r) \vee (r \wedge s) \vee (s \wedge n) = (n \vee r) \wedge (r \vee s) \wedge (s \vee n)$ for all $r,s \in L$.

The following results are well known C.f Gratzer [11, Theerem 9 P, 143] the supremum of two distributive elements are distributive; both the infimum and supremum of two standard elements are standard; both the infimum and supremum of two neutral elements are neutral. On the other land , the following example due to Gratzer [11,p144] shows that the infimum of two distributive elements is not necessarily distributive in Figure 3.1 both r and s are distributive whereas $r \wedge s$ is not.

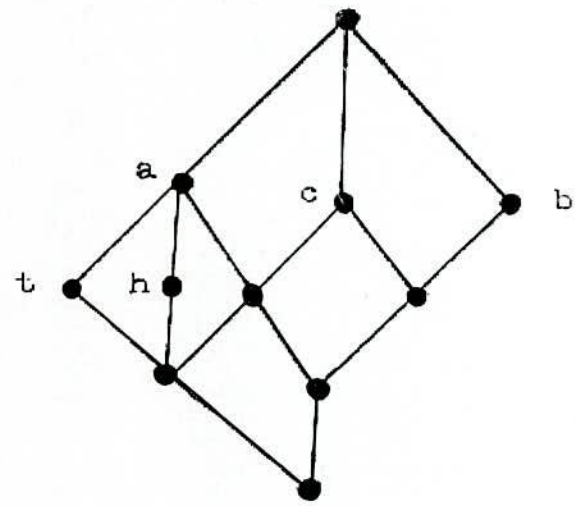


Figure 3.1

Cornish and Noor in [4] generalized the concepts of standard and neutral elements to lattices. They have also introduced the notion of new type of element. They preferred to call it as a strongly distributive element, as in case of lattices such an element stands between a distributive and a standard element.

In section 1 of this chapter we give a description on standard neutral and strongly distributive elements of a lattice,

In section 2 we discuss on standard elements in a weakly modular lattice. we show that in a weakly modules lattice , every strongly distributive element is neutral Thus in particular every standard element is neutral in a modular lattice.which is a generalization of [15, corr.2.3 and 2.4]

3.1. Standard and Neutral Elements of a Lattice.

3.1.1. Defination (Standard element): Let L be a lattice and s be an element of L.

Then s is said to standard if for all $x, y, t \in L$, $t \wedge [x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$

Obviously, any element of a distributive lattice is standard. Now suppose s is a standard element of a lattice L c.f Introduction, then for all

$x, y, t \in L, t \wedge [x \wedge y) \vee (x \wedge s)] = t \wedge [x \wedge (y \vee s)] = (t \wedge x) \wedge (y \vee s) = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.

This and a part of following proposition show that the two concepts coincide in a lattice.

Proposition 3.1.2:

The following two conditions on an arbitrary element s of a lattice L are equivalent.

(i) For any $x, y \in L, x \wedge (y \wedge s) = (x \wedge y) \vee (x \wedge s)$ whenever $y \vee s$ exists.

(ii) (a) if $x \vee s$ and $y \vee s$ exist for any $x, y \in L$ Then $(x \wedge y) \vee s$ exists and

$(x \wedge y) \vee x = (x \vee s) \wedge (y \vee s)$.

(b) For any $x, y \in L$ for which $x \vee s$ and $y \vee s$ exist $x \wedge s \geq y \wedge s$

and $x \vee s \geq y \vee s$ imply $x \geq y$.

Moreover both (i) and (ii) are necessary for L to be standaed but are not sufficient.

Proof: (i) inplies (ii) suppose $x, y \in L$ are such that $x \vee s$ and $y \vee s$ exist. Then $(x \wedge y) \vee s$ exists because of the upper bound property of L. Due to (i)

$(x \vee s) \wedge (y \vee s) = [(x \vee s) \wedge y] \vee [(x \vee s) \wedge s] = (x \wedge y) \vee (s \wedge y) \vee s = (x \wedge y) \vee s$.

Also if $x \wedge s \geq y \wedge s$ and $x \vee s \geq y \vee s$, then

$X = x \wedge (x \vee s) \geq x \vee ((y \vee s) = (x \wedge y) \vee (x \wedge s)$

by (i) $\geq (x \wedge y) \vee (y \wedge s) = y \vee (x \vee s) \geq y \wedge (y \vee s) = y$

(ii) implies (i) suppose $x, y \in L$ and $y \vee s$ exists.

Let $P = x \wedge (y \vee s)$ and $q = (x \wedge y) \vee (x \wedge s)$.

Now $p \wedge s = x \wedge s \leq q = (x \wedge y) \vee (x \wedge s) \leq x \wedge (y \vee s) = p$.

Hence $p \wedge s \leq q \wedge s \leq p \wedge s$.

That is $p \wedge s = q \wedge s$. Observe that as $p, s \leq y \vee s, p \vee s$ exists and since

$p = p \wedge (y \vee s), p \vee s = [p \wedge (y \vee s)] \vee s$

$= (p \vee s) \wedge (y \vee s)$ by (ii) (a) $= (p \wedge y) \vee s (by (ii)(a)) = (x \wedge y) \vee s$

$= (x \wedge y) \vee (x \wedge s) \vee s = q \vee s$. Then by (ii) (b),

$p = q$, that is (i) holds.

Now suppose S is standard in L, $x, y \in L$ and $y \vee s$ exists, then letting $y \vee s = r$.

We obtain $x \wedge (y \vee s) = x \wedge [(r \wedge y) \vee (r \wedge s)] = (x \wedge r \wedge y) \vee (x \wedge r \wedge s) = (x \wedge y) \vee (x \wedge s)$ as S is standard thus (i) and (ii) holds.

Finally, consider the lattice L in Figure 3.2. Here for all $x, y \in L$ the condition (i) holds but $d \wedge [(c \wedge a) \vee (c \wedge s)] > (d \wedge c \wedge a) \vee (d \wedge c \wedge s)$

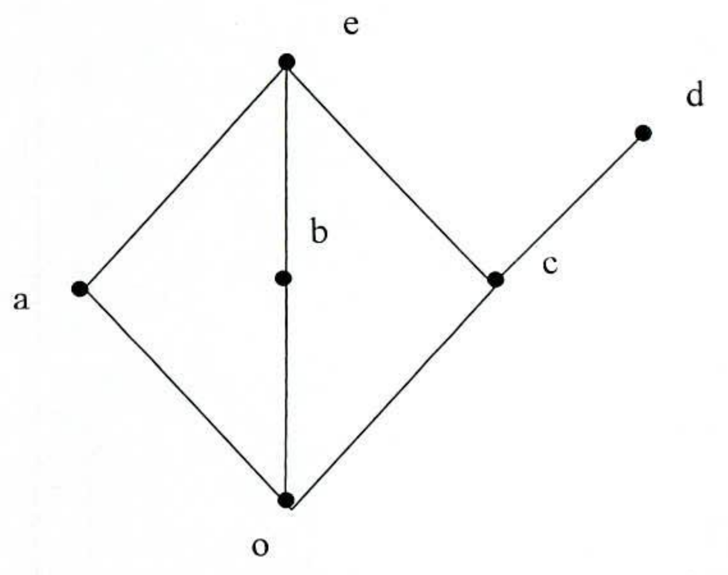


Figure 3.2

On many occasions we find that a long computation is required to prove that a given binary relation is congruence. Such computations are after facilitated by the following useful Lemma which is due to Cornish and Noor [4.lemma 2.3]. This is an extension of a characterization of lattice congruence. e.f

Gratzer [20 lemma 8,p-24] and also Gratzer and schimdt [15] to lattice.

Lemma 3.1.3: A reflexive symmetric binary relation θ on a lattice L is a congruence if and only if, for any $x, y, z, t \in L$,

(i) $x \equiv y(\theta)$ if and only if $x \wedge y = x(\theta)$ and $x \wedge y = y(\theta)$

(ii) $x \leq y \leq z$, $x \equiv y(\theta)$ and $y \equiv z(\theta)$ imply $x \equiv z(\theta)$

(iii) $x \leq y$ and $x \equiv y(\theta)$ imply that $x \wedge y \equiv y \wedge t(\theta)$ and

$x \vee t \equiv y \vee t(\theta)$ whenever $x \vee y$ and $y \vee t$ exist □

We now proceed to characterization of a standard element. The following theorem is a characterization of a standard element in a lattice, which is due to Greatzer and Schmidt [17]

The following conditions upon an element s of the lattice L are equivalent.

(i) s is a standard element.

(ii) The relation θ, defined by $X \equiv y(\theta)$ if and only if

$(x \wedge y) \vee s = x \vee y$, for some $s_1 \leq s$ is a congruence relation.

(iii) For each ideal K, $(s] \vee k = \{s_1 \vee k : s_1 \leq s, k \in K\}$

(iv) (s] is a standard element of the ideal lattice of L.

Theorem3.1.4: For an element s of a lattice L.

The following conditions are hold.

(i) s is a standard element.

(ii) The binary relation θ, which is defined by $x \equiv y(\theta)$

if and only if $X = (x \wedge y) \vee (x \wedge s)$ and $y = (x \wedge y) \vee (y \wedge s)$, is a congruence relation.

(iii) The binary relation ϕ which is defined by $x \equiv y(\phi)$ if and only if

$(x \wedge t) \vee (t \wedge s) = (y \wedge t) \vee (y \wedge s)$ for all $t \in L$ is a congruence relation.

(iv) For each ideal k, (s] k = {s k : s s, k and s k exists}

(v) $(s]$ is a standard element of the ideal lattice of L.

Moreover θ and ϕ of (ii) and (iii) respectively represent the some congruence $\vee iz$. θ_s. The smallest congruence of L having (s] as congruence class.

Proof: (i) implies (ii) Let θ be the binary relation, such that $x \equiv y(\theta)$ if and only if $x = (x \wedge y) \vee (x \wedge s)$ and $y = (x \wedge y) \vee (y \wedge s)$ clearly, θ is reflexive and symmetric. Now $x \equiv y(\theta)$ implies

$x = (x \wedge y) \vee (x \wedge s) = x \wedge (x \wedge y) \vee (x \wedge s)$. Also

$x \wedge y = (x \wedge (x \wedge y)) \vee ((x \wedge y) \wedge s)$ and so $x \equiv x \wedge y(\theta)$

similarly $y \equiv x \wedge y(\theta)$ conversely $x \wedge y \equiv x(\theta)$ and

$x \wedge y \equiv y(\theta)$. Certainly imply $x \equiv y(\theta)$.

Suppose $x \leq y \leq z$ and $x \equiv y(\theta); y \equiv z(\theta)$.

Then $z = y \vee (z \wedge s)$ and $y = x \vee (y \wedge s)$. So $z = x \vee (y \wedge s) \vee (z \wedge s)$

$= x \vee (z \wedge s)$. And it follows that $x \equiv z(\theta)$.

Now let $x \leq y, x \equiv y(\theta)$ and $x \vee t, y \vee t$ exist for some

$t \in L$. Then $y \vee t = (x \vee (y \wedge s)) \vee t = (x \vee t) \vee (y \wedge s)$; that

is $y \vee t = (x \vee t) \vee ((y \vee t) \vee s)$, which implies $x \vee t \equiv y \vee t(\theta)$.

Also for any $r \in L, r \wedge y = r \wedge ((x \wedge y) \vee (y \wedge s)) = (r \wedge x \wedge y) \vee (r \wedge y \wedge s)$

$= (r \wedge x) \vee (r \wedge y \wedge s)$. And so $r \wedge y = r \wedge x(\theta)$. Hence by 3.1.3

θ is a congruence relation.

(ii) implies (iii) suppose $x \equiv y(\theta)$ since θ is a congruence relation, $x \wedge t = y \wedge t(\theta)$ for any

$t \in L$. Then $x \wedge t = (x \wedge y \wedge t) \vee (x \wedge t \wedge s)$

and $y \wedge t = (x \wedge y \wedge t) \vee (y \wedge t \wedge s)$ and hence

$(x \wedge t) \vee (t \wedge s) = (x \wedge y \wedge t) \vee (t \wedge s) = (y \wedge t) \vee (t \wedge s)$.

This implies that $x \equiv y(\phi)$.

Conversely. Let $x \equiv y(\phi)$.

Then $(x \wedge t) \vee (t \wedge s) = (y \wedge t) \vee (t \wedge s)$, for all $t \in L$

By letting $t = x$ and $t = y$, we obtain $x = (x \wedge y) \vee (x \wedge s)$.

And $y = (x \wedge y) \vee (y \wedge s)$ respectively. Hence $x = y(\theta)$.

This implies that θ and ϕ are the one and the some congruence (iii) implies (iv). Then $T = \{s_1 \vee k : S_1 \leq S, k \in K \text{and } s \in K \text{ exist } \}$ is clearly closed under existent finite suprema.

Suppose $x \leq s_1 \vee k$ with $s_1 \leq s$ and $k \in K$.

Clearly, $s_1 \vee K \equiv K(\phi)$ and so $x = x \wedge (s_1 \vee k) \equiv x \wedge k(\phi)$.

Hence for all $t \in L, (x \wedge t) \vee (t \wedge s) = (x \wedge k \wedge t) \vee (t \wedge s)$.

Choosing $t = x$, we obtain $x = (x \wedge k) \vee (x \wedge s)$ and so $x \in T$.

Thus T is the ideal of L and it is clearly the supremum of (s] and K.

(iv) Implies (v).Let J and K be two ideals on L and

Suppose $x \in j \cap ((s)] \vee k)$. Then $x \in J$ and $x = s_1 \vee k$ for some

$s_1 \leq s$ And $k \in K$. So $x = (x \wedge s_1) \vee (x \wedge k)$ and thus $x \in (J \cap (s)]) \vee (J \cap k)$.

Consequently $J \cap ((s] \vee k) = J \cap (s] \vee (J \cap k)$, which implies

that (s] is standard in the ideal lattice of L

(v) implies (i) is trivial.

The last part is quite clear from the proof

of (ii) implies (iii) and of preliminaries □

In a lattice L an element n is called neutral if for any $t, x, y \in L$.

(i) $t \wedge ((x \wedge y) \wedge (x \wedge n)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge n)$ i.e., n is standard, and

(ii) $n \wedge ((t \wedge x) \vee (t \wedge y)) = (n \wedge t \wedge x) \vee (n \wedge t \wedge y)$.

Notice that a lattice is distributive if and only if each of its elements is neutral.

Also we already mentioned in the introduction that an element n in lattice L is neutral if and only if for all $x, y \in L, x \wedge (y \vee n) = (x \wedge y) \vee (x \wedge n)$ and

$n \wedge (x \vee y) = (n \wedge x) \vee (x \wedge y)$ □

The following lemma shows that the two concepts coincide in a lattice.

Lemma 3.1.5: Let a be an element of a lattice L. The following conditions are equivalent

(i) For all $t, x, y \in L,\ a \wedge ((t \vee x) \vee (t \wedge y)) = (a \wedge t \wedge x) \vee (a \wedge t \wedge y)$.

(ii) For all $x, y \in L$ for which $x \vee y$ exists, $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$.

(iii) For all ideals J and K of L $(a] \cap (j \vee k) = ((a] \cap j) \vee (a] \cap k)$.

Proof: When $x \vee y$ exists put $t = x \vee y$ in (i) to obtain (ii)

(ii) implies (iii). Let $x \in (a] \cap (j \vee k)$.

By the property of the supremum and infimum of the ideals $x \leq a$ and $x \in L_m$ for $m = 0,1,2..........$ when , $L_0 = J \cup K$,

$\mathrm{L_m} = \{\mathrm{t} \leq \mathrm{c\ \ d; c} \vee \mathrm{\ d\ exists\ and\ c, d} \in \mathrm{L_{m\text{-}1}}\}$.

Suppose $x \in L_0$. Then $x \in (a] \cap J, or\, x \in (a] \cap k$ and so $x \in ((a] \cap J) \vee ((a] \cap k)$.

Now we will use the induction, suppose $y \in L_{m-1}$, and $y \leq a$ implies that,

$y \in ((a] \cap J) \vee ((d] \cap k)$.

Since $x \in L_m$, $x \leq c \vee d$ for suitable $c, d \in L_{m-1}$.

Then $x \leq a \wedge (c \vee d) = (a \wedge c) \vee (a \wedge d)$. But $a \wedge c, a \wedge d \leq a$ and both belong to L_{m-1}.

Thus $x \in ((a] \cap J) \vee ((a] \cap k)$.

The reverse inclusion is obvious.

(iii) implies (i) is trivial □

The following result gives a characterization of a neutral element of a lattice which is immediate consequence of above lemma.

Theorem 3.1.6: An element n of a lattice L is neutral if and only if (n] in neutral in the lattice of ideals if L □

Theorem 3.1.7: Suppose n is an element of a lattice L. Such that (i) for all $x, y \in L$ for which $x \vee y$ exists,

$n \wedge (x \vee y) = (n \wedge x) \vee (n \wedge y)$. And (ii) for any $x, y \in L$ for which $y \vee n$ exists

$x \wedge (y \vee n) = (x \wedge y) \vee (x \wedge n)$. One may ask the question: ``Is n with the properties (1) and (ii) a neutral element of $L$'' Figure 2.3 show that, that answer is ``No'',

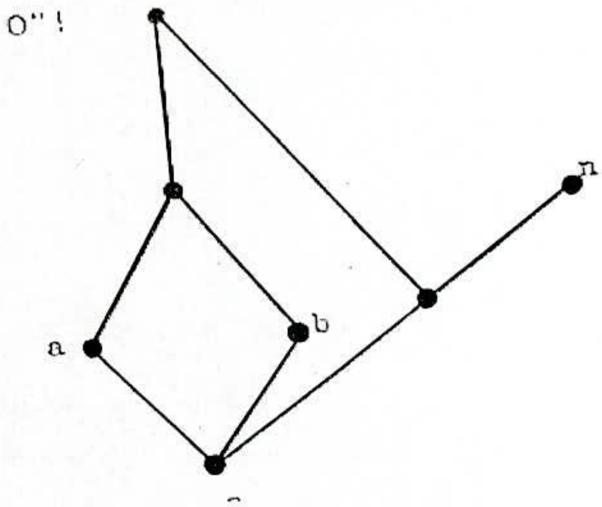


Figure 3.3

Notice that here n has both of the above properties. Yet

$b \wedge ((c \wedge a) \vee (c \wedge n)) > (b \wedge c \wedge a) \vee (b \wedge c \wedge n)$.

Thus n is not even a standard element □

3.2 Traces

Let s be an element in a lattice L. If t is any fixed element of L, then by the "trace of s in (t]"or simply "the trace of s". We mean the element $t \wedge s$ of (t].

Following proposition give characterizations of standard and neutral elements of a lattice which are due to [4]. Thus, in a lattice, these elements are trace invariant,

Proposition 3.2.1: An element s of a lattice L is standard if and only it its trace is standard in $(t]$, for each $t \in L$.

Proof: Suppose s is standard in L, let, $a,b,c \in (t]$. Then

$a \wedge [b \wedge c) \vee (b \wedge (t \wedge s)] = a \wedge [(b \wedge c) \vee (b \wedge s)] = (a \wedge b \wedge c) \vee (a \wedge b \wedge s)$

$= (a \wedge b \wedge c) \vee (a \wedge b \wedge (t \wedge s)$. Hence the trace of s is always standard.

Conversely, suppose the trace of S is standard in (t] for each $t \in L$

Let $x, y, z, \in L$ and

consider $x \wedge [(y \wedge z) \vee (y \wedge s)]$.

As $y \wedge s$ is standard in (y]

$x \wedge [(y \wedge z) \vee (y \wedge s)] = (x \wedge y) \wedge [(y \wedge z) \vee (y \wedge s)] = (x \wedge y \wedge z) \vee (x \wedge y \wedge s)$. And s is standard in L □

Proposition 3.2.2: An element n of a lattice L is neutral if and only if its trace is neutral in (t], for all $t \in L$.

Proof: Suppose n is neutral in L. Then by 3.2.1 the trace of n is standard in (t] for all $t \in L$ suppose $a,b,c \in (t]$.

Then $(t \wedge n) \wedge [(a \wedge b) \vee (b \wedge c)] = t \wedge [(a \wedge b \wedge n) \vee (b \wedge c \wedge n)]$

$= t \wedge [a \wedge b \wedge t \wedge n) \vee (b \wedge c \wedge t \wedge n)] = ((t \wedge n) \wedge (a \wedge b)) \vee ((t \wedge n) \wedge b \wedge c)$.

Thus $t \wedge n$ is neutral in (t], for all $t \in L$.

Conversely, suppose $t \wedge n$ is neutral in (t], for al $t \in L$ (by 3.2.1) n is standerd in L.

Let $x, y, z \in L$. Then

$n \wedge [(x \wedge y) \vee (y \wedge z)] = (y \wedge n) \wedge [(x \wedge y) \vee (y \wedge z)] = (x \wedge y \wedge n) \vee (y \wedge z \wedge n)$,

As $y \wedge n$ is neutral in (y]. Thus n is neutral in L □

Corollary 3.2.3: An element n of a lattice L is neutral if and only if the sublattice generated by $t \wedge n, t \wedge x$ and $t \wedge y$ are distributive for all t, x and y □

We tried to give definition of a distributive element for a lattice. But the concept does not seem appropriate in the context. From figure 3.1, it is fair enough to say that even for lattices, the notion of a distributive element is not trace invariant .From the idea of traces. Cornish and Noor [4] have introduced a new type of element, we start with the following proposition which is due to [4].

Proposition 3.2.4: Let L be a lattice and $s \in L$. Then the following condition is equivalent

(i) For any $x, y, t \in L$, $(t \wedge x \wedge y) \vee (t \wedge s) = [(t \wedge x) \vee (t \wedge s)] \wedge [(t \wedge y) \vee (t \wedge s)]$.

(ii) For any $x, y, t \in L$, $t \wedge ((x \wedge y) \vee (x \wedge s)) \vee (x \wedge s) = (t \wedge x \wedge y) \vee (x \wedge s)$.

Proof: (i) implies (ii) suppose (i) holds. Choose any t, x, y of L and let

$p = (x \wedge y) \vee (x \wedge s)$. Then $p \wedge x = p$ and so $(t \wedge [(x \wedge y) \vee (x \wedge s)] \vee (x \wedge s)$

$= (t \wedge p) \vee (x \wedge s) = (x \wedge t \wedge p) \vee (x \wedge s) = [(x \wedge t) \vee (x \wedge s)] \wedge [(x \wedge p) \vee (x \wedge s)]$ [by (i); here x, t and P play the roles of t, x and y respectively.

In (i)) $\Rightarrow [(x \wedge t) \vee (x \wedge s)] \wedge [(x \wedge y) \vee (x \wedge s)] = (x \wedge t \wedge y) \vee (x \wedge s)$ by a second application of (i) where x, t and y play the roles of x, t and y respectively in (i)

(ii) implies (i) suppose (ii) satisfies . Then for any

$t, x, y \in L, [(t \wedge x) \vee (t \wedge s)] \wedge [(t \wedge y) \vee (t \wedge s)] = ([(t \wedge x) \vee (t \wedge s)] \wedge (t \wedge y) \vee (t \wedge s)]) \vee (t \wedge s)$

$= ([t \wedge y) \vee (t \wedge s)] \wedge (t \wedge y) \vee (t \wedge s)$ $(by$ (ii) . Where $(t \wedge x) \vee (t \wedge s), t$ and y play the roles of t, x and y respectively in (iii).

Hence, $[(t \wedge x) \vee (t \wedge s)] \wedge [(t \wedge y) \vee (t \wedge s)] = ([[(t \wedge x) \vee (t \wedge s)] \wedge (t \wedge y) \vee (t \wedge s)$

$= (y \wedge [(t \wedge x) \vee (t \wedge s)]) \vee (t \wedge s) = (y \wedge t \wedge x) \vee (t \wedge s)$ by a second application of (ii). Where y, t, x play the role of t, x, y respectively of (ii) □

Proposition 3.2.5: Suppose L is a lattice and $s \in K$ holds the equivalent conditions of the above proposition.

Let $a, b \in L$. Put $t = a \vee b \vee s$ to obtain

$(a \wedge b) \vee s = (t \wedge a \wedge b) \vee (t \wedge s) = [(t \wedge a) \vee (t \wedge s)] \wedge [(t \wedge b) \vee (t \wedge s) = (a \vee s) \wedge (b \vee s)$.

Hence s is a distributive element of L. We have proved.

Proposition: If an element of a lattice satisfies the equivalent conditions of 2.1.10. Then it is distributive then it distributive □

Proposition 3.2.6: An element s of a lattice, which satisfies the equivalent conditions of prop. 3.1.10 is said to be strongly distributive. Clearly any standard element of lattice is strongly distributive.

Figure 3.1 produce (i) a distributive element in a lattice which is not strongly distributive and (ii) a strongly distributive element which is not standard.

Notice that in figure 3.1 b is distributive and a is strongly distributive. Observe that

$(a \wedge t \wedge h) \vee (a \wedge b) < [(a \wedge t) \vee (a \wedge b)] \wedge [(a \wedge h) \vee (a \wedge b)]$.

Which implies b is not strongly distributive. On the other hand $b \wedge (a \vee c) > (b \wedge a) \vee (b \wedge c)$, which implies that a is not standard.

Thus even for lattice, the notion of a strongly distributive element is strictly between the concepts of distributive and standard element.

The following proposition gives a sufficient condition for a distributive element to be strongly distributive □

Proposition 3.2.7: Any distributive atom of a lattice L with 0 is strongly distributive.

Proof: Suppose s is a distributive atom. For any $t \in L$.

Either $t \wedge s = 0 \; or \; t \wedge s = s$. if $t \wedge s = 0$ then obviously,

$(t \wedge x \wedge y) \vee (t \wedge s) = [(t \wedge x) \vee (t \wedge s)] \wedge [(t \wedge y) \vee (t \wedge s)]$. For any $x, y \in L$

If $t \wedge s = s$, then $(t \wedge x \wedge y) \vee (t \wedge s) = (t \wedge x \wedge y) \vee s = [(t \wedge x) \vee s] \wedge [(t \wedge y) \vee s]$

$= [(t \wedge x) \vee (t \wedge s)] \wedge [(t \wedge y) \vee (t \wedge s)$. As s is distributive.

An illustration of 3.2.7 considers the element c of the pentagonal lattice

$\{o, a, b, c, 1 : o \angle b \angle a \angle 1 : c \vee a = c \vee b = 1 : c \wedge a = c \wedge b = 0\}$. Here c is both distributive and an atom. Therefore, it is strongly distributive.

We conclude this section with the following characterization of a strongly distributive element. We omit the proof as it is immediate from its definition.

In fact, one might prefer this as the definition of a strongly distributive element. If is more easily to understood than the original definition □

Proposition 3.2.8: For an element s of a lattice L the following condition are equivalent.

(i) s is strongly distributive.

(ii) Its traces distributive in (t] for all $t \in L$.

(iii) Its trace is strongly distributive in (t] for all $t \in L$ □

3.3. Some properties of standard and neutral elements.

From [4] we know that the standard (neutral) elements of a lattice from a distributive sub lattice. Moreover, the map S $\rightarrow \theta_s$ is a lattice embedding of this sublattice into the distributive lattice of all congruence of L [4] also exhibited two examples to show that the strongly distributive elements may not be closed under infimum and supremum.

We are now about to generalize an interesting result of Gratzer and Schmidt [15. Theorem 5]; for any two strandard elements s_1 and s_2 of a lattice L, the sublattice generated s_1, s_2 and x is distributive for all $x \in L$.

Proposition 3.3.1: Suppose s_1 and s_2 are standard elements of a lattice L. Then the sub lattice generated by s_1, s_2 and x is distributive for all $x \in L$.

Proof: Let $x \in L$. Suppose L_1 is the sublattice generated by x, s_1 and s_2. As s_1 and s_2 are standard in L, thay are standard in L_1 by 2.1.5. $[s_1]_\lambda$ and $[s_2]_\lambda$ (principal ideals in A_1) are standard in the ideal lattice $I(L_1)$ of L_1. Hence by Gratzer and Schmidt [15] Theorem 5], the sub iattice p of $I(L_1)$ generated by $(x]_\lambda$, $(s_1]_\lambda$ and $(s_2]_\lambda$ is distributive. But as L_1 is generated by x_1 s_1 and s_2 $I(L_1) = P$. Thus, $I(L_1)$ is distributive and so, L_1 is distributive.

We already know from the introduction that an element *n* in a lattice L is neutral if and only if the sublattice generated by *x, y* and *n* is distributive for all $x, y \in L$. See also on 3.2.3 unfortunately, things are not the same in near lattices □

Theorem3.3.2: Let n be a neutral element of a lattice L. Then sublattice generated by x, y and n is distributive for all $x, y \in L$. But the converse is not necessarily true.

Proof: We omit the proof of first part as it is easily seen that this can be done in exactly the same way in which 3.3.1 was proved. To prove the converse we consider the lattice L of figure 3.4. Here, all the sub lattices generated by x, y and n for all $x, y \in L$ are distributive, yet $b \wedge [(t \wedge d) \vee (f \wedge n)] > (b \wedge f \wedge d) \vee (b \wedge f \wedge n)$.

Thus n is not even a standard elements of L ☐

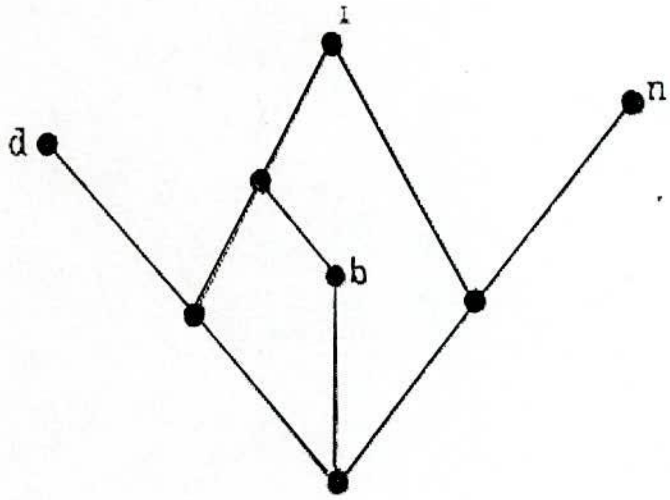


Figure 3.4

Now we prove the following results which generalize some of the results of [15].

Theorem 3.3.3: Let s and n be elements of a lattice L such that n is neutral, $s \leq n$ and s is standard in *(n]*.

Then s is a standard element of L ☐

Proof: Let *t, x, y* be the elements of L.

Then $[(x \wedge y) \vee (n \vee s)] \wedge [(x \wedge y) \vee (x \wedge n)]$

$= ([x \wedge y) \vee (n \wedge s)] \wedge (x \wedge y)) \vee \left(((x \wedge y) \vee (n \wedge s)) \wedge (x \wedge n)\right)$

$= (x \wedge y) \vee (x \wedge n) \wedge [(x \wedge y \wedge n) \vee (n \wedge s)))])$ as n is neutral.

$= (x \wedge y) \vee ((x \wedge y \wedge n) \vee (x \wedge n \wedge s)$ as s in standard in *(n]*

$=(x\wedge y)\vee(x\wedge n\wedge s)$

$=(x\wedge y)\vee(x\wedge s)$.

Hence using the neutrality of n

$t\wedge[(x\wedge y)\vee(x\wedge s)]$

$$
\begin{aligned}
&=t\wedge[(x\wedge y)\vee(n\wedge s)]\wedge((x\wedge y)\vee(x\wedge n)\\
&=\big((x\wedge y)\vee(n\wedge s)\big)\wedge t\wedge\big((x\wedge y)\vee(x\wedge n)\big)\\
&=\big((x\wedge y)\vee(n\wedge s)\big)\wedge\big((t\wedge x\wedge y)\vee(t\wedge x\wedge n)\big)\\
&=\big((x\wedge y)\vee(n\wedge s)\big)\wedge\big(t\wedge x\wedge y\big)\vee\big((x\wedge y)\vee(n\wedge s)\big)\wedge(t\wedge x\wedge n)
\end{aligned}
$$

As n is neutral.

$$
\begin{aligned}
&=(t\wedge x\wedge n)\vee[(t\wedge x)\wedge\big((x\wedge y\wedge n)\vee(s\wedge n)\big)]\\
&=(t\wedge x\wedge y)\vee(t\wedge x\wedge n)\wedge\big((x\wedge y\wedge n)\vee(s\wedge n)\big)\\
&=(t\wedge x\wedge y)\vee(t\wedge x\wedge y\wedge n)\vee(t\wedge x\wedge s\wedge n)
\end{aligned}
$$

Since s is standard in *(n]*

$=(t\wedge x\wedge y)\vee(t\wedge x\wedge s)$.

So s is standard element in L □

Theorem 3.3.4: Let s be a neutral element of *(n]* and n is neutral in *L*. Then s is a neutral element of *L*.

Proof: By the previous theorem s is standard in *L*.

To show that *s* is neutral, we need only to show that

$s\wedge[(x\wedge y)\vee(x\wedge t)]=(a\wedge x\wedge y)\vee(s\wedge x\wedge t)$ for all $x,y,t\in L$

Now, $s\wedge[(x\wedge y)\vee(x\wedge t)]=(s\wedge n)\wedge((x\wedge y)\vee(x\wedge t))$

$=s\wedge(x\wedge y\wedge n)\vee(x\wedge t\wedge n)$ As *n* is neutral

$=(s\wedge x\wedge y\wedge n)\vee(s\wedge x\wedge t\wedge n)$ As s is neutral in *(n]*

$=(s\wedge x\wedge y)\vee(s\wedge x\wedge t)$.

The proof is thus complete □

Theorem 3.3.5: An element n of a lattice L is neutral if and only if for all $t, x, y \in L$

$$(t \wedge n \wedge x) \vee (t \wedge n \wedge y) \vee (t \wedge x \wedge y) = \{(t \vee n) \vee (t \wedge x)\} \wedge \{(t \wedge n) \vee (t \wedge y)\} \wedge \{(t \wedge x) \vee (t \wedge y)\}.$$

Proof: When n is neutral its trace $t \wedge n$ is neutral in the lattice (t] and so the equality holds as $t \wedge n, t \wedge x$ and $t \wedge y$ then generated a distributive sub lattice of (t].

Conversely, the equality says that $t \wedge n$ is neutral in the lattice (t]. Then the proposition 3.2.2 does the rest.

We conclude here with two observations about strongly distributive elements □

CHAPTER FOUR

4.1 Prime ideals of a lattice.

Introduction: Prime ideal and pseudo complemented of a lattice have been studied by several authors including [32]. In this chapter we discuss prime ideals, minimal prime ideals and minimal prime n-ideals of a lattcies. In section one of these chapters we give some basic properties of prime ideals which will be needed in the next part.

In section two of this chapter we have given characterization of minimal prime ideals of a pseudocomplemented distributive lattice. Then we have show that every pseudocomplemented lattice is generalized stone

In section three we discuse the minimal prime n-ideals.

In section four of this chapter we have discussed lattice whose principal n-ideal form normal lattice.

Definition: (Dual ideal): A non empty subset I of a lattice L is called dual ideal of L if (1) $x, y \in I$ implies that $x \wedge y \in I$

(2) $d \in I, x \in L$ implies that $d \wedge x \in I$

Let $I = \{1, 2, 5, 10\}$ be the lattice under divisibility. Then $\{1o\}, \{5, 10\} \{2, 10,\}$ are all dual ideals of lattice L.

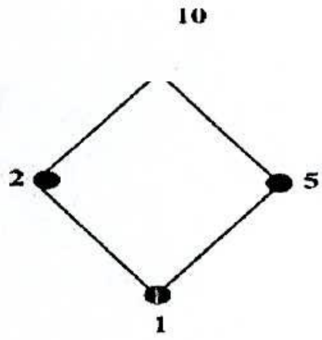


Figure 4.1

An *ideal* I of L is proper if $I \neq L$.

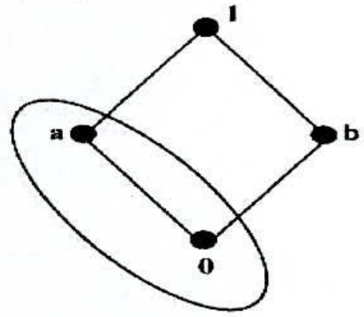


Figure 4.2

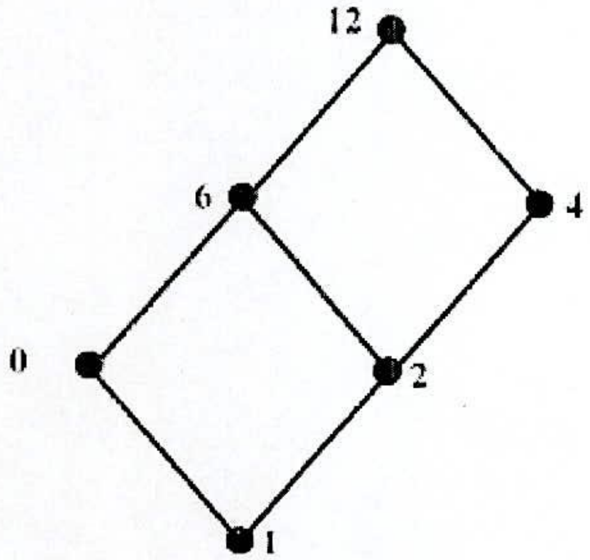


Figure 4.3

A proper ideal P of a lattice L is called a prime ideal if for any $x, y \in L$ and $x \wedge y \in p$ implies either $x \in P$ or $y \in P$. Let $L = \{1, 2, 3, 4, 6, 12\}$ of factors 12 under divisibility forms a lattice then $\{1, 2, 4\}$ be a prime ideal of L. But in the lattice {1, 2, 5, 10} under divisibility $\{1\}$ in not a prime ideal because $2 \wedge 5 = 1\{1\}$. But $2, 5\{1\}$.

Theorem 4.1.1: Every ideal of a lattice L is prime ideal if and only if the lattice L is chain.

Proof: Let L be a chain, let P be any proper ideal of L. If $a \wedge b \in P$ then as a, b are in a chain, they are comparable. Let $a \leq b$, then $a \wedge b = a$. Thus, $a \wedge b \in P \Rightarrow a \in I \Rightarrow P$ is prime.

Conversely, let every ideal in P be prime. To show that L in a chain, let $a, b \in L$ be any elements. Let $P = \{x \in L / x \leq a \wedge b\}$ then P is easily seen to be an ideal of L. Thus, P is a prime ideal.

Now $a \wedge b \in I, P$ is prime, thus $a \in P$ or $b \in P$ $\Rightarrow a \leq a \wedge b \Rightarrow$ or $b \leq a \wedge b$

$\Rightarrow a \wedge b \leq a \leq a \wedge b$ or $\Rightarrow a = a \wedge b$ or $b = a \wedge b$

$\Rightarrow a \leq b \Rightarrow$ or $b \leq a$. L is a chain □

Corollary 4.1.2: Let L be a distributive lattice. Let I be an ideal of L and let $a \in L$ and $a \in I$. Then there is a prime ideal P such that $P \supseteq I$ and $a \notin P$.

Theorem 4.1.3: Every ideal I of a distributive lattice is the intersection of all prime ideals containing it.

Proof: Let $I_1 = \bigcap\{P / P \supseteq I; P$ is a prime ideal of $L\}$, if $I \neq I_1$ then there is an element $a \in I_1 - I$ and so by Corollary 4.1.2. There in a prime ideal P, with $P \supseteq I$ and $a \notin P$. But then $a \notin P \supseteq I_1$ and is a contradiction. □

Theorem 4.1.4: Let P be a prime ideal of a lattice L, then $L-P$ is a dual prime ideal.

Proof: Since P is a prime ideal, therefore P is not empty.

$\therefore L-P$ is a proper subset of L.

Let $x, y \in L-P$. Then $x, y \in L, x, y \notin P$

$\Rightarrow x \wedge y \in L, x \wedge y \notin P$ (as $x \wedge y \in \Rightarrow x \in P$ or $y \in P$ as P is prime) $\Rightarrow x \wedge y \in L-P$.

Again, let $x \in L-P, I \in L$. Then $x \in L, x \notin P, I \in L \Rightarrow x \vee I \in L, x \notin P \Rightarrow x \vee I \in L, x \vee I \notin P$

(as $x \vee I \in P \Rightarrow x \in P$ as $x \leq x \vee I$). Thus, $x \vee I \in L-P$ i.e $L-P$ is dual ideal.

Now let $x \vee y \in L-P$, then $x \vee y \in L, x \vee y \notin P$

$\Rightarrow x, y \in L, x \notin P$ or $y \notin P$ (as $x, y \in P \Rightarrow x \vee y \in P$)

$\Rightarrow x \in L-P$ or $y \in L-P$

i.e., $L-P$ is a dual prime ideal □

4.2 Minimal prime ideal.

Def (Minimal prime ideal): A prime ideal P of a lattice L is called minimal if there does not exists a prime ideal Q such that $Q \subset P$.

The following lemma is an extension of a fundamental result in lattice theory e, f J.E. Kist [23]. Though our proof is similar to their proof, we include the proof for the convenience of the reader.

Theorem4.2.1: Let L be a lattice with 0. Then every prime ideal contains a minimal prime ideal.

Proof: Let P be a prime ideal of L and let R be the set of all prime ideals Q contained in P. Then R is nonvoid, since $P \in R$ If C is a chain in R and $Q = \bigcap(x : x \in C)$ then Q is nonvoid, since $0 \in Q$ and Q is an ideal ; infact Q is prime. In deed if $r \wedge s \in Q$ for some $r, s \in L$. Then $r \wedge s \in X$, for all $x \in C$, since X is prime , either $r \in X$ or $s \in X$. Thus, either $Q = \cap(X : r \in X)$ or $Q = \cap(X : s \in X)$.

Proving that either r or $s \in Q$.

Therefore, we can apply to R the dual form of Zorn's lemma to conclude the existence of minimal member of R □

Theorem 4.2.2: Let L be a distributive lattice with 0, the following conditions are equivalent.

(i) L is normal.

(ii) Each prime ideal of L contains a unique minimal prime ideal.

(iii) Each Prime filter of L is contained in a unique ultrafilter of L.

(iv) Any two distinct minimal prime ideals are comaximal.

(v) For all $x, y \in L, x \wedge y = 0$ implies $(x]^* \vee (y]^* = L$.

(vi) $(x \wedge y]^* = (x]^* \vee (y]^*$ for all $x, y \in L$ □

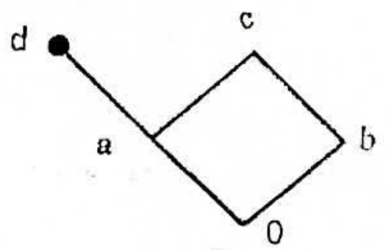
Remark: Here $(x]^*$ we means relatively pseudo complement of $(x]$.

Dense set: $D(L)=\{a\in L: a^*=0\}, D(L)$ is called the dense set.

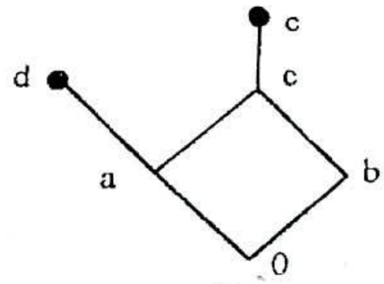
Theorem 4.2.3: Let L be a sectionally pseudo complemented distributive lattice and P be a prime ideal in L. Then the following conditions are equivalent

(i) P is minimal, (ii) $x\in P$ implies $(x]^*\notin P$, (iii) $x\in P$ implies $(x]^{**}\subseteq P$, (iv) $P\cap D(L)=\varphi$.

Remarks: Consider the following distributive lattice with 0. Observe that in both L_1 and L_2, $(b]$ and $(d]$ are distinct minimal prime ideals.



S_1



S_2

Figure- 4.4

Moreover, $(b]\vee(d]=S_1$ but $(b]\vee(d]\neq S_2$. Therefore, L_1 is normal but L_2 is not. Also observe that in $L_{2,}$, $\{0,a,b,c,d\}$ is a prime ideal which contains two prime ideals $(b]$ and $(d]$, and so L_2 is not normal □

Definition (Stone lattice): A distributive pseudo complemented lattice L is called a stone lattice if for each $a\in I$, $a^*\vee a^{**}=I$.

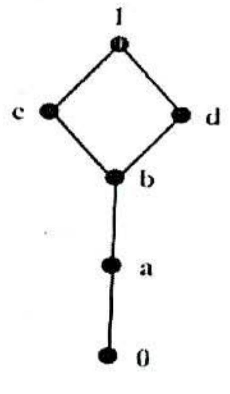


Figure - 4.5

Theorem 4.2.4: If L is a complete stone lattice, then ideal of L is also complete stone lattice.

Proof: Let $I^* = (0]$, where $a = (x^* : x \in L)$ and let $x \in I \cap I^*$, then $x \in I$ and $x \in I^* = (a]$ implies that $x \in I$ and $x \in A$ implies that $x \in I$ and $x \le y^*$ for all $y \in I$ implies that $x \le x^*$ implies that $x = x \wedge y^* = 0$, implies that $I \wedge I^* = (0]$.

Let $I \wedge J$, choose any $j \in J$, then $i \wedge j = 0$ for all $i \in I$, implies that $j \le i^* i \in I$ j ≤ i *, implies that $j \le \wedge(I^* : i \in I)$ implies that $j \le a$ implies that $j \in I^*$ implies that $J \le I^*$ implies that I^* is a pseudocomplemented.

Since $0 \in L$, so ideal of L is complete. Finally, we have to show that $I^* \vee I^{**} = L$.

Now $I^* \vee I^{**} = (a] \vee (a]^* = (a]^{**} \vee (a]^*$

$$= (a^{**}] \vee (a^*]$$

$$= (a^{**} \vee a]$$

$$= L$$

Hence ideal of L is a stone. Thus ideal of L is a complete stone lattice □

Definition (Generalized stone lattice): A lattice L with O is called generalized stone lattice if $(x]^* \vee (x]^{**} = L$ for each $x \in L$.

Theorem 4.2.5: A distributive lattice L with 0 is a generalized stone lattice if and only if each interval $[0, x], 0 < x \in L$ is a stone lattice.

Proof: Let L with 0 be a generalized stone and let $P \in [0, x]$. Then $(P]^* \vee (P]^{**} = L$.

So $x \in (P]^* \vee (P]^{**}$ implies $x = r \vee s$ for some $r \in (P]^*, s \in (P]^{**}$. Now $r \in (P]^*$ implies $r \wedge p = 0$ also $0 < r < x$. Suppose $t \in [0, x]$ such that $t \wedge p = 0$, then $t \in (P]^*$ implies $t \wedge s = 0$. Therefore, $t \wedge x = t \wedge (r \vee s) = (t \wedge r) \vee (t \wedge s) = (t \wedge s) \vee 0 = t \wedge r$ implies

$t = t \wedge r$ implies $t \le r$. So r is the relative pseudo complement of P in $[0, x]$, i.e. $r = P^*$ since $s \in (P]^{**}$ and $r \in (P]^*$. So $s \wedge r = 0$. Let $q \in [0, x]$. Such that $q \wedge r = 0$. Then as $x = r \vee s$, so $q \wedge x = (q \wedge r) \vee (q \wedge s)$ implies $q = q \wedge s$ implies $q \le s$. Hence, s is the relative pseudo complement of $r = p^*$ in $[0, x]$ i.e., $s = p^{**}$ implies $x = r \vee s = p^* \vee p^{**}$. Thus $[0, x]$ is a stone lattice.

Conversely, suppose $[0, x], 0 < x \in L$ [o, x]. is a stone lattice. Let $p \in L$, then $p \wedge x \in [0, p]$. Since $[0, p]$ is a stone lattice, then $(p \wedge x)^* \vee (p \wedge x)^{**} = p$, where $(p \wedge x)^*$ is the relative pseudo complement of $p \wedge x$ in $[0, p]$.

Therefore, $P \in ((p] \cap (p \wedge x]^* \vee ((p] \cap (p \wedge x)^{**}$. So, we can take $p = r \vee s$, for $r \in (p \wedge x]^*, s \in (p \wedge x]^{**}$. Now, $r \in (p \wedge x]^*$ implies $r \wedge p \wedge x = 0$ implies $r \wedge x = 0$ implies $r \in (x]^*$ and $s \in (p \wedge x)^{**}$. Now $p \wedge x \le x$ implies $(p^{**} x] \subseteq (x]^{**}$, and so $s \in (x]^{**}$ Therefore, $p = r \vee s \in (x]^* (x]^{**}$ and so, $L \subseteq (x]^* \vee (x]^{**}$. But $(x]^* \vee (x]^{**} \subseteq L$ is obvious. Hence $(x]^* \vee (x]^{**} = L$ and so L is generalization stone □

Theorem 4.2.6: Let L be a distributive lattice with 0. If L is generalized stone, then it is normal.

Proof: Let P and Q be two minimal prime ideals of L.

Then P, Q are unordered. Let $x \in P - Q$. Then $(x] \wedge (x]^* = (0] \subseteq Q$ implies $(x]^* \subseteq Q$. Since P is minimal, so by Theorem 4. 2.3 above $(x]^{**} \subseteq P$. Again as L is generalized stone, so $(x]^* \vee (x]^{**} = L$

This implies $P \vee Q = L$, and so by Theorem 4.2.2, L is normal □

Def (Co- maximal ideal): Two Ideals I and J of a lattice L are called Co-maximal if $I \vee J = L$

Theorem 4.2.7: A sectionally pseudocomplemented distributive lattice L is generalized stone if and only if any two minimal prime ideals are co-maximal.

Proof: Suppose L is generalized stone. So by theorem 4.2.6 any two minimal prime ideals are co-maximal.

To prove the converse, let P, Q be two minimal prime ideals of L. We need to show that $[0,x]$ is stone, for each $x \in L$. Let P_1 Q_1, be two minimal prime ideals in $[0,x]$.

We know if L_1 is a sub lattice of a distributive lattice L and P_1 . is minimal prime ideal in L_1, then there exists a minimal prime ideal P in L, such that $P_1 = L_1 \wedge P$.

So there exist minimal prime ideal P, Q in L such that $P_1 = P \wedge [0,x], Q = Q \wedge [0,x]$.

Therefore, $P_1 \vee Q_1 = (P \cap [0,x] \vee (Q \cap [0,x] = [P \vee Q] \cap [0,x] = L \cap [0,x] = [0,x]$.
Therefore, $[0,x]$ is stone. So by Theorem3. 2.5 above L is generalized stone □

Theorem 4.2.8: Let L be a distributive with 0 and 1 for an ideal I of L.

We set $I^* = \{x : x \wedge i = 0$ for all $i \in I\}$. Let P be a prime ideal of L. Then P is minimal prime ideal if and only if $x \in P$ implies that $(x]^* \subseteq P$.

Proof : By the definition of $I^*, (x]^* = \{y : y \wedge x = 0\}$ as $x^* \wedge x = 0$ implies that $x^* \in (x]^*$ implies that $(x^*] \subseteq (x]^*$, again let $z \in (x]^*$, then $z \wedge x = 0$ implies that $z \leq x^*$ implies that $z \in (x]^*$ implies that $(x]^* \subseteq (x^*]$ implies that $(x]^* = (x^*]$. Now suppose P be a minimal prime ideal and $x \in P$, then by $x^* \notin P$ implies $(x^*] \not\subset P$ implies $(x^*] \subseteq P$. Conversely, if for $x \in P, (x]^* \not\subset P$ and if possible, let P is not minimal then there exist a prime ideal Q such that $Q \subseteq p$. Let $x \in P - Q$.

Now $x^* \wedge x = 0 \in Q$ implies that $x^* \in Q$ implies that $x \in P$ implies $(x^*] \subseteq P$ implies $(x]^* \subseteq P$ which is a contradiction. Hence the proof □

Theorem 4.2.9: A prime ideal P of a stone algebra L is minimal if and only $P = (P \cap S(L))_L$.

Proof: Suppose P is minimal, let $a \in (P \cap S(L))_L$. Then $a \le r$ for some $r \in (P \cap S(L))_L$

$\Rightarrow r \in P$ and $r \in S(L) \Rightarrow a \in P \Rightarrow r \in P$ and $r \in S(L)$ implies $r \in P$ implies $a \in P$.

Implies that $(P \cap S(L))_L \subseteq P$ . . .. (i)

Again let $a \in P$, since P is minimal so, $a^{**} \in P \cap S(L)$, since $a \le a^{**}$. So $a \in (P \cap S(L))_L$

implies that $P \le (P \cap S(L))_L$ ....(ii)

From (i) and (ii) we have $a \in (P \cap S(L))_L$.

Conversely let $P = (P \cap S(L))_L$ and $a \in P$ then $a \le r$ for some $r \in P \cap S(L) \Rightarrow a^{**} \le r^{**} = r$

$\Rightarrow x^{**} \in P$. Hence P is minimal □

4.3 MINIMAL PRIME n-IDEALS

A prime n-ideal P is a minimal prime n-ideal belonging to n-ideal I if

(i) $I \subseteq P$ and

(ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

Thus a prime n-ideal P of a lattice L is minimal prime n-ideal if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. i.e., minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$

Definition (Medial): An element n of a lattice L is medial if $m(x,n,y)$ exists for all $x, y \in L$.

Since for the definition of a prime n-ideal of L, the medial property of n is essential, so any distribution about prime n-ideals of L we will always assume n as a medial element. We start this section with the following result which is a generalization of a well known result in lattice theory.

Theorem 4.3.1: Let L be lattice with medial element n. Then every prime n ideal contains a minimal prime n-ideal.

Proof: Let P be a prime n-ideal of L and let R be the set of all prime n-ideal Q contained in p. Then R is non-void, since $P \in R$. If C is a chain in R and $Q = \cap(x : x \in C)$, then Q is a non-empty as $n \in Q$ and Q is an n-ideal, in fact, Q is prime.

Indeed, if $m(a,n,b) \in Q$ for some $a, b \in L$, then $m(a,n,b) \in X$ for all $X \in C$. Since X is prime, either $a \in X$ or $b \in X$. Thus, either $Q = \cap(X : a \in X)$ or $Q = \cap(X : b \in X)$, proving that $a \in Q$ or $b \in Q$. Therefore, we can apply to R the dual form of zorns lemma to conclude the existence of a minimal member of R.

If L is a distributive lattice with $n \in L$, then we already know that $F_n(L)$ is a distributive lattice with {n} as the smallest element. So we can talk on the sectionally pseudocomplementeness of $F_n(L)$ is called sectionally pseudo- complementted if each interval $[\{n\}, < a_1 < \ldots\ldots\ldots\ldots\ldots a_r > n]$ is pseudo-complemented.

That is for $\{n\} \subseteq < b_1 < \ldots\ldots\ldots b_r > n] \subseteq < a_1 < \ldots\ldots\ldots\ldots\ldots\ldots a_r > n$. relative pseudo-complement $< b_1 < \ldots\ldots\ldots\ldots\ldots\ldots\ldots b_r > n$ in $[\{n\}, < a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots a_r > n$ belongs to $F_n(L)$

Now we give a characterization of minimal prime n-ideals of a distributive lattice L, when $F_n(L)$ is seasonally pseudo complemented. To do this we establish the following theorems □

Theorems 4.3.2: Let L be a distributive lattice and $n \in L$ be a medial element. Then for any $i, j \in I_n(L), (I \cap J)^* \cap I = J^* \cap I$.

Proof: Since $I \cap J \subseteq J$. So R H. S $\subseteq$ L. H. S .

To prove the reverse inclusion, let $x \in L$. H. S Then $x \in I$ and $m(x, n, t) = n$

for all $t \in I \cap J$. Since $x \in I$, so $m(x, n, j) \in I \cap J$.

Thus $m(x, n, m(x, n, j)) = n$.

But it can be easily seen that $m(x, n, m(x, n, j)) = m(x, n, j)$. Thus implies $m(x, n, j) = n$ for all $i \in J$. Hence $x \in R.H.S$ and so $L.H.S \subseteq R.H.S$ Thus $(I \cap J)^* \cap I = J^* \cap I$ □

Theoren 4.3.3: Suppose n is medial element of a lattice L. If $I \subseteq J, I, J \in I_n(L)$.

Then (i) $I^* = I^* \cap J$ and (ii) $I^{**} = I^{**} \cap J$.

Proof: (i) am trivial. For (ii), using (i) we have, $I^{**} = (I^*)^* \cap J = (I^* \cap J)^*$.

Thus by Theorem 3.2, $I^{**} = I^{**} \cap J$ □

Theorem 4.3.4: Let n be a sesqui - medial element of a distributive lattice L. Suppose $F_n(L)$ is sectionally pseudo-complemented distributive lattice and P is a prime n-ideal of L. Then the following conditions are equivalent.

(i) P is minimal. (ii) $x \in P$ implies $< x >_n^* \not\subset P$.

(iii) $x \in P$ implies $< x >_n^{**} \subseteq P$.

(iv) $P \cap D(< t_n >) = \varphi$ for all $t \in L - P$, where $D(< t_n >) = \{x \in L < t_n >:< x >_n^* = \{n\}\}$

Proof: (i)$\rightarrow$ (ii), suppose P is minimal. If (ii) fails, then there exists $x \in P$,

such that $< x >_n^* \subseteq P$. Since P is a prime n-ideal, then P is a prime ideal or a prime filter. Suppose P is a prime ideal. Let $D = (L - P) \vee [x)$. We claim that $n \notin D$. if $n \in D$, then $n = q \wedge x$ for some $q \in L - P$.

Then $< q >_n \cap < x >_n =< (q \wedge x) \vee (q \wedge n) \vee (x \wedge n) >_n = \{n\}$ implies $< q >_n \subseteq < x >_n^* \subseteq P$. Thus, $q \in P$, which is contradiction. Hence $n \notin D$. Then there exist a prime n- ideal Q with $Q \cap D = \varphi$. Then $Q \subseteq P$ as $Q \cap (L - P) = \varphi$ and $Q \neq P$, since $x \notin q$. But this contradicts the minimality of P.

Hence $< x >_n^* \subseteq P$. Similarly, we can prove that $< x >_n^* \leq P$ if P is a prime fitter.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds and $x \in P$. Then $< x >_n^* \not\subset P$.

Since $< x >_n^* \cap < x >_n^{**} = \{n\} \subseteq P$ and P is prime, so $< x >_n^{**} \subseteq P$.

(iii) $\Rightarrow$ (iv). Suppose (iii) holds and $t \in L - P$. Let $x \in P \cap D(< t >_n)$

Then $x \in P, x \in D(< t >_n)$. Thus $< x >_n^* = \{n\}$ and so $< x >_n^{**} =< t >_n$.

By (iii) $x \in P$ implies $< x >_n^{**} \subseteq P$. Also by Theorem 3.3.3, $< x >_n^{**} \cap < x >_n^{**} \cap < t >_n$.

Hence $< x >_n^{**} \cap < t >_n =< t >_n$ and so $< t >_n \subseteq < x >_n^{**} \subseteq P$. That is $t \in P$, which is a contradiction.

Therefore, $P \cap D(< t >_n) = \varphi$ for all $t \in L - P$.

(iv) $\Rightarrow$ (i). Suppose P is not minimal. Then there exists a Prime n-ideal $Q \subset P$.

Let $x \in P - Q$. Since $< x >_n \cap < x >_n = \{n\} \subseteq Q$ So $< x >_n^* \subseteq Q \subset P$

Thus, $<x>_n \vee <x>_n^* \subseteq P$.

Choose any $t \in L - P$. Then $<t>_n \cap (<x>_n \vee <x>_n^* \subseteq P$.

Now $<t>_n \cap (<x>_n \vee <x>_n^*) = <t>_n \cap <x>_n) \vee (<t>_n \cap (<x>_n^*$

$=< m(t,n,x)>_n \vee \; <t>_n \cap <x>_n)^* \cap (<t>_n)$($<t>_n$ (by Theorem3. 3.2)

$=< m(t,n,x)>_n \vee \; (m(t,n,x)>_n^* \cap <t>_n)$

$=< m(t,n,x)>_n \vee \; (< m(t,n,x)>_n^*$ (by Theorem 3.3.3), where $< m(t,n,x)>_n$ is the relative pseudo-complement of $< m(t,n,x)>_n$ in $<t>_n$. Since $F_n(L)$ is sectionally pseudo-complemented $< m(t,n,x)>_n^*$ is finitely generated and so $(< m(t,n,x)>_n \vee m(t,n,x)_n^*$ is a finitely generated n-ideal contained in $<t>_n$.

Therefore, $< m(t,n,x)>_n \vee m(t,n,x)>_n^* =< r>_n$ for some $r \in <t>_n$.

Moreover, $<r>_n^* = < m(t,n,x)>_n \vee m(t,n,x)_n^{**} \{n\}$. Thus, $r \in P \cap D <t>_n$,

which is a contradiction. Therefore, P must be minimal □

4.4 Lattice whose principal n-ideal form normal lattice.

Definition(Normal lattice): A distributive lattice L with 0 is normal if every ideal of L contains a unique minimal prime ideal.

Definition(Central element): Any element of a lattice which is standard and complemented in each interval containing it is actually neutral, then the element is called central.

We already known that for a central element $n \in L, P_n(L) \cong (n)^d x[n)$.

Thus, we have the following result.

Lemma 4.4.1: Suppose n is a central element of a distributive lattice L.

Then $P_n(L)$ is normal if and only if $(n]^d$ and $[n)$ are normal.

A distributive lattice L with 1 is called dual normal if its every prime filter is contained in a unique ultra-filter (maximal and proper). In a general lattice, this condition is also equivalent to the condition of normality, that is, every prime ideal contains a unique minimal prime ideal. Thus obviously the concept of dual normality coincides with the normality in case of bounded distributive lattices.

Therefore, from above lemma $P_n(L)$ is normal if and only if $[n)$ is a normal lattice and $[n)$ is a dual normal lattice. Following theorem is needed to prove the main results of this chapter.

Theorem 4.4.2: Suppose L be a distributive lattice and $n \in L$.

Let $x, y \in L$ with $<x>_n \cap <y>_n = \{n\}$. Then the following conditions are equivalent.

(i) $<x>_n^* \cap <y>_n^* = L$.

(ii) For any $t \in L$, $< m(x,n,x) >_{n+} \vee < m(y,n,t) >_{n+} < t >_n$, where $< m(x,n,x) >_{n+}$ denote the relative pseudocomplement of $< m(x,n,t) >_n$ in$[\{n\}, < t >_n]$.

Proof: (i) $\Rightarrow$ (ii). Suppose (i) hold. Then for any $t \in L$,

$$< m(x,n,t) >_{n+} \vee < m(y,n,t) >_{n+} = (< x >_n \cap < t >_n)^+ \vee (< y >_n \cap < t >_n)^+$$

$$= ((< x >_n \cap < t >_n)^* \cap < t >) \vee < y >_n) \cap < t >_n)^* \cap < t >_n) \text{ [by Lemma 4.3.3.]}$$

$$= ((< x >_n \cap < t >_n) \vee < y >_n^*) \cap < t >_n) \text{ [by Lemma 4.3.2.]}$$

$$= (< x >_n \vee < y >_n)^* \cap (< t >_n)$$

$$= S \cap < t >_n$$

$= < t >_n$. Hence (ii) holds.

(ii) $\Rightarrow$ (i). Suppose (ii) holds and $t \in L$.

By (ii), $< m(x,n,t) >_{n+} \vee < m(y,n,t) >_{n+} = < t >_n$. Then using Lemmas 4.3.2 and 4.3.3 and the calculation of (i) $\Rightarrow$ (ii) above, we get,

$(< x >_n^* \vee < y >_n^*) \cap < t >_n = < t >_n$. This implies $< t >_n \subseteq < x >_n^* \vee < y >_n^*$ and $t \in < x >_n^* \vee < y >_n^*$ so. Therefore, $< x >_n^* \vee < y >_n^* = L$ □

Conish in [4] has given some characterizations of normal lattices. Then [30] extended those results for lattices [30] has given the following characterizations for normal lattices.

Theorem 4.4.3: Let L be a distributive lattice and n be a central element of L. The following conditions are equivalent.

(i) $P_n(L)$ is normal.

(ii) Every prime n-ideal of L contains a unique minimal prime n-ideal.

(iii) For any two minimal prime n-ideals P and Q of L, $P \vee Q = L$.

Proof: (i) $\Rightarrow$ (ii). Let $P_n(L)$ be normal, since $P_n(L) \cong (n]^d \times [n)^d$, so both $(n]^d$ and $[n)$ are normal. Suppose P is any prime n-ideal of L. Then either $P \supseteq (n]$ or $P \supseteq [n)$. Without loss of generality, suppose $P \supseteq (n]$. Then P is prime ideal of L. Hence $P_1 = P \cap [n)$ is a prime ideal of $[n)$. Since $[n)$ is normal, so by definition P_1 contains a unique minimal prime ideal R_1 of $[n)$. Therefore, P contains a unique minimal Prime ideal R of L where $R_1 = R \cap [n)$. Since $n \in R_1$ so $n \in R$ and hence R is a minimal prime n-ideal of L. Thus (ii) holds.

(ii)$\Rightarrow$(i). Suppose (ii) holds. Let P_1 be a prime ideal in $[n)$. Then $P_1 = P \cap [n)$ for some prime ideal P_1 of L. Since $n \in P_1 \subseteq P$, so P is prime n-ideal .

Therefore, P_1 contains a unique minimal prime n-ideal R of L. Thus, P_1 contains the unique minimal prime ideal $R_1 = R \cap [n)$ of [n). Hence by definition $[n)$ is normal. Similarly, we can prove that $(n]^d$ is also normal. Since $P_n(L) \cong (n]^d \times [n)^d$, so $P_n(L)$ P_n is normal.

(ii)$\Leftrightarrow$ (iii) is trivial by Stone's separation Theorem.

Recall tha: for a prime ideal P of a distributive lattice L with 0, [31] has defined $0(P) = \{x \in L : x \wedge y = 0 \text{ for some } y \in L - P\}$. Clearly, $0(P)$ is an ideal and $0(P) \subseteq P$. [31] has shown that $0(P)$ is the intersection of al the minimal prime ideals of L, which are contained in P.

For a prime n-ideal P of a distributive lattice L, we write $n(P) = \{y \in P : m(y,n,x) = n \text{ for some } x \in L - P\}$. Clearly, $n(P)$ is an n- ideal and $n(P) \subseteq P$ □

Lemma 4.4.4: Let n be a medial element of a distributive lattice L and P be a prime n-ideal in L. Then each minimal prime n-ideal belonging to $n(P)$ is contained in P.

Proof: Let Q be a minimal prime n-ideal belonging to $n(P)$. If $Q \not\subset P$, then choose $y \in Q - P$. Since Q is a prime n-ideal, so we have Q is either an ideal or a filter. Without loss of generality, suppos Q is an ideal. Now let $T = \{t \in L, m(y,n,t) \in n(P)\}$. We show that $T \not\subset Q$. If not, let $D = (L-Q) \vee [y)$. Then $n(p) \cap D = \varphi$.

For otherwise, $y \wedge r \in n(P)$ for some $r \in L - Q$.

Then by convexity, $y \wedge r \leq m(y,n,r) \leq (y \wedge r) \vee n$ implies $m(y,n,r), n(P)$.

Hence $r \in T \subseteq Q$, which is a contradiction. Thus there exists a prime n- ideal R containing $n(P)$ disjoint to D. Then $R \subseteq Q$.

Moreover, $R \neq Q$ as $y \in R$, this shows that Q is not a minimal prime n- ideal belonging to $n(P)$, which is a contradiction.Therefore, $T \not\subset Q$. Hence there exists $z \notin Q$ such that $m(y,n,z) \in n(P)$. Thus $m(m(y,n,z),n,x) = n$ for some $x \in L - P$. It is easy to see that $m(m(y,n,z),n,x) = n = m(m(y,n,x),n,z)$.

Hence $m(m(y,n,x),n,z) = n$. Since P is prime and $y, z \in P$ so $m(y,n,x) \notin P$. Therefore, $z \in n(P) \subseteq Q$, which is a contradiction. Hence $Q \subseteq P$ □

Proposition 4.4.5: If n is a medial element of a distributive lattice and P is a prime n-ideal in L, then n(p is the intersection of all minimal prime n - ideals contained in P.

Proof: Clearly, $n(P)$ is contained in any prime n- ideal which is contained in P. Hence $n(P)$ is contained in the intersection of all minimal prime n- ideals contained in P. Since L is distributive, so $n(P)$ is the intersection of all minimal prime n- ideals belonging to it. Since each prime n- ideal contains a minimal prime n- ideal, above remarks and Lemma 4.4.4 establish the proposition □

Theorem 4.4.6: Let L be a distributive lattice and let n be central element in L. Then the following conditions are equivalent.

(i) $P_n(L)$ is normal.

(ii) Every prime n- ideal contains a unique minimal prime n- ideal.

(iii) For each prime n- ideal P, $n(P)$ is prime n- ideal.

(iv) For all $x, y \in L, <x>_n \cap <y>_n = \{n\}$

Implies $<x>_n^* \cap <y>_n^* = L$.

(v) For all $x, y \in L, (<x>_n \cap <y>_n) = <x>_n^* \vee <y>_n^*$.

Proof: (i) $\Rightarrow$ (ii) holds by theorem 4.4.3

(ii) $\Rightarrow$ (iii) is a direct consequence of proposition 4.4.5.

(iii) $\Rightarrow$ (iv). Suppose (iii) is a direct consequence of proposition 4.4.5

(iii) (iv). Suppose (iii) holds.

Consider $x, y \in L$ with $<x>_n \vee <y>_n = \{n\}$

If $<x>_n^* \vee <y>_n^* \neq L$, then by Theorem I.4.7 there exiats a prime n- ideal P.

Such that $<x>_n^* \vee <y>_n^* \subseteq P$, then $<x>_n^* \subseteq P$ and $<y>_n^* \subseteq P$, imply $x \in n(P)$.

And $y \notin n(P)$. But $n(P)$ is prime and so $m(x, n, y) = n \in n(P)$ is contradictory.

Therefore, $<x>_n^* \vee <y>_n^* = L$.

(iv) $\Rightarrow$ (v). Obviously, $<x>_n^* \vee <y>_n^* \subseteq (<x>_n \cap <y>_n)^*$.

Conversely, let $w \in (<x>_n \cap <y>_n)^*$.

Then, $<w>_n \cap (<x>_n \cap <y>_n) = \{n\}$ or, $<m(w, n, x)>_n \cap <y>_n = \{n\}$.

So by (iv), $<m(w, n, x)>_n^* \cap <y>_n^* = L$.

So, $w \in <m(w, n, x)>_n^* \cap <y>_n^*$.

Threfore, $w \wedge n, w \vee n \in <m(w, n, x)>_n^* \vee <y>_n^*$.

Here $w \vee n$ exists as n is an upper element.

Then $w \vee n = r \vee s$ for some $r \in <m(w, n, x)>_n$ and $s \in <y>_n^*$ with $r, s \geq n$.

Now $r \in <m(w, n, x)>_n^*$.

CHAPTER FIVE

Introduction: In lattice theory there are different classes of lattice known as verity of course the most powerful variety. Throughout this capter we will be concerned with another large veriety known as the class opseudocomplemented lattice. Pseudocomplemented lattices have been studied by several authors [16],[17], [25], [26], [27], [28]

In this chapter we have studies relatively pseudocomplemented lattice and Multipler Extension of Sectionally Pseudocomplemented Distributive lattices.

5.1. Relatively Pseudoeomplemented of lattice.

Definition (Pseudoeomplemented element): Let L be a lattice with o and 1 for an element $x \in L$, element $x^{*} \in L$ is called pseudo complement of x if $x \wedge x^{*} = 0$ and $x \wedge y = 0 (y \in L)$ implies $y \leq x^{*}$.

Definition (Pseudoeomplemented lattice): Let L be a bounded distributive lattice, let $a \in L$, an element $a^{*} \in L$ is called a pseudocomplemented of a in L if the following conditions hold:(i) $a \wedge a^{*} = 0$ (ii) $\forall\ x \in L$, $a \wedge x = 0$ implies that $x \leq a^{*}$

Also A lattice L is called pseudocomplemented if its every element has a peudocomplement.

For a lattice L with o, we can talk about sectionally pseudocomplemented lattice,

A lattice L with 0 is called sectionally Pseudocomplemented if the interval $[0, x]$ for each $x \in L$ is pseudocomplemented. Of course every finite distributive lattice is Sectionally pseudocomplemented.

Definition(Relatively Pseudocomplement lattice): A lattice L is called relatively pseudocomplemented if interval $[a,b]$ for each $a,b \in L$, $a < b$ is pseudocomplemented.

Theorem 5.1.1: If L is a distributive sectionally pseudocomplemented lattice, then S_F is a distributive pseudocomplemented lattice.

Proof: Suppose L is sectionally pseudocomplemented. Since L_F is a distributive lattice. Let $[x] \in L_F$, then $[0] \subseteq [x] \leq F$. Now $0 \leq x \wedge f \leq f$ for all $f \in F$. Let y be the pseudocomplement of $x \wedge f$ in $[0, f]$, then $y \wedge x \wedge f = 0$ implies $[y \wedge f] \wedge [x] = [0]$, that is $[y] \wedge [x] = [0]$.

Suppose $[z] \wedge [x] = [0]$, for some $[z] \in L_F$, then $z \wedge x = 0(\psi_F)$.

This implies $z \wedge x \wedge f^1 = 0$ ----------- (i) for some $f^1 \in F$. Since $z = z \wedge f(\psi_F)$.

So $z \wedge f^1 = z \wedge f \wedge f^{11}$.................... (ii) for some $f^{11} \in F$ from (i) and (ii) .

We get $z \wedge x \wedge f^1 \wedge f^{11} = 0$. Setting $g = f^1 \wedge f^{11}$

we have $z \wedge g = z \wedge g \wedge f$ which implies $z \wedge g \leq f$ and $z \wedge g \wedge x \wedge f = 0$.

So $0 \leq z \wedge g \wedge x \leq f$ and
$z \wedge g \leq y$. Hence, $[z \wedge g] \subseteq (y)$. But $[z] = [z \wedge g]$ as $g \in F$. Therefore, $[z] \subseteq [y]$ and so L_F is pseudocomplemented distributive lattice □

Lemma 5.1.2: Let L be a distributive relatively pseudocomplemented lattice.

Let $x \leq y \leq z$ in L and $s \in L$ is the relative pseudocomplemeted of y in $[x, z]$.

Then for any $r \in L, s \in r$ is the relative pseudocomplemet of $y \wedge r$ in $[x \wedge r, z \wedge r]$.

Proof: Suppese $t \wedge r$ is the relative pseudocomplemet of $y \wedge r$ in $[x \wedge r, z \wedge r]$. Since s is the relative pseudocomplemented of y in $[x, z]$. So $s \wedge y = x$, Thus,

$(s \wedge r) \vee (y \wedge r) = x \wedge r$.

This implies $s \wedge r \leq t \wedge r$. Again, $x \leq s \vee (t \wedge r) \leq z$ and

$y \wedge (s \vee (t \wedge r) = (y \wedge s) \vee (y \wedge r) \vee (t \wedge r) = x \vee (x \wedge r)$ implies $s \vee (t \wedge r) \leq s$

i.e $s = s \vee (t \wedge r)$.

Hence $t \wedge r \leq s$, and so $t \wedge r \leq s \wedge r$.

This implies $t \wedge r \leq s \wedge r$. Therefore, $s \wedge r$ is the relative pseudocomplemet of $y \wedge r$ in $[x \wedge r, z \wedge r]$ □

Theorem 5.1.3: If L is a distributive relatively pseudocomplemeted lattice, then L_F is a distributive relatively pseudocomplemeted lattice.

Proof: Since L_F is a distributive lattice. Let $[x],[y],[z] \in L_F$ with $[x] \subseteq [y] \subseteq [z]$.

Then $[x] = [x \wedge y] and [y] = [y \wedge z]$. Thus $x = x \wedge y (\psi_{F)}$ and $y = y \wedge z (\psi_{F)}$.

This implies $x \wedge f = x \wedge y \wedge f$ and $y \wedge g = y \wedge z \wedge g$ for some $f, g \in F$.

Then $x \wedge f \wedge g = x \wedge y \wedge f \wedge g$ and $y \wedge f \wedge g = y \wedge z \wedge f \wedge g$, and so

$x \wedge f \wedge g \leq y \wedge f \wedge g \leq z \wedge f \wedge g$, that is $x \wedge h \leq y \wedge h \leq z \wedge h$ where $h = f \wedge g \in F$.

Suppose t is the relative pseudocomplemented of $y \wedge h$ in $[x \wedge h, z \wedge h]$.

Then $t \wedge y \wedge h = x \wedge, h$ and so $[t] \wedge [y \wedge h] = [x \wedge h]$. That is $[t] \wedge [y] = [x]$ as

$y = y \wedge h(\psi_F)$ and $x = x \wedge h(\psi_F)$.

Moreover, $[t] \wedge [z] = [t] \wedge [z \wedge h] = [t \wedge z \wedge h] = [t]$ implies $[x] \subseteq [y] \subseteq [z]$. We claim that [t] is the relative pseudocomplement of $[y]$ in $[[x],[y]]$ in L_F.

Suppose $[s] \wedge [y] = [x]$ for some $[s] \in [[x],[z]]$. Then $x \wedge y = x(\psi_F)$ and

so $s \wedge y \wedge f^1$ for some $f^1 \in F$. Again $[s] =\subseteq [z]$ implies $s \equiv s \wedge z(\psi_F)$

and so $s \wedge g^1 = s \wedge z \wedge g^1$ for some $g^1 \in F$. Then $s \wedge y \wedge f^1 \wedge g^1 = x \wedge f^1 \wedge g^1$

and $s \wedge f^1 \wedge g^1 = s \wedge z \wedge f^1 \wedge g^1$ · Thus, $s \wedge y \wedge k = x \wedge k$ and $s \wedge k = s \wedge z \wedge k$

where $k = f^1 \wedge g^1$. These imply $x \wedge h \wedge k \leq s \wedge h \wedge k \leq z \wedge h \wedge k$ and

$(s \wedge h \wedge k) \wedge (y \wedge h \wedge k) = x \wedge h \wedge k$. Then by above lemma, $s \wedge h \wedge k \leq t \wedge k$.

Hence $[s] = [s \wedge h \wedge k] \subseteq [t \wedge k] = [t]$ and so $[t]$ is relative pseudocomplemet of $[y]$ in $[[x],[y]]$. Therefore, L_F is relatively pseudocomplemet □

Definition (Stone algebra): A pseudocomplemented distributive lattice L is called a stone algebra if and only if it satisfies the condition for each $a^* \vee a^{**} = I$ which is known as stone identity.

5.2 Multiplier Extension of Sectionally Pseudo-complemented Distributive Lattices.

Introduction: A lattice is a meet semilattice together with the property that any two elements possessing a common upper bound have a supremum. Here, the authors study multipliers on distributive lattices which are sectionally in B_n, $-1 \le n \le \omega$. They have showed that a distributive lattice L is sectionally in B_n if and only if its set of all multipliers $M(L)$ is in B_n. Moreover, for $1 \le n \le \omega$, the above conditions are also equivalent to the condition that L is sectionally pseudo-complemented, and for any $n+1$ minimal prime ideals $P_1,\ldots, P_{n+1}$, $P_1 \wedge \ldots \wedge P_{n+1} = L$.

Let L be a lattice and σ a mapping of L into itself. Then σ is called a multiplier on L, if $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$ for each $x, y \in L$. Each multiplier on L has the following properties:

$\sigma(x) \le x$, $\sigma(=\sigma(x)\sigma(x))$ and $x \le y$ implies $\sigma(x) \le \sigma(y)$.

Each $a \in L$ induces a multiplier μ_a defined by $\mu_a(x) = a \wedge x$ for each $x \in L$, which is called an inner multiplier. The identity function on L, which will be denoted by i, is always a multiplier. $M(L)$ denotes the set of all multipliers on L. It is obvious that $M(L)$ has a zero denoted by ω if and only if L has a 0.

If σ and λ are multipliers on a lattice L, then $\sigma \wedge \lambda$ and $\sigma \vee \lambda$ are defined by $(\sigma \wedge \lambda)(x) = \sigma(x) \wedge \sigma(x)$ and $(\sigma \vee \lambda)(x) = \sigma(x) \vee \sigma(x)$. Note that $\sigma(x) \vee \sigma(x)$ always exists by the upper bound property of L, as $\sigma(x), \sigma(x) \le x$, although $\sigma \vee \lambda$ is not necessarily a multiplier.

Also, $\sigma(\lambda(x) = \sigma(\lambda(x \wedge x)) = \sigma(\lambda(x) \wedge x) = \sigma(x) \wedge \sigma(x)$. If L is a distributive lattice, then $M(L)$ is a distributive lattice.

A distributive lattice L with 0 is called sectionally pseudo-complemented if each interval $[0, x], x \in L$ is pseudo-complemented.

Let L be a sectionally pseudo-complemented distributive lattice and $\sigma \in M(L)$. We define the pseudo-complement of σ (denoted by σ^*) by $\sigma^*(x) = \sigma(x)^*$, where $\sigma(x)^*$ is the relative pseudo-complement of $\sigma(x)$ in $[0, x]$ for each $x \in L$. In fact, we have given a proof of this in Proposition 5.2.2

In this section, we study multipliers on sectionally pseudo-complemented distributive lattices and also on distributive latices which are sectionally in B_n, $-1 \le n \le \omega$. Then we generalize a number of results of [1]. We show that L is sectionally in B_n if and only if $M(L)$ is in B_n. We also show that, for $1 \le n \le \omega$, the above conditions are also equivalent to the conditions that L is sectionally pseudo-complemented and for any $n+1$ minimal prime ideals $P_1, \ldots, P_{n+1}$, $P_1 \wedge \ldots \wedge P_{n+1} = L$.

Multipliers on Distributive lattices which are Sectionally in B_n

Lee [4] has determined the lattice of all equational subclasses of the class of all pseudo-complemented distributive lattices. They are given by $B_{-1} \subset B_0 \subset B_1 \subset \ldots \subset B_n \subset \ldots \subset B_\theta$, where all the inclusions are proper and B_θ is the class of all pseudo-complemented distributive lattices, B_{-1} consists of all one element algebras, B_0 is the variety of Boolean algebras while B_n, for $1 \le n \le \omega$, consists of all algebras satisfying the equation

$$(x_1 \wedge x_2 \wedge \ldots \wedge x_n)^* \vee \bigvee_{i=1}^{n} \left(x_1 \wedge \ldots \wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge \ldots \wedge x_n\right)^* = 1,$$

where x^* denotes the pseudo-complement of x . Thus B_1 consists of all Stone algebras.

A lattice L is sectionally complemented if $[0, x]$ is complemented for each $x \in L$. L is semiboolean if it is sectionally complemented and distributive.

Recall that a distributive lattice L with 0 is sectionally pseudo-complemented if each interval $[0, x]$, $x \in L$ is pseudo-complemented.

Theorem 5.2.1: A lattice L is distributive if and only if $M(L)$ is a distributive lattice.

Proposition 5.2.2: If L is a sectionally pseudo-complemented distributive lattice with 0, then $M(L)$ is pseudo-complemented.

Proof: For each $\sigma \in M(L)$ and $x \in L$, $\sigma(x) \in [0,x]$. Suppose $\sigma(x)^*$ denotes the pseudo-complement of $\sigma(x)$ in $[0,x]$. Define $\sigma^* : L \to L$ by $\sigma^*(x) = \sigma(x)^*$ for each $x \in L$. If $a, b \in L$,

then $(\sigma^*(a) \wedge b) \wedge \sigma(a \wedge b) = \sigma^*(a) \wedge b \wedge \sigma(a) \wedge b = 0$ implies

$(\sigma^*(a) \wedge b) \leq \sigma(a \wedge b)^* = \sigma^*(a \wedge b)$. On the other hand,

$(\sigma^*(a \wedge b) \wedge \sigma(a) = \sigma^*(a \wedge b)^* \wedge \sigma(a) = \sigma(a \wedge b)^* \wedge \sigma(a) \wedge b = 0$

implies $\sigma^*(a \wedge b) \leq \sigma(a)^* \sigma^*(a)$.

Since $\sigma^*(a \wedge b) \leq b$, so $\sigma^*(a \wedge b) \leq \sigma(\sigma^*(a) \wedge b$.

Therefore, $\sigma^*(a \wedge b) = \sigma^*(a) \wedge b$ and so $\sigma^* \in M(L)$.

Now, $(\sigma \wedge \sigma^*)(x) = \sigma(x) \wedge \sigma^*(x) = 0 = \omega(x)$ implies $\sigma \wedge \sigma^* = \omega$. If

$\sigma \wedge \tau = \omega, \quad \sigma(x) \wedge \tau(x) = 0$.

Then for each $x \in L$. Since $\sigma(x), \tau(x) \in [0,x]$, so $\tau(x) \leq \sigma(x)^* = \sigma^*(x)$.

This implies $\tau \leq \sigma^*$ and so σ^* is the pseudo-complement of σ in $M(L)$. Therefore, $M(L)$ is pseudo-complemented □

Proposition 5.2.3: For a distributive lattice L with 0, if $M(L)$ is pseudo-complemented then L is sectionally pseudo-complemented.

Moreover, for each $\sigma \in M(L)$ and $x \in L$, the element $\sigma^*(x)$ is the relative pseudo-complement of σ (x) in $[0,x]$.

Proof: Consider any interval $[0,y]$ in L. Suppose $x \in [0,y]$.

Then $0 = \omega(y) = (\mu_x \wedge \mu_x^*)(y) = \mu_x(y) \wedge \mu_x^*(y) = x \wedge y \wedge \mu_x^*(y) = x \wedge \mu_x^*(y)$. Now, if $x \wedge t = 0$ for some $t \in [0,y]$, then for all $p \in L$, $(\mu_x \wedge \mu_t)(p) = x \wedge t \wedge p = 0$, and so $\mu_x \wedge \mu_t = \omega$. This implies $\mu_t \leq \mu_x^*$. Thus, $\mu_t(y) \leq \mu_x^*(y)$, and so $t = t \wedge y \leq \mu_x^*(y)$. Hence, $\mu_x^*(y)$ is the relative pseudo-complement of x in $[0, y]$. Therefore, L is sectionally pseudo-complemented.Finally, for each $x \in L$, $\sigma(x) \wedge \sigma^*(x) = 0$. Also $\sigma^*(x) \in [0, x]$. Now let $t \wedge \sigma(x) = 0$ for some $t \in [0, x]$. Then for any $p \in L$, $(\mu \wedge \sigma)(p) = \mu(p) \wedge \sigma(p) = t \wedge p \wedge \sigma(p) = t \wedge \sigma(p) = t \wedge x \wedge \sigma(p)$

$= t \wedge p \wedge \sigma(x) = 0 = \omega(p)$. This implies $\mu \wedge \sigma = \omega$ and so $\mu \leq \sigma^*$. Then $\mu(x) \leq \sigma^*(x)$

Thus, $t = t \wedge x \leq \sigma^*(x)$. This shows that $\sigma^*(x)$ is the pseudocomplement of $\sigma(x)$ in $[0, x]$ □

Corollary 5.2.4: Suppose L is a sectionally pseudo-complemented distributive lattice with 0. If x^* is the pseudo-complement of x in $[0, y]$, then $x^+ = \mu^*_x(y)$.

Theorem 5.2.5: Let L be a distributive with 0. For a given n such that $-1 \leq n \leq \omega$, the following conditions are equivalent:

(i) L is sectionally in B_n;

(ii) $M(L)$ is in B_n.

Proof: (i) implies (ii). The case $n = -1$ is trivial. The case $n = \omega$ follows from Proposition 4.2.2

For $n = 0$, L is semiboolean. Then by Proposition 4.2.2, $M(L)$ is pseudo-complemented and for $\sigma \in M(L)$, $\sigma^*(x) = \sigma(x)^*$ for each $x \in L$, where $\sigma(x)^*$ is the pseudo-complement of $\sigma(x)$,$[0, x]$. Since L is semiboolean, $\sigma(x)^*$ is also the relative complement of $\sigma(x)$ in $[0, x]$. Then $(\sigma \vee \sigma^*)(x) = \sigma(x) \vee \sigma^*(x) = \sigma(x) \vee \sigma^+(x)x = \iota(x)$.

This implies $\sigma \wedge \sigma^* = \iota$ and so σ^* is also the complement of σ in $M(L)$.

Therefore, $M(L)$ is Boolean.

Now, suppose L is sectionally in B_n. $1 \leq n \leq \omega$. For $\sigma_1 ....\sigma_n \in M(L)$ and for each $x \in L$, using Proposition 5.2.2

$$\left[(\sigma_1 \wedge \ldots \wedge \sigma_n)^* \vee \bigvee_{i=1}^{n} \left(\sigma_1 \wedge \ldots \wedge \sigma_i^* \wedge \ldots \wedge \sigma_n\right)^*\right](x)$$

$$= (\sigma_1 \wedge \ldots \wedge \sigma_n)^*(x) \vee \bigvee_{i=1}^{n} \left(\sigma_1 \wedge \ldots \wedge \sigma_i^* \wedge \ldots \wedge \sigma_n\right)^*(x)$$

$$= ((\sigma_1 \wedge \ldots \wedge \sigma_n)(x))^+ \vee \bigvee_{i=1}^{n} ((\sigma_1 \wedge \ldots \wedge \sigma_i^* \wedge \ldots \wedge \sigma_n)(x))^*$$

$$= (\sigma_1(x) \wedge \ldots \wedge \sigma_n(x))^+ \vee \bigvee_{i=1}^{n} \left(\sigma_1(x) \wedge \ldots \wedge \sigma_i^*(x) \wedge \ldots \wedge \sigma_n(x)\right)^*$$

$$= (\sigma_1(x) \wedge \ldots \wedge \sigma_n(x))^+ \vee \bigvee_{i=1}^{n} \left(\sigma_1(x) \wedge \ldots \wedge \sigma_i(x)^+ \wedge \ldots \wedge \sigma_n(x)\right)^*$$

$$= x = \iota(x).$$

Hence, $(\sigma_1 \wedge \ldots \wedge \sigma_n)^* \vee (\sigma_1^* \wedge \ldots \wedge \sigma_n)^* \vee \ldots \vee (\sigma_1 \wedge \ldots \wedge \sigma_n^*)^* = \iota$ and so $M(L)$ is in B_n

(ii) implies (i). The case $n = \omega$ follows from Proposition 5.2.3. For $n = 0$, $M(L)$ is Boolean. Then by Proposition 5.2.3, L is sectionally pseudo-complemented.

Suppose $x \in [0, y]$. Then the pseudo-complement μ_x^* of μ_x is also the complement of μ_x. Thus, $\mu_x \vee \mu_x^* = \iota$. If x^+ is the pseudo-complement of x in $[0, y]$, then by Corollary 5.2.4

$y = \iota(y) = (\mu_x \vee \mu_x^*)(y) = \mu_x(y) \vee \mu_x^*(y) = \mu_x(y) \vee \mu_x^*(y) = (x \wedge y) \vee x^* = x \vee x^*$.This implies x^+ is the relative complement of x in $[0, y]$ and hence, L is semiboolean.

Now, suppose $M(L)$ is in B_n, $1 \le n \le \omega$. Let $x_1 \ldots\ldots\ldots x_n \in [0, y]$.

Then using Proposition 5.2.2

$$y = \iota(y) = \left[(\mu_{x_1} \wedge \ldots \wedge \mu_{x_n})^* \vee \bigvee_{i=1}^{n} (\mu_{x_1} \wedge \ldots \wedge \mu^*_{x_1} \wedge \ldots \wedge \mu_{x_n})^*\right](y)$$

$$= ((\mu_{x_1} \wedge --- \wedge \mu_{x_n})^* \quad \square$$

Lemma 5.2.6:

(i) Let L be a distributive lattice with 0. If $0 \le x \le L$ and the interval $[0,x]$ is pseudo-complemented, where y^{+} is the pseudo- complement of $x \in [0,x]$ then in the lattice of ideals of L . $(y]^{*} \cap (x]$ and $(y^{++} = (y)^{**} \cap (x]$.

(ii) If L is a distributive near lattices with 0 and $0 \le x \le L$ is such that $(y]^{*} \cap (x]$ is principal for each $y \in [0,x]$, then $[0,x]$ is pseudo-complemented and $(y]^{*} \cap (x] = (y^{+}]$.

Lemma 5.2.7: Let L be a distributive lattice with *0*. For any $r \in L$ and any ideal I,

$((r] \cap I)^{*} \cap (r] = I^{*} \cap (r]$.

Proof: Obviously, RHS $\subseteq$ LHS. To prove the reverse inclusion, let

$t \in ((r] \cap I) \cap (r]$.

Then $t \le r$ and $t \wedge r \wedge i = 0$ for all $i \in I$. This implies $t \wedge i = 0$ and so $t \in I^{*}$. Thus, $t \in I^{*} \cap (r]$ and this completes the proof □

Lemma 5.2.8: If S is sub lattice of a distributive lattice L and is a prime ideal in L_L, then there exists a prime ideal P in L such that $P_1 = S_1 \cap P$.

Proof: Let I be the ideal generated by P_1 in L . Then $I = (H]$ $]$ where H is the hereditary subset of L generated by P_1 . Suppose $x \in I \cap (L_I\text{-}P_1)$. Then $x \in I$ and $x \in S_1 - P_1$. Then by Theorem 11 in [2], $x = h_1 \vee .... \vee h_i$ for some $h_1...............h_i \in H$. Again, $h_i \in H$ implies $h_1 \le t_1 \le t_i$ for some $t_i \in P_i$, i = 1,2........n .

Then $x = (x \wedge h_1) \vee ....(x \wedge h_n) \le (x \wedge t_i) \vee ....(x \wedge t_n) \le x$ (this exists by the upper bound property). Thus, x = ($x = (x \wedge t_1 \;) \vee ..... \vee (x \wedge t_n \;) \in p_1$) which gives a contradiction. Therefore, $I \cap (L_1 - P_1) = \varphi$. Then as $L_1 - P_1$ is a filter in L_1, $I \cap (L_1 - P_1) = \varphi$, where $(L_1 - P_1)$ is the filter generated by $L_1 - P_1$ in L. Then by Theorem in [7], there is a prime ideal P in L such that $I \subseteq P$ and $(L_1 - P_1) \cap P = \phi$. Then $P_1 \subseteq I \cap L_1 \subseteq P \cap L_1$ and $P \cap L_1 \subseteq P_1$ .Hence, $P_1 = L_1 \cap P$

A prime ideal P of a near lattice L with 0 is a minimal prime ideal if there exists no prime ideal Q such that $Q \subseteq P$. Thus, we have the following corollary. We omit the proof as it can be done in a similar way.

Corollary 5.2.9: If L_1 is a sub lattice with a smallest element of a distributive lattice L_1 with O and P_1 is a minimal prime ideal in L_1, then there exists a minimal prime ideal P in L such that $P_1 = L_1 \cap P$.

We conclude this paper with the following theorem which is a nice extension of Theorem4.5 in [1].

Theorem 5.2.10: Let L be a distributive lattice with 0. For a given n such that $1 \le n \le \omega$ the following conditions are equivalent

i) L is sectionally in B_n

ii) $M(L)$ is in B_n

iii) For any $y \in L$ and for $x_1 \ldots\ldots\ldots x_n \in (y]$,

$(y] \subseteq ((x_1] \wedge \ldots\ldots \wedge (x_n]^*) \vee ((x_1])^* \vee \ldots\ldots \vee ((x_1])^* ((x_1] \wedge \ldots\ldots \wedge (x_n]^*)^*$

(iv) For any $x_1 \ldots\ldots\ldots x_n \in L$

$((x_1] \wedge \ldots \wedge (x_n])^* \vee ((x_1]^* \wedge \ldots \wedge (x_n])^* \vee \ldots \vee ((x_1] \wedge \ldots \wedge (x_n]^*)^* = L$

(v) L is sectionally pseudocomplemented and each prime ideal contains at most n minimal prime ideals;

(vi) L is sectionally pseudocomplemented and for any n+1 distinct minimal prime ideals $P_1, \ldots\ldots, P_{n+1}, P_1 \vee \ldots\ldots \vee P_{n+1} = L$

$$((r \wedge x_1] \wedge ....\wedge (r \wedge x_n])^* \wedge (r]) = ((x_1] \wedge .....\wedge (x_n])^* \wedge (r]$$

Again, for each $1 \le i \le n, r \wedge x_i \le x_i$ implies $(r \wedge x_i]^* \supseteq (x_i]^*$ Thus.

$$((r \wedge x_1] \wedge ....\wedge (r \wedge x_i])^* \wedge ....(r \wedge x_n]) \supseteq ((r \wedge x_1] \wedge .....\wedge (r \wedge x_i]^* \wedge .....(r \wedge x_n]).$$

Psendocomplemented distributive lattices

and so

$$((r \wedge x_1] \wedge ....\wedge (r \wedge x_i])^* \wedge ....(r \wedge x_n])^* \wedge (r]$$

$$\subseteq ((r \wedge x_1] \wedge .....\wedge (r \wedge x_i]^* \wedge .....(r \wedge x_n])^* \wedge (r])$$

$$= ((x_1] \wedge .....\wedge (x_i])^* \wedge ....\wedge (x_n])^* \wedge (r].$$

By using Lemma 5.2.7 again.

Therefore, $(r] \subseteq ((x_1] \wedge ....\wedge (x_n]^* \vee ((x_1]^* \wedge ......\wedge (x_n])^* \vee ..... \vee ((x_1 \wedge ......x_n]^*)*$. Which

implies that,

$$((x_1] \wedge ....\wedge (x_n]^* \vee ((x_1]^*) \wedge ......\wedge (x_n])^* \vee ..... \vee ((x_1 \wedge ......x_n]^*)* = L$$

If $n = 1$, then for any r, we have by (iii) that

$$(r] \subseteq (c \wedge x_1]^* \vee (r \wedge x_1]^{**}.$$

Thus,

$$(r] = ((r \wedge x_1]^* \cap (r] \vee ((r \wedge x_1]^{**} \cap (r])$$

$$= ((x_1]^* \cap (r]) \vee ((r \wedge x_1]^{**} \cap (r]) \quad \text{(by Lemma 2.7)}$$

$$\subseteq (x_1]^* \vee (x_1]^{**}$$

and hence $(x_1]^* \vee (x_1]^{**} = L$

(iv) implies (i) following exactly from the same proof of Theorem 5.2.5 (iv) $\Rightarrow$ (i) in [1].

(v) implies (vi). Suppose (v) holds, and $P_1, ..., P_{n+1}$ are distinct minimal prime ideals.

If $P_1 \vee .... \vee P_{n+1} \neq L$, them by Theorem 5.2.6 , there exists a prime ideal P

containing $P_1, ..., P_{n+1}$ which contradicts (v),

(vi) implies (v). Suppose (vi) holds. If (v) does not hold then there exists a prime ideal P which contains more then n minimal prime ideals. They by (vi), $P = L$ which is impossible.

(v) implies (vi). We omit this proof as it can be prove exactly in a similar way

(iv) implies (vi) in Theorem 5.2.5 in [1].

(vi) implies (i). Suppose (vi) holds and $a \in L$. Let $Q_1, \ldots\ldots\ldots\ldots, Q_{n+1}$ be $n+1$ distinct minimal prime ideals in $[0, a]$ .By Corollary 5.2.9, there are minimal prime ideals P_i in L, such that $Q_i = [0, a]P_i$ for each $1 \le i \le n+1$. Since Q_i are distinct, all P_i's are also distinct. By(vi),

$(a] = (a] \wedge (P_1 \vee \ldots\ldots\ldots\ldots \vee P_{n+1}) = ((a] \vee P_1) \vee \ldots\ldots\ldots\ldots \vee ((a] \wedge P_{n+1}) = Q_1 \vee \ldots\ldots\ldots\ldots \vee Q_{n+1}$.
Since each interval $[0, a]$ is pseudocomplemented, so $[0, a]$ = B_n by Theorem 1 in [4], and hence, L is section ally in B_n □

CHAPTER SIX

Homomorphism and standard ideals

Introduction: In this Chapter we studies extensively standard ideal and homomorphism kernels. The idea of standard ideals in lattice was first introduced by [15], [21]. It had extended the ideal to convex sub lattices and proved many results on homomorphism by [10], [33] also [7] and [8]

A congruence φ of a lattice L is called standard if $\phi = \Theta_{(s)}$ for some standard ideal S of L. For any two lattice L_1 and L_2, a map $\phi : L_1 \to L_2$ is called an isotone if for any $x, y \in L$, with $x \leq y$ implies $\phi(x) \leq \phi(y)$. Also the above mapping is called a meet homomorphism if for all $x, y \in L_1$, $\phi(x \wedge y) = \phi(x) \wedge \theta(y)$.

Therefore meet homomorphism is an isotone, and hence ϕ is isotone.

$\therefore \phi(x) \vee \phi(y) \leq \phi(x \vee y)$. Therefore $\phi(x) \vee \phi(y)$ exists by upper bound property of L.Latif in his thesis has introduced the concept of standard n- ideals of a lattice. We conclude this section with some more properties of standard and neutral ideals, For the background meterial on standard ideals we refer the reader to consult the text of Gratzer [18]. We also extended the result of Cornish and A.S.A Noor [4] , we also show that If I is an arbitrary ideal and S is standard ideal then $I \wedge S$ and $I \vee S$ are principal, then I itself principal.

Secondly, we have discussed homomorphism, kernels and stadard ideals. Gratzer and Schmitd in [15] were translated several thorem of Group theory to lattice theory. Here we have generalized so of their result, we have shown that if S is a standard ideal of a lattice L, then Θ_S the extension of Θ (S) to $I(L)$ and Θ (S) is the restriction of Θ_S to the lattice L. Then we have shown that in a sectionally complemented lattice all congruences are standard. We also show that in a relatively complemented lattice L with 0, if every standard ideal of L is generated by a finite number of standard elements, then the congruence lattice $C(L)$, is Boolean.

Finally, we have generalized two results of [5] and [6] regarding lattices all of whose congruence are standard. We know that the set of all standard ideals of a lattice L is a sub lattice of $I(L)$. Also the congruence Θ_S where S is standard form a sub lattice ofΘ $(I(L))$, and $S \to \Theta_S$ is an isomorphism. Suppose Θ is a congruence relation of L. θ defines in the natural way a homomorphism of $I(L)$ under which $I = J(I, J \in I(L))$ if and only if to any $x \in I$ there exists $a, b \in J$ such that $x \equiv y\Theta$ and conversely. We call this congruence relation the extension of Θ to I (L). On the other hand any congruence relation φ of $I(L)$ induces a congruence relation of A under which $x \equiv y$ if and only if$(x] \equiv (y](\varphi)$. This is called the restriction of φ to L.

Thirdly in [15] Gratzer and Schmidt have proved Isomorphism theorem for standard ideals in lattices. In their paper they have translated several theorems of group theory to lattice theory using ideal, standard ideal, factor group and group operation. Here we shall generalize isomorphism theorem.

We refer the reader to [6], [7], [8], [9] for a necessary background on this section.

6.1 Charecterization of standard ideals

We start with the following characterization of standard ideals in a lattice, which is due to [4]. We prefer to include the proof for the convenience of the reader.

Theorem:6.1.1 Let S be an ideal in a lattice L. Then the following conditions are equivalent.

(i) S is a standard ideal.

(ii) The binary relation $\Theta_{(s)}$, defined by $x \equiv y\Theta_{(s)}$, if and only if

$x = (x \wedge y) \vee (x \wedge a) \vee y = (x \wedge y) \vee (y \wedge b)$ for some $a, b \in S$ is a congruence relation.

iii) The binary relation ϕ defined by $x \equiv y(\phi)$, if and only if

For all $t \in L, (x \wedge t) \vee (t \wedge c) = (y \wedge t) \vee (t \wedge c)$ for some $c \in S$ is a congruence.

iv) For each ideal $K, S \vee K = s \vee k : s \vee k$ exists, and $s \in S$ and $k \in K$. Moreover, (ii) and (iii) represent the same congruence, viz. $\Theta(S)$, the smallest congruence of L having S as congruence class.

Proof: (i) implies (ii). If (i) holds, then the relation

$J \equiv K(\Theta s)(J, K \in I(L))$ if and only if $J = (J \cap K) \vee (J \cap S)$ and $K = (J \cap K) \vee (K \cap S)$ is a congruence on $I(L)$. Then $\Theta s / L$, restriction to L, is a congruence on L and $x \equiv y(\Theta s / L)$ if and only if $(x) = (x \wedge y) \vee (x \cap S)$ and $(y] = (x \wedge y)((y] \cap S\ )$.

Thus to prove (ii), it is sufficient to prove that $(x\ ] = (x \wedge y\ ] \wedge ((x\ ] \cap S\ )$

implies $x = (x \wedge y) \vee (x \wedge a)$ for some $a \in S$. This is proved by induction. By the property of the supremum of two ideals, $(x \wedge y] \vee ((x] \cap S = \bigcup_{n=0}^{\alpha} Ln$, where $L_0 = (x \wedge y\ ] \cup ((x\ ] \cap S$ and

$Ln = \{t \in L : t \leq p \vee q; p \vee q$ exists and $p, q \in A_{n-1}\}$ for n=1,2,............

Indeed, we show by induction that $(x \wedge y\ ] \vee ((x] \cap S) = \{t : t \leq (x \wedge y] \vee (x \vee a)$

$(x \wedge y\ ] \vee ((x] \cap S) = \{t : t \leq (x \wedge y] \vee (x \vee a)$ for some $a \in S$ }.

If $t \in L$ then $t \in (x \wedge y]$

Or, $t \in (x] \cap S$. In the first instance, $t \leq x \wedge y \leq (x \wedge y) \vee (x \wedge s)$ for any $s \in S$ and in the second instance $t = t \wedge x \leq (x \wedge y) \vee (x \wedge t)$ and $t \in S$. Thus the result holds for $n = 0$. Suppose the result hold for $n-1$ for some $n \geq 1$. Let $t \in L$, then $t \leq p \vee q$ with $p, q \in L_{n-1}$.

So $p \leq (x \wedge y) \vee (x \wedge s_1)$ and $q \leq (x \wedge y) \vee (x \wedge s_2)$ for some $s_1, s_2 \in S$

Then $t \leq (x \wedge y) \vee (x \wedge s_1) \vee (x \wedge s_2) = (x \wedge y) \vee (x \wedge s)$.

For some $s \in S$ since $(x \wedge s_1) \vee (x \wedge s_2) \leq S$ and is in S, it is of the form $x \wedge s$ for some $s \in S$. Thus, we have $(x \wedge y] \vee ((x] \cap s) = \{t : t \leq (x \wedge y) \vee (x \wedge s)$ for some $s \in S$; in fact, $x \leq (x \wedge y) \vee (x \wedge a)$ for some $a \in S$ and so $x = (x \wedge y) \vee (x \wedge a)$; as required.

ii) Implies (iii), Let $x \equiv y(\Theta(S))$. Since $\Theta(S)$ is a congruence , $x \wedge t = y \wedge t(\Theta(S))$ for any $t \in L$, and so $x \wedge t = (x \wedge y \wedge t) \vee (x \wedge t \wedge a)$ and $\wedge t = (x \wedge y \wedge t) \vee (x \wedge t \wedge b)$ for some $a, b \in S$ .Then

$(x \wedge t) \vee [t \wedge [t \wedge a) \vee (t \wedge b)] = (x \wedge t) \vee (t \wedge a) \vee (t \wedge b)(x \wedge y \wedge t) \vee (t \wedge a) \vee (t \wedge b)$

$= (y \wedge t) \vee (t \wedge a) \vee (t \wedge b) = (y \wedge t) \vee (t \wedge [(t \wedge a) \vee (t \wedge b)$.

Observe taht $(t \wedge a) \vee (t \wedge b) \in S$. Thus, $x \equiv y(\phi)$

Conversely, if $x \equiv y(\phi)$ then for any $t \in L$, $(x \wedge t) \vee (t \wedge c) = (y \wedge t) \vee (t \wedge c)$ for some $c \in S$. Choosing $t = x$ and $t = y$, we have $x = (x \wedge y) \vee (x \wedge t)$ and $y = (x \wedge y) \vee (y \wedge c)$ respectively. Thus, $x \equiv y(\Theta(S))$ and ϕ is the congruence $\Theta(S)$.

iii) Implies (iv). Let $T = \{s \vee k : s \vee k$ exixts and $s \in S$ and $k \in K\}$. Suppose $x \leq s \vee k$, $s \in S, k \in K$. Clearly $s \vee k \equiv k(\Theta(S)$ and so $x = x \wedge (s \vee k) \equiv (x \wedge k)(\Theta(S))$.

Hence for all $t \in L$,

$(x \wedge t) \vee (t \wedge c) = (x \wedge k \wedge t) \vee (t \wedge c)$ for some $c \in S$ Choosing $t = x$, we obtain $x = (x \wedge k) \vee (x \wedge c)$ and so $x \in T$. But T is closed under existent finite suprima. It follows that T is an ideal of L and $T = S \vee K$.

iv) Imlies (i) Let $x \in J \cap (S \vee K)$ .Then $x \in J$ and $x \in (s \vee k)$. So $x = s \vee k$ for suitable $s \in S$. And $k \in K$ Then, $x = (x \wedge s) \vee (x \wedge k)$ and so $x \in (J \cap S) \vee (J \cap K)$. The reverse inclusion is obvious. Thus $J \cap (S \cap K) = (J \cap S) \vee (J \cap K)$; S is a standard ideal. The last part is clear from the proof of (ii) implies (iii). Now we give another characterization of standard ideals of a lattice. This is a generalization of [15, Theorem 2] □

Theorem 6.1.2: For an ideal S of a Lattice L, the following conditions are equivalent;

(i) S is a standard ideal.

ii) The equality $I \cap (S \vee K) = (I \cap S) \vee (I \cap K)$ holds if I and K are principal ideals.

(iii) If for the principal ideals I and J the inequality $J \subseteq S \vee I$ holds, then

$J = (J \cap S) \vee (J \cap I)$.

iv) The relation Θ [S] of L defined by $x \equiv y(\Theta(S))$ hold if and only if

$x = (x \wedge y) \vee (x \wedge a)$,

$y = (x \wedge y) \vee (y \wedge b)$ for some $a, b \in S$ is a congruence realtion.

Proof: (i) Implies (ii) is obvious, from the definition of the standard ideal.

(ii) Implies (iii) is clear.

iii) Implies (iv). Obviosly the relation is an equivalence relation. Let $x \leq y$ and $x \equiv y(\Theta(S))$ then $y = x \vee (y \wedge b)$ y for some $b \in S$. Suppose for some $t \in L$, $y \vee t$ exists. Then $y \vee t$ exists.

Hence, $y \vee t = (x \vee t) \vee (y \wedge b) \leq (x \vee t) \vee ((y \vee t) \wedge b) \leq y \vee t$

thus $y \vee t = (x \vee t) \vee (y \vee t) \wedge b)$. So $x \vee t \equiv y \vee t(\Theta S))$.

Now, $y \wedge t \leq x \vee (y \wedge b) \in (x] \vee S$, so $(y \wedge t] \subseteq (x] \vee S$

Then by (iii) $(y \wedge t] = (x \wedge y \wedge t) \vee (s \wedge y \wedge t],0 = (x \wedge t] \vee (s \wedge (y \wedge t])$. Then a similar proof of (i) implies (ii) of Theorem 5.1.1 shows that $y \wedge t = (x \wedge t) \vee (y \vee t) \wedge a)$; for some $a \in S$.

Thus (S) is a congruence relation.

(iv) implies (i) holds by theorem 5.1.1 □

We conclude this section with the following result, which is a generalization of a well known result of lattice theory of [15, Lemma 8]

Theorem: 6.1.3: Let I be an arbitrary and S be a standard ideal of the lattice L.

If $I \vee S$ and $I \cap S$ are principal, then I itself is principal.

Proof: Let $I \vee S = (a]$ and $I \cap S = (b]$. Then by Theorem 6.1.1, $a = x \vee s$ for some $x \in I$ and $s \in S$. Since $b \leq a$ and $x \leq a$, so $x \vee b$ exists by the upper bound property of L. We claim that $I = (x \vee b)$. Of course $(x \vee b] \subseteq I$. For the reverse inequality, let $t \in I$. Since $t, x \vee b \leq a$ so again by the upper bound property of L, $w = t \vee x \vee b$ exixts and $w \in L$.

Then $(a] \supseteq S \vee ((w] \supseteq S \vee (x \vee b] \supseteq S \vee (x] = (a\}$, i.e., $S \vee (w] = S \vee (x \vee b]$.

Further, $(b] = S \cap I \supseteq S \cap (x \vee b] = s \cap (b] = (b]$, and so $S \cap (w] = s \cap (x \vee b]$.

This two equalities imply that $(w] = (x \vee b]$ as S is standard and so $w = x \vee b \in (x \vee b]$.

Since $t \leq w, t \in (x \vee b]$ and hence $I = (x \vee b]$, which completes the proof □

6.2 Standard ideals and Homomorphism kernels

Gratzer and Schmidt in [15] proved many results on homomorphism kernels and standard ideals of a lattice. Their main aim was to translate several theorems of Group theory to lattice theory. In this chapter we have Generalized some of their results. We have also given the charecterizations of those lattices whose all congruences are standard, which are generalizations of two papers [5] and [6].

A congruence φ of a lattice L is called a standard if $\varphi = \Theta(S)$ for some standard ideal S of L.

Definition (Isotone): For any two lattice L_1 and L_2, a map $\varphi : L_1 \to L_2$ is called an istone if for any $x, y, \in L_1$ with $x \leq y$ implies $\varphi(x) \leq \varphi(y)$.

Definition(Meet homomorphism): For any two lattices L_1 and L_2, a map $\varphi : L_1 \to L_2$ is called a meet homomorphism if for all $x, y \in L_1$, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$.

Therefore, it is clear that every meet homomorphism is an istone.

Defination(Join homomorphism): $\varphi : L_1 \to L_2$ is called Join homomorphism if $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for all $x, y \in L_1$.

Since φ is istone $\varphi(x) \vee \varphi(y) \leq \varphi(x \vee y)$. Therefore, $\varphi(x) \vee \varphi(y)$ exists by the upper bound property of L_2.

In chapter 1, we have given homomorphism theorem for lattice. Now we generalize two isomorphism theorems of [9] for lattice.

Definition: If $\theta : L_1 \to L_2$ be an onto homomorphism .The set $\{x \in L_1 / \theta(x) = o^1\}$ where 0^1 is least element of L_2 is called kernel of θ and is denoted by Kerθ if L_2 does not have the zero element , ker ϕ does not exist.

Definition(Sublattice): A non empty subset A of a lattice L is called a sub lattice of L if $a,b \in A$ implies that $ab, a \vee b \in$ A. If L is any lattice and $a \in L$ then {a} is sublattice of L.

Theorem 6.2.1: Homomorphic image of relatively complemented lattice is relatively complemented.

Proof: $\varphi: L_1 \to L_2$ be an onto homomorphism and suppose L_1 is relatively complemented.

Let $[x^1, y^1]$ be any interval in L_2 since θ is onto homomorphism ,

$\exists$ Pre-images x and y for x^1, y^1 respectively such that $\varphi(x) = x^1$

$\varphi(y) = y^1$ and $x < y (ax^1 \angle y^1)$.

Thus [x, y] is an interval in L_1.

Let $b \in [x^1, y^1] = [\phi(x), \phi(y)]$ be any element. Then as before $\exists$ a pre-image a of b, such that $\theta(a) = b$ and $x \le a \le y$.

Now L_1 relatively complemented implies that a has a complement a^1 relative to $[x^1, y^1]$

i.e $$a \wedge a^1 = x, a \vee a^1 = y$$

$$\Rightarrow \varphi(a) \wedge \varphi(a^1) = \varphi(x), \varphi(a) \vee \varphi(a^1) = \varphi(y)$$

$$\Rightarrow b \wedge \varphi(a^1) = x^1, b \vee \varphi(a^1) = y^1$$

$\Rightarrow \varphi(a^1)$ is complement of b relative to $[x^1, y^1]$.

Thus each element in any interval in L_2 has complement, going us the required result □

Theorem 6.2.2: $\theta: L_1 \to L_2$ is an onto homomorphism where L_1 , L_2 are lattices and 0^1 is least element of L_2, then kernel θ is an ideal of L.

Proof: Since θ is onto, $0^1 \in L_2$ thus $\ker\theta \ne \phi$ as pre-mage of 0^1 exists in L_1.

Now a, b$\in \ker\theta \Rightarrow$ $\theta(a) = o^1 = \theta(b)$ $\theta(a \vee b) = \theta(a) \vee \theta(b) = 0^1 \vee 0^1 \Longrightarrow a \vee b \in \ker\theta$.

Again $a \in \ker\theta$, $1 \in$ L gives $\theta(a) = 0^1$, $\theta(a \wedge l) = \theta(a) \wedge \theta(l) = 0^1 \wedge l = 0^1$

$\Rightarrow a \wedge l \in \ker\theta$

Hence $\ker\theta$ is an ideal of L □

Theorem 6.2.3: Let S be a standard ideal. Then Θ_S is the extension of $\Theta_{(S)}$ to $I(L)$ and Θ (s) is the restriction of Θ_S to the lattice L.

Proof: Let Θ_S be the extension of Θ_S to $I(L)$ and $I = J(\Theta_{(s)})$.

We suppose $I \subseteq J$. Choosing $a, y \in I$. We can find an $x \in I (y \geq x)$ with $x \equiv y(\Theta_{(s)})$ and so there exists an S_{xy} with $y = x \vee (y \wedge s_{xy})$. The ideal S^1 generated by the $y \wedge S_{xy}$ satisfies $S^1 \subseteq S$ and $I \vee S^1 = J$. Hence $I \equiv J(\theta)$. On the other hand, if $I \equiv J(\Theta_s)$ then $I \vee S^1 = J$ with a suitable $S^1 \subseteq S$. Then for any $y \in J$ it follows that $y \in I \vee S$ and so for any $y \in J$ it follows that $y \in I \vee S$ and so $y = x \vee s = x \vee (y \wedge s)$ for some $s \in S$ as S is standard .Thus $x \equiv y(\Theta_{(s)})$ [Theorem 6.1.1] and hence $\Theta_{(S)} = \Theta_s$.

To prove the 2nd assertion, suppose $(a] \equiv (b](\Theta_s)$.

Then $(a] = (a] \wedge (b]\Theta_s = (a \wedge b](\Theta_s)$ and hence $(a] = (a \wedge b] \vee S^1$ for suitable $s^1 \subseteq S$.

Then $a \in (a \wedge b) \vee S$ and since S is standard [Theorem 6.1.1].

So $a = (a \wedge b) \vee (a \wedge s_1)$ for some $s_1 \in S$.

Similarly, we can show that $b = (a \wedge b) \vee (b \wedge s_2)$ for some $s_2 \in S$. Thus $(a] \equiv (b](\Theta_s)$.

Hence $\Theta_{(s)}$ is the restriction of Θ_s to L □

Recall that a congruence Θ of L is a standard congruence of $\Theta = \Theta(S)$ for some standard ideal S of L. Thus we have the following corollary.

Corollary 6.2.4: The correspondence $\Theta(s) \rightarrow \Theta_S$ is a isomorphism between the lattice of all standard congruence relations of L and the lattice of all principal standard congruence relations of $I(L)$. **If S is a standard ideal of a lattice L, then $\theta_{(S)}$ is the congruence relation**

defined in [Theorem 6.1.1. The congruence induced by S and S is the kernel of the homomorphism induced by $\theta_{(s)}$. Thus in a lattice every standard ideal is a homomorphism kernel of at least one congruence relation.

Following figure shows that even in lattice theory, the converse of this is not true.

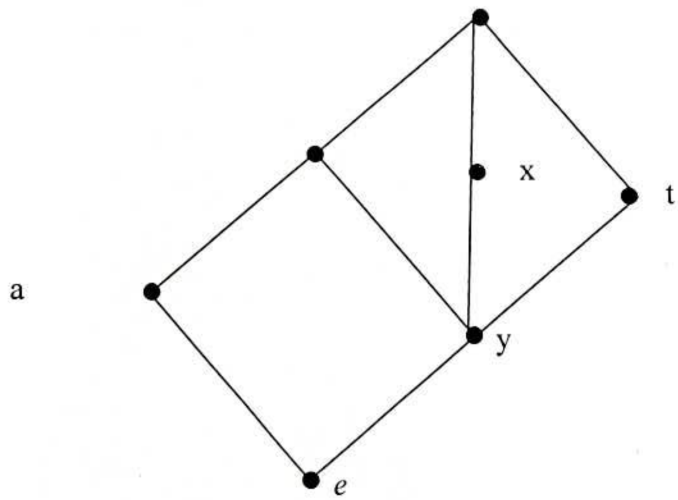


Figure- 6.1

The principal ideal (a] of this lattice is a homomorphism kernel, but it is not standard for $x \wedge (a \vee t) = x$ but $(x \wedge a) \vee (x \wedge y) = y$.

Recall that a lattice L with 0 is sectionally complemented if $[0, x]$ is a complemented sub lattice for each $x \in L$.

Theorem 6.2.5: Let L be a sectionally complemented lattice. Then every homomorphism kernel of L is a standard ideal and every standard ideal is the kernel of precisely one congruence- relation.

Proof: Suppose the ideal I of L is homomorphism kernel induced by the congruence relation θ. Let $a \equiv b(\theta), a, b \in L$, then $a \wedge b \equiv a(\theta)$ and $0 \leq a \wedge b \leq a$. Since L is sectionally complemented, so there exists c; such that $a \wedge b \wedge c = 0$ and $(a \wedge b) \vee c = a$.

This implies $0 = (a \wedge b) \wedge c = a \wedge c = c(0)$ .Since I is a homomorphism kernel, so $c \in I$. Moreover, $a = (a \wedge b) \vee c = (a \wedge b) \vee (a \wedge c)$; similarly,we can show that $b = (a \wedge b) \vee (b \wedge d)$ for some $d \in I$. Therefore, I is a standard ideal □

At the same time we have proved that if I is the kernel of the homomorphism induced by θ, then $\theta = I(\theta)$. Hence every standard ideal is the kernel of precisely one congruence relation. Thus we have the following corollary;

Corollary 6.2.6: In a sectionally complemented lattice all congruencies are standard.

We know, an ideal (s] is standard if and only if s is a standard element . Moreover, we can easily show that an ideal generated by finite number of standard elements is standard. □

Theorem 6. 2.7: Let L be a relatively complemented lattice with 0. If every standard ideal of L is generated by a finite number of standard elements, then $C(L)$ the congruence lattice is Boolean. Moreover, the converse of this is not true.

Proof: Let $\varphi \in C(L)$ with $w < \varphi < w_1$ where w and w$_1$ are the smallest and the largest congruence. Since L is relatively complemented, so by Corollary 5.1.6 above, $\varphi = \Theta_{(s)}$ for some standard ideal S. Then $(0] \subset S \subset L$. Since every standard ideal is generated by a finite number of standard elements, so there exist standard elements $a_1 ....a_m$ and $b_1 ....b_n$, such that $s = (a_1 ....a_m]$ and $L = (b_1 ....b_n]$. Then $(0] \subset (a_1 ....a_m) \subset (b_1 ....b_n)$. Since $(a_1 ....a_m] \subset (b_1 ....b_n]$, at least one of $b \notin (a_1 ....a_m]$.

Suppose $b_{1_1}, b_{1_2}, ..., b_{1_r}$ are the only elements $\{b_1 ....b_n\}$, such that they do not belong to $(a_1 ....a_m]$. Then of course $(a_1 ....a_m] \vee (b_{1_1}, b_{1_2}, ..., b_{1r}] = L$.

Set $C_{lk} = (a_1 \wedge b_1)_k \vee ...... \vee (a_m \wedge b_1)_K$ for each k, $1 \le k \le r$,

then $0 \le c_{l\,K} \le b_{l_K}$ and each c_{l_K} is standard. Since L is sectionally complemented, there exist d_{l_K}, such that $c_{l_K} \wedge d_{l_K} = 0$ and $c_{l_K} \vee d_{l_K} = b_{l_K}$. Since each d_{l_K} is standard.

Now $c_{I_K} \in (a_1........a_m)$ for each k.

Thus $(a_I......a_m) \vee (d_{I_I}.......d_{I_r}] \geq (c_{1_I}........c_{I_r}] \vee (d_{I_I}......d_{1_K}]$

$= (c_{I_I} \vee d_{I_I}] \vee .... \vee (c_{I_r} \vee d_{I_r}] = (b_{I_I}] \vee .... \vee (b_{I_r}] = (b_{I_I}...........b_{I_r}]$, and so

$(a_1.........a_m] \vee (d_{I_I}...........d_{I_r}]$

$= (b_{I_I}.............b_{I_r}] \vee (a_I..........a_m] = A$

Also as each a_1 is standard.

So $(a_I..........a_m] \cap (d_{I_K}] = ((a_i] \vee ....(a_m]) \cap (d_{I_K}] \cap (b_{I_K}]$

$= ((a_I \wedge b_{I_K}] \vee .... \vee (a_m \wedge b_{I_K}]) \cap (d_{I_K}], = ((a_I \wedge b_{I_K}) \vee .... \vee (a_m \wedge b_{I_K})] \cap (d_{I_K}]$

$= (c_{I_K}] \cap (d_{I_K}] = (o]$. Then using the standardness of each d_{I_K}, we have

$(a_I.....................a_m] \cap (d_I...............d_{I_K}] = (0]$

Thus we obtain a standard ideal $T = (d_{I1}...........d_{I_r}]$ of L,

such that θ(T) is the complement of φ. Therefore, $C(L)$ is Boolean. For the converse statement, consider the following lattice L Here it is easy to see that $C(L)$ is Boolean.

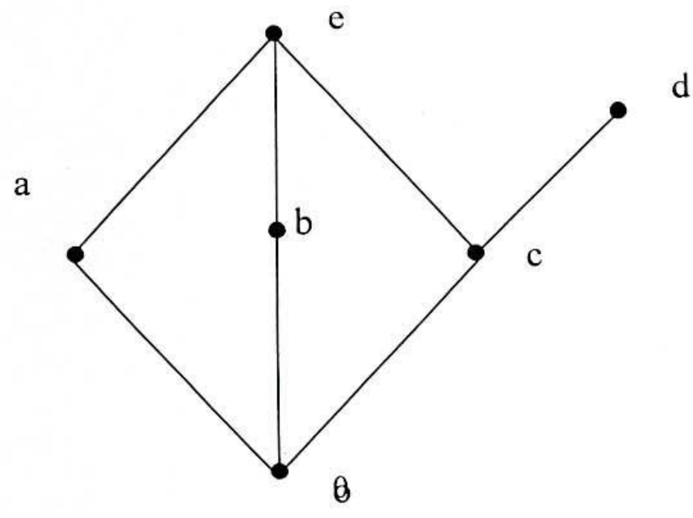


Figure 6.2

$L=(a,d]$, which is of course a standard ideal. But both a and d are not standard elements of L □

[8] and [9] have characterized those lattices whose all congruence is standard and neutral. Our following theorems give characterizations to those lattices whose all congruence are standard and neutral. These are certainly generalizations of above authors work.

Theorem 6.2.8: Let L be a lattic. Then the following conditions are equivalent.

(i) All congruence of L is standard.

(ii) L has a zero and for $x,y\in L$ there exists $a\in L$, such that $x=(x\wedge y)\vee(x\wedge a)$, $a\equiv \dot{\Theta}(x\wedge y,x)$.

Proof: (i) imply (ii). Since the smallest congruence w of L is standard. L must have a zero.

Let $x,y\in L$, then $\Theta(x\wedge y,x)=\Theta(I)$, for some standard ideal I.

i.e., $x=(x\wedge y(\Theta(I)$, where I is standard , hence $x=(x\wedge y)\vee(x\wedge a)$ for some $a\in I$.

Hence $a\equiv 0(\Theta(x\wedge y,x))$.

(ii) Implies (i). Let ϕ be a congruence and $I=[0]\phi$. Suppose $x\equiv y(\phi)$. Then by (ii) there exists $a\in L$ such that $x=(x\wedge y)\vee(x\wedge a)$ and $a\equiv 0(\Theta(x\wedge y,x)$. Since $\Theta(x\wedge y,x)\leq\varphi$, so $a\equiv 0(\phi)$ and hence $a\in I$. Similarly $y=(x\wedge y)\vee(y\wedge b)$ for some $b\in I$.

Thus I is a standard ideal and $\varphi=\Theta(I)$, and so (i) holds □

Theorem 6.2.9 : Let L be a lattice. Then the following conditions are equivalent.

(i) All congruence of L is neutral.

(ii) L has a zero and satisfies the condition:

$x\leq(t\wedge y)\vee(t\wedge z)$; $t,x,y,z\in L$, implies the existence of $a\in L$,such that $x\vee(t\wedge a)=(a\wedge t\wedge y)\vee(a\wedge t\wedge z)\vee(x\wedge y)$, $a\equiv o\Theta(x\wedge y,x)$

(iii) L has a zero and satisfies the condition

$x \le (t \wedge y) \vee (t \wedge z)$; $t, x, y, z \in L$, implies the existence of $a \in L$, such that $x \vee (t \wedge a) = (t \wedge a \wedge y) \vee (y \wedge ((t \wedge a) \vee x))$, $a \equiv 0 \Theta (x \wedge y, x)$.

Proof: L must have a zero of $w = \Theta(\{0\})$.

Let $x \le (t \wedge y) \vee (t \wedge z)$; $t, x, y, z \in L$.

Then $\Theta(x \wedge y, x) = \Theta(I)$ for some neutral ideal I. since I is standard , by the above theorem there exists $a \in I$ such that $x = (x \wedge y) \vee (x \wedge a_I)$.

Now $x \wedge a_I \le (t \wedge y) \vee (t \wedge z), a_I \in I, (x \wedge a_I] \subseteq I$, and $(x \wedge a_1 \le ((t \wedge y) \vee (t \wedge z)]$.

Hence $\subseteq I \cap (t \wedge y] \vee (t \wedge z]) = (I \cap (t \wedge y] \vee (I \cap (t \wedge z])$ $(x \wedge a_1)$ as I is neutral.

Therefore, $x \wedge a_1 \le p \wedge q$, for some $p \in I \cap (t \wedge y]$ and $q \in I \cap (t \wedge z]$.

Thus, $p \le t \wedge y, q \le t \wedge z$ $p \le t \wedge y$, and $p, q \in I$. Hence $p = p \wedge t \wedge y$ and $q = q \wedge t \wedge z$.

Let $p \vee q = a$, then $x \wedge a_1 \le a = (p \wedge t \wedge y) \vee (q \wedge t \wedge y) \le (a \wedge t \wedge y) \vee (a \wedge t \wedge z) \le a$.

Hence $a = (a \wedge t \wedge y) \vee (a \wedge t \wedge z), a \in I$.

But $a \vee (x \wedge y) = a \vee (x \wedge a_1) \vee (x \wedge y) = a \vee (x \wedge a_1) \vee (x \wedge y) = a \vee x$.

Thus. $(a \wedge t) \vee x = a \vee x = a \vee (x \wedge y) = (a \wedge t \wedge y) \vee (a \wedge t \wedge z) \vee (x \wedge y)$ and

$a \equiv 0(\Theta(x \wedge y, x))\Theta(I)$ as $a \in I$

(ii) implies (iii). Let $x, y, z, t \in L$ and $x \le (t \wedge y) \vee (t \wedge z)$, then there exists $a \in L$, such that $a \equiv 0(\Theta(x \wedge y, x))$ and $x \vee (t \wedge a) = (a \wedge t \wedge y) \vee (a \wedge t \wedge z) \vee (x \wedge y)$.

Now

$x \vee (t \wedge a) = (a \wedge t \wedge y) \vee (a \wedge t \wedge z) \vee (y \wedge (x \wedge (t \wedge a))) \le (a \wedge t \wedge y) \vee (a \wedge t \wedge z) \wedge (y \wedge (x \wedge y)$
$= x \vee (t \wedge a)$,
hence $x \vee (t \wedge a) = (a \wedge t \wedge z) \vee (y \wedge (x \vee (t \wedge a))))$. Thus (iii) holds. (iii) implies (i). Let φ be any congruence of L . Suppose $x \ge y$ and $x \equiv y(\phi)$. Let $I = [0]\phi$. Since $x \ge y$ $x = y \vee x$ so by (iii) with $t = z = x$ there exists an $a \in L$, such that

$x \vee (x \wedge a) = (a \wedge x \wedge x) \vee (y \wedge (x \vee (x \wedge a))) = (x \wedge a) \vee y.$ *i.e* $x = (x \wedge a) \vee y$, where $a \equiv 0(\Theta(x \wedge t, x)) \leq \phi$ [since Theorem 6.1.1].

Hence I is a standard ideal and $\phi = \Theta(I)$.

Now it suffices to show that all standard ideals of L are neutral. Let I be a standard ideal of L and $x \in L \cap (J \vee K)$ for some ideals J and K. Then $x \in I$ and $x \in J \vee K$. Then $x \in L_m$ for some m=0,1, 2........., where $A_0 = J \cup K$.

$L_m = \{t \leq p \vee q : p \vee q \in L_{m-1}\}$.

Suppose $x \in L_0$. Then $x \in I$ and $x \in J$ or $x \in K$, and so $x \in (I \cap J) \vee (I \cap K)$. Now we will use the induction. Suppose $y \in L_{m-1}$, and $y \in I$ implies that

$y \in (I \cap J) \vee (I \cap K)$. Since $x \in L_m, x \leq p \vee q$ for suitable $p, q \in L_{m-1}$. Set $t = p \vee q$. Then $x \leq (t \wedge p) \vee (t \wedge q)$. Then by (iii) there exists $b \in L$ such that

$x \vee (t \wedge b) = (b \wedge t \wedge q) \vee (p \wedge (t \wedge b) \vee x))$, $b \equiv 0(\Theta(x \wedge p, x))$. Since $x, x \wedge p, 0 \in I$ and I is a homomorphism kernel, we get $b \in I$.

Hence $x \vee (t \wedge b) \in I$.

Further $(x \vee (t \wedge b)) \wedge p) \vee ((x \vee (b \wedge t) \wedge q) \geq (x \vee (t \wedge b)) \wedge p) \vee (b \wedge t \wedge q) = x \wedge (t \wedge b)$.

Putting $a = x \vee (t \wedge b)$, we get $x \leq a = (a \wedge p) \vee (a \wedge q)$ with $a \in I$.

Now both $a \wedge p, a \wedge r$ are members of I and L_{m-1}. Thus both $a \wedge p, a \wedge q$ a$\wedge$p, a$\wedge$q belongs to $(I \cap J) \vee (I \cap K)$, and so $x \in (I \cap J) \vee (I \cap K)$.

Hence I is neutral □

Theorem 6.2.10: Let k be an ideal in a lattice. Then following conditions are hold.

(i) K is a standard ideal

(ii) The binary relation $\Theta(K)$, defined by $x = y\Theta(x)$, if and only if

$x = (x \wedge y) \vee (x \wedge a), y = (x \wedge y) \vee (y \wedge b)$ for some $a, b \in K$ is a lattice congruence.

(iii) The binary relation φ, defined by $x = y(\phi)$ if and only if for all $t \in S$

$(x \wedge t) \vee (t \wedge c) = (y \wedge t) \vee (t \wedge c)$, for some $c \in K$, is lattice congruence.

(iv) For each ideal $H, K \vee H = \{k \vee h : k \vee h$ exists and $k \in K$ and $h \in H\}$.

In Chapter 3 Theorem 3.1.4 proved the theorem.

Theorem 6.2.11: Let L be a lattice with a smallest element o in which each initial segment is a complemented lattice. Then the map $K \rightarrow \Theta_{(k)}$ is a lattice - isomorphism of the lattice of standard ideals of L on to the lattice - congruence of L.

Proof: Let ϕ be a lattice - congruence of L and $J = \{x \in L : x \equiv o(\varphi)\}$. Of course

J is an ideal. Suppose $a \equiv b(\phi)$ and let c and d be respective complements

of $a \wedge b$ in $(a]$ and $(b]$. Then $c = c \wedge a = c \wedge a \wedge b = 0(\Theta)$ and $d = d \wedge b \equiv 0(\phi)$.

Also $a = (a \wedge b) \vee (a \wedge c)$ and $b = (a \wedge b) \vee (b \wedge d)$ with $c, d \in J$.

Conversely, these last relations imply $a = b(\phi)$.

Hence by the above theorem J is a standard ideal and $\varphi = (H)(J)$.

The remainder follows from corollary: The standard ideals of a lattice L form a distributive sub lattice of the ideal - lattice $J(L))$ and the map $K \rightarrow \Theta(K)$ is a lattice - embedding of this sub lattice into the distributive lattice of all lattice congruence on L.

The situation is more complex when it comes to permutability. We close this section with some result in this direction.

A lower semi lattice $(L: \wedge)$ is called medial if the supermum $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists for all $x, y, z \in L$. This is equivalent to saying the supremum of any three elements exists when the suprema of each pair exist. Thus a medial lower semi lattice is a lattice and so will be referred to as dedial lattice □

6.3. Isomorphism's Theorem

Definition(Isomorphism's): Let (P,R) and (Q,R) be two posets . A one-one and on to map $f:P\to Q$ is called an isomorphism's if $xRy \Rightarrow f(x)R^1 f(y), x,y \in P.$

If we use the same symbol $\leq$ for both the relations R and R^1 and thus our definition translates to a one-one, onto map $f:p\to a$ is called an isomorphism's if $x \leq y \Leftrightarrow f(x) \leq f(y)$. We write in that case $P \cong Q$. It is easy to show that the relation of isomorphism's is an equivalence relation.

In fact a map $f:P\to Q$ is called isotone if $x \leq y \Leftrightarrow f(x) \leq f(y)$.

Gratzer and schmidt have proved isomorphism theoerms for standerd ideals in lattices. In their paper they have transleted serveral theoerms of group theory to lattice theory using ideal, standard ideal, factor lattice and join operation for subgroup, invariant subgroup, factor group and group operation respectively. In this section we generalize two isomorphism theorem for standard ideals of lattices.

Definition(Congruence classes): Set of all congruence classes of a lattics L for any congruence Θ on L, L/Θ denotes the set of all congruence classes of L.

We define $\wedge$ on L/Θ by $[a]\Theta \wedge [b]\Theta = [a\wedge b]\Theta$ .If for any $a,b \in L, a\vee b$ exists, then we define $[a]\Theta \vee [b]\Theta = [a \vee b]\Theta$.

Theorem 6.3.1: A mapping $f:L\to M$ is an isomorphism iff f is isotone and has an isotone inverse.

Proof: Let $f:L\to M$ be an isomorphism.Then f being one-one, onto f^{-1} exists and is one-one onto. Again by definition of isomorphism, f will be isotone. We show $f^1;M\to L$ is also isotone. Let $y_1, y_2 \in M$, where $y_1 \geq y_2$. Since f is onto,

$\exists\ x_1, x_2 \in L$ s.t $f(x_1)y_1, f(x_2) = y_2 \Leftrightarrow x_1 = f^{-1}(y_1), x_2 = f^{-1}(y_2)$.

Now $y_1 \leq y_2, f(x_1) \leq f(x_2) \Rightarrow x_1 \leq x_2$ [from the definition of isomorphism]

$\Rightarrow \quad f^{-1}(y_1) \le f^{-1}(y_2) \Rightarrow f^{-1}$ is isotone.

Conversely, let f be isotone such that f^{-1} is also isotone, since f^{-1} exists, f is one-one, onto. Again, as f is isotone $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2), x_1, x_2 \in L$.

Also f^{-1} is isotone implies $f(x_1) \le (x_1) \le f(x_2) \Rightarrow f^{-1}f(x_1) \le f^{-1}(f(x_2) \Rightarrow x_1 \le x_2$.

Thus $x_1 \le x_2 \Leftrightarrow f(x_1) \le f(x_2)$. Hence f is an isomorphism □

Theorem 6.3.2: L/Θ is a lattice.

Proof: Of course L/Θ is a meet semilattice. We need only to show that it has the upper bound property.

Let $[a]\Theta, [b]\Theta \le [c]\Theta$, then $[a]\Theta = [a]\Theta \wedge [c]\Theta = [a \wedge c]\Theta$

$[b]\Theta = [b]\Theta \wedge [c]\Theta = [b \wedge c]\Theta$.

Now, $(a \wedge c) \vee (b \wedge c)$ exists by the upper bound property of L.

Hence $[a \wedge c]\Theta \vee [b \wedge c]\Theta = [(a \wedge c) \vee (b \wedge c)]\Theta$ and so $[a]\Theta \vee [b]\Theta$ exists.

Therefore, L/Θ is a lattice.

If Θ is a congruence of a lattice L, then the map $\phi : L \to L/\Theta$ defined by $\phi(a) = [a]\Theta$ is the natural homomorphism. This is known as the homomorphism induced by Θ. For a standard ideal S of L, we denote the quotient lattice $L/\Theta_{(s)}$, simply by L/S □

Now we give the homomorphism theorem for lattices which is a generalization of [Lattice Theory First Concepts by Gratzer Theorem 11 p-26]

Theorem 6.3.3: Every homographic image of a lattice L is isomorphic to a suitable quotient lattice L. In fact if $\phi : L \to M$ is a homomorphism of L onto M and if ϕ is the congruence relation of L defined by $x \equiv y\Theta$ if and only if $\phi(x) = \phi(y)$, then $L/\Theta = L_1$; is **an isomorphism given by $\psi : [x]\Theta \to q(x), x \in L$.**

Proof: Since φ is a homomophism then it is easy to check that Θ is a congruence relation.

To prove that Ψ is an isomorphism, we have to check that (i) Ψ is well defined.

Let $[x]\Theta - [y]\Theta$. Then $x \equiv y(\Theta)$;thus $\phi(x) = (p(y)) \Rightarrow ([x]\Theta\psi = [y]\Theta\psi$. i.e.$\Psi$ is wel defined.

(ii) Ψ is one – one. $\psi([x])\Theta = \psi(y)\Theta \Rightarrow \varphi(x) = \varphi(y)$ then $x = y(\Theta)$ and so $[x](\Theta) = [y](\Theta)$, i.e., Ψ is one - one.

(iii) Ψ is onto : Let $x \in L$ since φ is onto There is any $y \in L$ with $\varphi(y) = x$.

Thus, $([y]\Theta)y\Psi = x$, i.e., Ψ is onto .

(iv) Preserves the operations. i.e. Ψ is homomorphism. Let $[x]\Theta, [y]\Theta \in L/\Theta$.

Therefore, $\psi([x]\Theta \wedge ([y]\Theta) = \Psi([x \wedge y]\Theta = \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$

$= \Psi([x]\Theta) \wedge \psi([y]\Theta)$, and finally for $\vee$. Suppose $[x]\Theta \vee [y]\Theta$ exists.

Then $[x]\Theta \vee [y]\Theta = [t]\Theta$ for some $t \in L$. So $[x]\Theta \subseteq [t]\Theta$ and $[y]\Theta \subseteq [t]\Theta$.

This implies $[x]\Theta = [x]\Theta \wedge [t]\Theta = [x \wedge t]\Theta$.

Similarly $[y]\Theta = [y \wedge t]\Theta$.

Then $\psi([x]\Theta \vee [y]\Theta)$

$= \psi([x \wedge t]\Theta \vee [y \wedge t])\Theta = \psi([(x \wedge t) \vee (y \wedge t)]\Theta)$

$= \varphi((x \wedge t) \vee (y \wedge t)) = \varphi(x \wedge t) \vee \varphi(y \wedge t)$

$= \psi([x \wedge t]\Theta \vee \psi([y \wedge t]\Theta)$

$= \psi([x]\Theta \vee \psi([y]\Theta)$.

Hence Ψ is a lattice homomorphism and so it is an isomorphism □

Theorem 6.3.4 (First isomorphism theorem for standard ideals)**:** Let L be a lattice, S be a standard ideal and I an arbitrary ideal of L. Then $S \cap I$ is a standard ideal of I and $(I \cup S)/S \cong I/(I \cap S)$.

Proof: 1st part mentioned here in Theorem 6.1.3

For the 2nd part, we can use the first isomorphism theorem for universal algebra [38 Theorem 1.2]. Then it remains to prove that every congruence class of $I \vee S$ may be represent by an element of I. So let $x \in I \vee S$. Then [since Theorem 6.1.1]

$x = i \vee s$ for some $i \in I, s \in S$. Moreover $x = i \vee s \equiv i\Theta(s)$.

Hence congruence class that contains x may be represented by $i \in I$. That is $[x] \equiv [i]\Theta(s)$.

Therefore $(I \cup S)/S \cong I/(I \cap S)$ □.

For the 2nd isomorphism theorem we need the following results. We omit the proofs as they are very trivial.

Lemma 6.3.5: Let the correspondence $x \to \bar{x}$ be lattice homomorphism of lattice L onto a lattice $\bar{L}$. If s is a standard element of L , then $\bar{s}$ is a standard element of $\bar{L}$ □

Corollary 6.3.6: Let $x \to \bar{x}$ be a lattice homomorphism of L onto $\bar{L}$.

Let s be an ideal of L, and denote by $\bar{S}$ the homomorphic image of S under this homomorphism .If S is standard in L then $\bar{S}$ is standard in $\bar{L}$ □

Theorem 6. 3.7 (Second isomorphism theorem for standard ideals)**:**

Let L be a lattice, s be an ideal and T be a standard ideal of L. $S \subseteq T$. Then S is a standard ideal of L if and only if S/T is a standard ideal in L/T and in this case $L/S \cong (A/T)/(S/T)$.

Proof: First suppose that s is a standard ideal of L. Let $\varphi : L \to L/T$ $\varphi :$ be the natural mapping. Then $x \to \bar{x}$ is a lattice homomorphism and onto.So by Corollary 5.2.6,

$\bar{S}$ is a standard ideal of L/T.

Now $\overline{S} = S/T$. Hence S/T is a standard ideal of L/T.

Conversely, suppose that S/T is a standard ideal of L/T.

We are to show that s is a standard ideal of L.

Let us define a relation $\Theta(S)$ by 6.1.1 (ii), suppose $x \geq y$ with $x \equiv y(\Theta(s))$.

Then $x = y \vee s$ x= for some $s \in S$.

Thus, for any $u \in L$, if $x \vee u$ exists, then $x \vee u = (y \vee u) \vee s$. This implies $x \vee u \equiv y \vee u(\Theta(s))$.

To prove substitution property for $\wedge$, suppose $\overline{a}$ denotes the image of the element a under the homomorphism $L \to L/T$. Suppose $\overline{x} \equiv \overline{y}(\Theta(S/T))$. Since S/T is standard in L/T, there is a suitable $\overline{s} \in S/T$, such that $\overline{x} \wedge \overline{u} = (\overline{y} \wedge \overline{u}) \vee \overline{s}$.

Further, since T is standard in L we can find $a \in T$ such that $x \wedge u = [(y \wedge u) \vee s] \vee t$.

We put $s_1 = s \vee t$ and get $x \wedge u = (y \wedge u) \vee s_1, s_1 \in S$. Hence $\Theta(s)$ is a congruence relation of L, and so by 6.1.1, S is standard.

In above proof we have also shown that the congruence classes of L/T under $\Theta(S/T)$ are the homomorphism image of those of L under $\Theta(s)$. Then the second isomorphism theorem for universal algebra [38, Theorem 1.4] finishes the proof □

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