Numerical Study of the Effects of Temperature Dependent Physical Properties on MHD Natural Convection Flow along a Vertical Wavy Surface with Heat Generation

by

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Declaration

This is to certify that the thesis work entitled "**Numerical Study of the Effects of Temperature Dependent Physical Properties on MHD Natural Convection Flow along a Vertical Wavy Surface with Heat Generation**" has been carried out by Md. Shahidul Alam in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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Abstract

In this thesis the numerical study of the effect of temperature dependent physical properties on MHD natural convection flow of viscous incompressible fluid along a uniformly heated vertical wavy surface with heat generation has been investigated. The governing boundary layer equations are first transformed into a non-dimensional form using suitable set of dimensionless variables. The resulting nonlinear system of partial differential equations are mapped into the domain of a vertical flat plate and then solved numerically employing the implicit finite difference method, known as Keller-Box scheme. The numerical results of the surface shear stress in terms of skin friction coefficient and the rate of heat transfer in terms of local Nusselt number have been presented graphically for some selected values of appeared parameters consisting of thermal conductivity variation parameter γ□, heat generation parameter Q, magnetic parameter M, amplitude to the length ratio of wavy surface α and Prandtl number Pr. Some numerical results of the skin friction coefficient and the rate of heat transfer also have been presented in tabular forms.

Contents

Nomenclature

$\vec{B}$	Magnetic induction vector
C_{fx}	Local skin friction coefficient
C_p	Specific heat at constant pressure [$Jkg^{-1}K^{-1}$]
$\vec{D}$	Electron displacement vector [m]
$\vec{E}$	Electric field vector
$\vec{F}$	Body force per unit volume [$kgms^{-2}$]
f	Dimensionless stream function [$f(x, \eta)$]
g	Acceleration due to gravity [ms^{-2}]
Gr	Grashof number [$g\,\beta\,(T_w - T_\infty)\,L^2 / \nu^2$]
$\vec{J}$	Current density vector [kgm^{-3}]
k	Thermal conductivity of the fluid [$Wm^{-1}K^{-1}$]
k_∞	Thermal conductivity of the ambient fluid [$Wm^{-1}K^{-1}$]
L	Wavelength associated with the wavy surface [m]
M	Magnetic parameter [$\sigma_0 \beta_0^2 L^2 / \mu\, Gr^{1/2}$]
n	Wave number indicator
$\bar{n}$	Unit vector normal to the wavy surface [$\frac{\bar{i} f_x + \bar{j} f_y}{\sqrt{f_x^2 + f_y^2}}$]
Nu_x	Local Nusselt number [$-(\partial\theta/\partial y)_{y=0}$]
P	Dimensional pressure of the fluid [Nm^{-2}]
P_∞	Pressure of the ambient fluid [Nm^{-2}]
p	Dimensionless pressure of the fluid
Pr	Prandtl number [$\mu\, C_P / k$]

$\vec{q}$	Velocity vector [ms^{-1}]
q_w	Heat flux at the surface [$-k(\overline{n}.\nabla T)_{y=0}$] [$Wm^{-2}$]
T	Temperature of the fluid in the boundary layer [^0K or ^0C]
T_w	Temperature at the surface [^0K or ^0C]
T_∞	Temperature of the ambient fluid [^0K or ^0C]
u , v	Dimensionless velocity components in x, y direction
x , y	Dimensionless Cartesian co-ordinates
α	Amplitude-to-length ratio of the wavy surface
β	Volumetric coefficient of thermal expansion [K^{-1}]
B_0	Applied magnetic field strength
γ	Thermal conductivity variation parameter [$\gamma = \gamma^* (T_w - T_\infty)$]
γ^*	Thermal conductivity gradient due to film temperature [$\frac{1}{k_f}\left(\frac{\partial k}{\partial T}\right)_f$]
∇	Vector differential operator
ε	Viscosity variation parameter [$\varepsilon = \varepsilon^* (T_w - T_\infty)$]
ε'	Electric permeability of the medium
η	Dimensionless similarity variable [$y\,x^{-1/4}$]
θ	Dimensionless temperature function [$\theta(x, \eta)$]
μ	Dynamic coefficient of viscosity [$kgm^{-1}s^{-1}$]
μ_∞	Dynamic viscosity of the ambient fluid [$kgm^{-1}s^{-1}$]
μ_e	Magnetic permeability of the medium
ν	Kinematic coefficient of viscosity [$m^2 s^{-1}$]
ρ	Density of the fluid [kgm^{-3}]
ρ_e	Charge density [kgm^{-3}]

σ_0	Electrical conductivity of the fluid
τ_w	Shearing stress $[(\mu \bar{n}.\nabla U)_{y=0}]$
ψ	Stream function $[x^{3/4} f(x, \eta)]$ $[m^2 s^{-1}]$
σ_x	Non dimensional surface profile function
$\bar{\sigma}$	Surface profile function
Prime (′)	Differentiation with respect to η

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CHAPTER I

Introduction

The characteristics of natural convection flow of electrically conducting fluid in the presence of magnetic field along a wavy surface is important from the technical point of view and such type of problems have received much attention of many researchers. Natural convection occurs due to the variations in density, which is caused by the non-uniform distribution of temperature or/and concentration of a dissolved substance. The natural convection procedures are governed essentially by three features namely the body force, the temperature difference in the flow field and the fluid density variations with temperature. The manipulation of natural convection heat transfer can be deserted in the case of large Reynolds number and very small Grashof number. Alternately, the natural convection should be the governing aspect for large Grashof number and small Reynolds number. The analysis of natural convection has been of considerable interest to engineers and scientists since it is important in many industrial and natural problems. There are many physical processes in which buoyancy forces resulting from thermal diffusion play an important role in the convective transfer of heat. Few examples of the heat transfer by natural convection can be found in geophysics and energy related engineering problems such as natural circulation in geothermal reservoirs, refrigerator coils, hot radiator used for heating a room, transmission lines, porous insulations, solar power collectors, spreading of pollutants etc. A very common industrial application of natural convection is free air cooling without the aid of fans, which can happen on small scales to large scale process equipment.

Some times it is necessary to study the heat transfer from an irregular surface. If the surface is roughened the flow is disturbed by the surface and this alters the rate of heat transfer. Irregular surfaces are often present in many applications. It is often encountered in heat transfer devices to enhance heat transfer. Laminar natural convection flow from irregular surfaces can be used for transferring heat in different heat transfer devices, for examples, flat-plate solar collectors, flat-plate condensers in refrigerators, heat exchanger, functional clothing design, geothermal reservoirs and other industrial applications. They are widely used in space heating, refrigeration, air conditioning, power plants, chemical plants, petrochemical plants, petroleum refineries and natural gas processing. One common example of a heat exchanger is the radiator used in car/vehicles, in which the heat generated from engine transferred to air flowing through the radiator. Heat exchanger also widely used in industry both for cooling and heating large scale industrial processes. Another industrial application of wavy surface is injection molding system. Injection molding is used to create many things such as wire spools, packaging, bottle caps, automotive dashboards, pocket combs and plastic products available now a days.

In heat transfer, sinusoidal wavy surface can be shown approximately in practical geometries. A good example is a cooling fin. Since cooling fins have a larger area than a flat surface, they are better heat transfer devices. Another example is a machine-roughened surface for heat transfer enhancement. The interface between concurrent or countercurrent two-phase flow is another example remotely related to this problem. Such an interface is always wavy and momentum transfer across is by no means similar to that across a smooth, flat surface, and neither is the heat transfer. Also a wavy interface can have an important effect on the condensation process.

The word magnetohydrodynamics (MHD) is derived from magneto- meaning magnetic field, hydro-meaning liquid and dynamics meaning–movement. Magnetohydrodynamics (MHD) is the branch of continuum mechanics, which deals with the flow of electrically conducting fluids in presence of electric and magnetic fields. Probably the advance towards an understanding of such phenomena comes from the field of astrophysics and geophysics. It has long been assumed that most of the matter in the universe is in the plasma or highly ionized state and much of the basic knowledge in the area of electromagnetic fluid dynamics evolved from these studies.

The motion of the conducting fluid across the magnetic field induced electric currents which change the magnetic field and the action of the magnetic field on these currents give rise to mechanical forces, which modify the fluid flow. The interaction of the magnetic field and the moving electric charge carried by the flowing fluid induces a force, which tends to oppose the fluid motion near the leading edge. The velocity is very small, so that the magnetic force that is proportional to the magnitude of the longitudinal velocity and acts in the opposite direction is also very small. Consequently, the influence of the magnetic field on the boundary layer is exerted only through induced forces within the boundary layer itself without additional effects arising from the free stream pressure gradient. Thus there is a two-way interaction between the flow field and the magnetic field, the magnetic field exerts force on the fluid by producing induced currents and induced currents change the original magnetic field.

Many natural phenomena and engineering problems are susceptible to MHD analysis. It is useful in astrophysics. Geophysical encounter MHD phenomena in the interactions of conducting fluids and magnetic fields those are present in and around heavenly bodies. Engineers employ MHD principles in the design of heat exchanger, pumps and flow meters, in space vehicle propulsion, control and re-entry, in creating novel power generating systems and developing confinement schemes for controlled fusion. The most important application of MHD are in the generation of electrical power with the flow of an electrically conducting fluid through a transverse magnetic field, electromagnetic pump, the MHD generator using ionized gas as an armature, electromagnetic pumping of liquid metal coolants in nuclear reactors. Other potential applications for MHD include electromagnets with fluid conductors, various energy conversion or storage devices and magnetically controlled lubrication by conducting fluids etc.

As a branch of plasma physics, the field of MHD consists of the study of a continuous electrically conducting fluid under the influence of electromagnetic fields. A related application is the use of MHD acceleration to shoot plasma into fusion devices or to produce high-energy wind tunnels for simulating hypersonic flight. Originally, MHD included only the study of strictly incompressible fluid, but today the terminology is applied to studies of partially ionized gases as well.

Most of the liquids and gases are poor conductors of electricity. In the case when the conductor is either a liquid or gas, electromagnetic forces will be generated which may be of the same order of magnitude as the hydrodynamical and inertial forces. Thus the electromagnetic force will have to take into account with the other forces in the equation of motion.

Joule heating is the heating effect of conductors carrying currents. Joule heating occurs when electric current passes through a material and material's resistivity to the current causes heat generation. When current flows in an electrical conductor such as wire, electrical energy is lost due to the resistance of the electrical conductor. This lost electrical energy is converted into thermal energy called Joule heating. One common example of Joule heating is light bulb where electrical energy converts to thermal energy.

Joule heating is caused by interactions between the moving particles that form the current (usually, but not always, electrons) and the atomic ions that make up the body of the conductor. Joule heating is also referred to as Ohmic heating or Resistive heating because of its relationship to Ohm's law.

It was first studied by James Prescott Joule in 1841. It is the process by which the passage of an electric current through a conductor releases heat. Joule's first law is also known as Joule effect. It states that heat generated by a constant current passing through a resistive conductor for a time whose unit is Joule. It is also related to Ohm's first law. The SI unit of energy was subsequently named the Joule and given the symbol *J*. The commonly known unit of power, the watt, is equivalent to one joule per second.

Physical properties like viscosity and thermal conductivity may be changed significantly with temperature. The viscosity of liquids decreases and the viscosity of gases increases with temperature. The viscosity of air is 1.3289 kg $m^{-1}s^{-1}$, 2.671 kg $m^{-1}s^{-1}$ and 3.625 kg $m^{-1}s^{-1}$ at 100^0C, 500 ^{0}C and 800^0C temperature respectively. The viscosity of water is 1006.523 poise, 471.049 kg $m^{-1}s^{-1}$, 282.425 kg $m^{-1}s^{-1}$ and 138.681 kg $m^{-1}s^{-1}$ at 20^0C, 60^0C, 100^0C and 200^0C temperature respectively (Cebeci and Bradshow (1984)). For a liquid, it has been found that the thermal conductivity k varies with temperature in an approximately linear manner in the range from 0 to 400^0F (Kays (1966)). To predict the behavior of flow accurately, it is necessary to take into account viscosity and thermal conductivity.

A scalar function whose contour lines define the streamlines is known as the stream function. The stream function ψ is constant along a streamline.

Fluids, which obey Newton's law of viscosity, are called as Newtonian fluids. Common fluids like water, air, and mercury are all Newtonian fluids. Fluids, which do not obey Newton's law of viscosity, are called as non-Newtonian fluids. For such fluids the shear stress is not proportional to the velocity gradient. Fluids like blood, Paints, coal tar, liquid plastics and polymer solution are all non-Newtonian fluids.

Taking the x-axis to be horizontal and the y-axis to be vertically upwards, a motion in which the equation of the vertical section of the free surface is of the form $Y_w = \bar{\sigma}(X) = \alpha \sin\left(\frac{n\pi X}{L}\right)$ when $\alpha = 0$, the profile is $y = 0$ which is the mean level. The maximum value of y, namely α, is known as the amplitude-to-length ratio of the wave. The elevation is known as crest. The distance between two consecutive crests is known as the wavelength and is denoted by $L = 2\pi/n$, where n is the wave number.

1.1 Temperature Dependent Physical Properties

The viscosity and thermal conductivity of the fluid to be proportional to a linear function of temperature. Two semi-empirical formulae were proposed by Charraudeau (1975). Arunachalam and Rajappa (1978) studied thermal boundary layer in liquid metals with variable thermal conductivity. Gray et al. (1982) studied the effect of significant viscosity variation on convective heat transfer in water–saturated porous media. Transient free convection flow with temperature dependent viscosity in a fluid saturated porous media has shown by Mehta and Sood (1992). As per their investigation the flow characteristics substantially change when the effect of temperature dependent viscosity considered. Mehta and Sood (1993) extended their works by considering effect of temperature dependent viscosity on the free convective flow across an impermeable partition. The effect of temperature dependent viscosity on the free convective laminar boundary layer flow past a vertical isothermal flat plate in the region near the leading edge have been studied by Kafoussius and Williams (1995). Kafoussius and Rees (1995) also studied numerical study of the combined free and forced convective laminar boundary layer flow past a vertical isothermal flat plate with temperature dependent viscosity. Hady, Bakier et al. (1996) studied mixed convection boundary layer flow on a continuous flat plate with variable viscosity. Chaim (1998) investigated heat transfer in a fluid with variable thermal conductivity over a linearly stretching sheet. Elbashbeshy (2000) analyzed the free convection flow along a vertical plate, taking into account the variation of the viscosity and thermal diffusivity with temperature in the presence of the magnetic field. Hossain et al. (2000) investigated the natural convection flow past a permeable wedge with uniform surface heat flux for the fluid having temperature dependent viscosity and thermal conductivity. They considered the various configurations of wedge from Blasius flow to Hiemenz flow. They used three distinct methodologies; namely, the perturbation method for small values of the transpiration parameter ξ, the asymptotic solutions for large values ξ and an implicit finite difference method for all values of ξ to solve the equations. They concluded that the dimensionless dynamic viscosity as well as the thermal conductivity of the fluid approach unity at the outer edge of the boundary layer for values of all the pertinent parameters, which was trivial. Hossain and Munir (2000) presented mixed convection flow from a vertical flat plate with temperature dependent viscosity. Unsteady flow of viscous incompressible fluid with temperature dependent viscosity due to a rotating disc in presence of transverse magnetic field and heat transfer is studied by Hossain and Wilson (2001). Hossain and Munir (2001) have studied numerically natural convection flow of a viscous fluid about a truncated cone with temperature dependent viscosity and thermal conductivity. They used the perturbation method to obtain the solution in the regimes near and far away from the point of truncation. They also used the implicit finite difference method for solving the governing equations numerically. They compared the perturbation solutions with the finite difference solutions and found in excellent agreement. Hossain et al. (2001) investigated the effect of radiation on the free convection flow of fluid with variable viscosity from a porous vertical plate. Munir et al. (2001a) studied natural convection of a viscous fluid with viscosity inversely proportional to linear function of temperature from a vertical wavy cone. They considered the boundary-layer regime when the Grashof number was very large and assumed that the wavy surfaces have $O(1)$ amplitude and wavelength. They also considered the buoyancy forces assist the flow for various values of the viscosity variation parameter ε, with the Prandtl number Pr = 0.71 and 7.0 which are appropriate for air and water respectively. They found the difference between the flow and heat transfer characteristics over a flat cone and a wavy one, respectively. For a wavy cone the isotherms showed a sinusoidal behavior, while for a flat cone these are parallel lines. Considering natural convection with variable viscosity and thermal conductivity from a vertical wavy cone Munir et al. (2001b) extended their works using the Kellar box method. The problem of natural convection of fluid with temperature dependent viscosity from a heated vertical wavy surface has been studied by Hossain et al. (2002). Mamun et al. (2005) investigated natural convection flow from an isothermal sphere with temperature dependent thermal conductivity. Molla et al. (2005) considered natural convection flow from an isothermal horizontal circular cylinder with temperature dependent viscosity. They considered the effects of viscosity variation parameter ε and Prandtl number Pr on the velocity and viscosity distribution of the fluid as well as on the local rate of heat transfer in terms of the local Nusselt number Nu and the local skin-friction for fluids having Prandtl number, Pr ranging from 1.0 to 30.0. They concluded that the assumption of the constant fluid properties might introduce severe errors in the prediction of surface friction factor and heat transfer rate. Rahman et al. (2008) investigated the effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate with heat conduction. The numerical calculation was proceeding in finite-difference method and the effect of various parameters on the velocity, temperature, local skin friction co-efficient and surface temperature profiles were shown by Rahman and Alim (2009) considering the numerical study of magnetohydrodynamic free convective heat transfer flow along a vertical plate with temperature dependent thermal conductivity. Nasrin and Alim (2009) investigated MHD free convection flow along a vertical flat plate with thermal conductivity and viscosity depending on temperature. Numerical study on a vertical plate with variable viscosity and thermal conductivity has been investigated by Palani and Kim (2009). They assumed that the viscosity of the fluid is an exponential function and the thermal conductivity is a linear function of the temperature. They considered the unsteady boundary layer equations on neglecting the viscosity and thermal conductivity variation and they found substantial errors and concluded that the effects of the variation viscosity and thermal conductivity should be considered to predict more accurate results.

1.2 Computational Method

The heat transfer and its governing equations including conservation forms of the Navier-Stoke's system of equations and energy equation derived from the first law of thermodynamics, expressed in terms of the control volume/surface integral equations, which represents various physical phenomena. In order to visualize these thermo fluid flow scenarios, an approximate numerical solution is needed, which can be obtained by the Computational Fluid Dynamics (CFD) code.The governing equations of fluid mechanics and convective heat transfer are discretized in order to obtain a system of approximate algebraic equations, which then can be solved on a computer. The approximate values are applied to small domain in space and / or time so the numerical solution provides results at discrete locations in space and time. The accuracy of the experimental data depends on the quality of the tools used; the accuracy of numerical solution is dependent on the quality of discretization used. The CFD computation involves the creation of a set of numbers that constitutes a realistic approximation of a real life system. The result of the computation work improves the understanding of the behavior of a system. So, the CFD codes are very useful tools by which researcher can produce physically realistic result with good accuracy in simulation with finite grids. The broad field of Computational Fluid Dynamics are the activities which cover the range from automation of well established engineering design methods to the use of detailed solution of the Navier-Stokes equations as substitutes for experimental research into the nature of complex flows. A wide range of Fluid Dynamics problems have been solved using CFD codes. CFD codes are more frequently used in the field where the geometry is complicated or some important feature that can not be dealt with standard methods. The complete Navier-Stokes equations are considered as the correct mathematical description of the governing equations of fluid motion. Most of the accurate numerical computations in fluid dynamics come from solving the Navier –Stokes equations, since the Navier-Stokes equations represent the conservation of momentum. There are four discretization methods available for the high performance numerical computation of CFD.

- Finite Volume Method (FVM)
- Finite Element Method (FEM)

- Finite Difference Method (FDM)
- Boundary Element Method (BEM)

1.3 Prandtl's Boundary Layer Theory

An important contribution to fluid dynamics was made by L. Prandtl in 1904 by introducing the concept of boundary layer. He clarified the essential influences of viscosity in flows at high Renold's numbers by showing how the Navier-Stokes equation could be simplified to yield approximate solutions for overcoming the limitation.

For convenience, consider laminar two-dimensional flow of fluid of small viscosity (large Renold's number) over a fixed semi infinite plate. It is observed that, unlike an ideal (non viscous) fluid flow, the fluid does not slide over the plate, but "sticks" to it. Since the plate is at rest, the fluid in contact with it will also be at rest. As we move outwards along the normal, the velocity of the fluid will gradually increase and at a distance far from, consider the plate the full stream velocity U is attained. However, it will be assumed that the transition from zero velocity at the plate to the full magnitude U takes place within the thin layer of the fluid in contact with the plate. This is known as the boundary layer.

1.4 Some Useful Dimensionless Number

Besides the inertia force, there always exist some additional forces which are responsible for fluid motion. The required conditions for dynamic similarity can always be obtained by considering the ratio of the inertia force and any one of the remaining forces (e.g., viscous force, gravity force, pressure force, elastic force and so on). Since ratios of two forces will be considered, we obtain some dimensionless number as discussed below:

1.4.1 Reynold's Number

The Reynold's number Re is defined as

$$\text{Re} = \frac{\text{Inertia force}}{\text{Viscous force}}$$

$$= \frac{\text{Mass} \times \text{Accleration}}{\text{Shear stress} \times \text{Crosssectional area}}$$

$$= \frac{\text{Volume} \times \text{Density} \times (\text{Velocity/Time})}{\text{Shear stress} \times \text{Crosssectional area}}$$

$$= \frac{\text{Cross sectional area} \times \text{Linear dimension} \times \rho \times \dfrac{\text{Velocity}}{\text{Time}}}{\text{shear stress} \times \text{crosssectional area}}$$

$$= \frac{(\text{Velocity})^2 \times \rho}{\mu(\partial u/\partial y)} = \frac{V^2 \rho}{\mu(V/L)} = \frac{VL\rho}{\mu} = \frac{VL}{\nu}$$

where L and V denote the characteristic length and characteristic velocity respectively so that velocity will be proportional to V and $\partial u/\partial y$ will be proportional to V/L. If for any flow problem Re is small then we can ignore the inertia force, whereas if Re is large then we can neglect the effect of the viscous force and consequently the fluid may be treated as non-viscous fluid. When the viscous-force is the predominating force, Reynolds number must be the same for dynamic similarity of two flows.

1.4.2 Grashof Number

The Grashof number is a dimensionless quantity used in analyzing the velocity distribution in free convection systems. In free convection, the driving force is a buoyancy force caused by a temperature gradient, as the fluid would be at rest in the absence of temperature variations. The Grashof number is analogous to the Reynolds number in forced convection.

Essentially, the Grashof number is a ratio of buoyant forces to viscous forces.

$$\text{Gr} = \frac{\text{Buoyancy force}}{\text{Viscous force}}$$

$$=\frac{g^{1}\rho^{v}}{\rho v^{2}}$$

$$=\frac{g\beta\Delta TV}{v^{2}}\ .$$

When Gr >> 1, the viscous force is negligible compared to the buoyancy and inertial forces. When buoyant forces overcome the viscous forces, the flow starts a transition to the turbulent regime. For a flat plate in vertical orientation, this transition occurs around $Gr = 10e{+}9$.

In terms of viscosity, the Grashof number can be defined as in the following 2 ways:

$$\mathrm{Gr}=\frac{\rho g B^{3}\Delta\rho}{\mu^{2}}\ ,$$

where $\Delta\rho$ = total change in density between regions of high temperature and low temperature. This can be obtained by expanding ρ in a Taylor series about the mean temperature. The most useful form for Mat lab application is:

$$\mathrm{Gr}=\frac{B^{3}\rho^{2}\Delta T\beta g}{\mu^{2}}$$

where,

β = the inverse of the film (mean) temperature

ρ = the density evaluated at the mean temperature

g = the gravitational constant

ΔT = the temperature difference

B = the distance between regions of high temperature and low temperature

μ = the viscosity of the convective fluid

1.4.3 Prandtl Number

The Prandtl number Pr is defined by

$$\Pr = \frac{\text{Viscous discipation rate}}{\text{Thermal discipation rate}}$$

Or, $\Pr = \frac{\mu g c_p}{k}$,

where C_p is the specific heat at constant pressure and k is the thermal conductivity. Evidently Pr depends only on the properties of the fluid. For air Pr = 0.71 approx. and for water (at 15°c) Pr = 7 approx., whereas for oils it is of the order of 1000 due to large values of μ for oils. Prandtl number is the ratio of viscous force to the thermal force. It throws light on the relative importance of viscous dissipation to the thermal dissipation.

1.4.4 Nusselt Number

In heat transfer at a boundary (surface) within a fluid, the Nusselt number is the ratio of convective to conductive heat transfer across (normal to) the boundary. Named after Wilhelm Nusselt, it is a dimensionless number. The conductive component is measured under the same conditions as the heat convection but with a (hypothetically) stagnant (or motionless) fluid.

A Nusselt number close to unity, namely convection and conduction of similar magnitude, is characteristic of "slug flow" or laminar flow. A larger Nusselt number corresponds to more active convection, with turbulent flow typically in the 100-1000 range. The convection and conduction heat flows are parallel to each other and to the surface normal of the boundary surface, and are all perpendicular to the mean fluid flow in the simple case.

$$Nu_L = \frac{\text{Convective heat transfer}}{\text{Conductive heat transfer}} = \frac{hL}{k}$$

where:

- L = characteristic length
- k = thermal conductivity of the fluid
- h = convective heat transfer coefficient of the fluid

Selection of the characteristic length should be in the direction of growth (or thickness) of the boundary layer. Some examples of characteristic length are: the outer diameter of a cylinder in (external) cross flow (perpendicular to the cylinder axis), the length of a vertical plate undergoing natural convection, or the diameter of a sphere. For complex shapes, the length may be defined as the volume of the fluid body divided by the surface area. The thermal conductivity of the fluid is typically (but not always) evaluated at the film temperature, which for engineering purposes may be calculated as the mean-average of the bulk fluid temperature and wall surface temperature. For relations defined as a local Nusselt number, one should take the characteristic length to be the distance from the surface boundary to the local point of interest. However, to obtain an average Nusselt number, one must integrate said relation over the entire characteristic length. Typically, for free convection, the average Nusselt number is expressed as a function of the Rayleigh number and the Prandtl number, written as Nu = f(Ra, Pr). Else, for forced convection, the Nusselt number is generally a function of the Reynolds number and the Prandtl number as Nu = f(Re, Pr). Empirical correlations for a wide variety of geometries are available that express the Nusselt number in the aforementioned forms. The mass transfer analog of the Nusselt number is the Sherwood number.

CHAPTER II

Literature Review

Natural convection heat transfer has gained considerable attention because of its numerous applications in the areas of energy conservations, cooling of electrical and electronic components, design of solar collectors, heat exchangers and many others. The most important application of MHD is in the generation of electrical power with the flow of an electrically conducting fluid through a transverse magnetic field. In case of natural convection flows, now a days, MHD analysis is playing a vital role. Sparrow and Cess (1961) investigated the effect of magnetic field on free convection heat transfer. Kuiken (1970) investigated MHD free convection in a strong cross field. Gebhart and Pera (1971) investigated the nature of vertical natural convection flows resulting from the combined buoyancy effects of thermal and mass diffusion. They indicated that buoyancy effects from concentration gradients could be as important as those from temperature gradients. There are applications of interest in which combined heat and mass transfer by natural convection, such as design of chemical processing equipment, design of heat exchangers, formation and dispersion of fog, distributions of temperature and moisture over agricultural fields, pollution of the environments and thermoprotection systems. Wilks (1976) presented MHD free convection about a semi-infinite vertical plate in a strong cross field. Ingham (1978) investigated free convection boundary layer on an isothermal horizontal cylinder. Raptis and Kafoussius (1982) analyzed MHD free convection flow and mass transfer through a porous medium bounded by an infinite vertical porous plate with constant heat flux. Pozzi and Lupo (1988) explored the coupling of conduction with laminar convection along a flat plate. By means of two expansions, the entire thermo-fluid dynamic field was studied. The first one, describing the field in the lower part of the plate, was a regular series. The radius of convergence of which was determined by means of approximant techniques. The second expansion, an asymptotic one required a different analysis because of the presence of eigensolutions. Hossain and Ahmed (1990) considered MHD forced and free convection boundary layer flow near the leading edge. Hossain (1992) analyzed the viscous and Joule heating effects on MHD free convection flow with variable plate temperature and found that temperature varied linearly with the distance from the leading edge in presence of uniformly transverse magnetic field. The equations governing the flow were solved and the numerical solutions were obtained for small Prandtl numbers, appropriate for coolant liquid metal, in the presence of a large magnetic field. Hossain et al. (1997) considered MHD forced and free convection boundary layer flow along a vertical porous plate. Hossain et al. (1998) studied heat transfer response of MHD free convection flow along a vertical plate to surface temperature oscillation. Al-Nimr and Hader (1999) studied MHD free convection flow in open-ended vertical porous channels. Chowdhury and Islam (2000) presented MHD free convection flow of visco-elastic fluid past an infinite porous plate. The conjugate conduction-natural convection heat transfer along a thin vertical plate with non-uniform internal heat generation. presented by Mendez and Trevino (2000). El-Amin (2003) analyzed combined effect of viscous dissipation and Joule heating on MHD forced convection over a non isothermal horizontal cylinder embedded in a fluid saturated porous medium. Ahmed and Zaidi (2004) presented magnetic effect on overback convection through vertical stratum. Molla et al. (2006) also investigated MHD natural convection flow on a sphere with uniform heat flux in presence of heat generation. Viscous dissipation effects on MHD natural convection flow over a sphere in the presence of heat generation have been investigated by Alam et al. (2007). Alim et al. (2007) investigated Joule heating effect on the coupling of conduction with MHD free convection flow from a vertical flat plate. Combined effects of viscous dissipation and Joule heating on the coupling of conduction and free convection along a vertical flat plate have also studied by Alim et al. (2008). Entropy generation during fluid flow in a channel under the effect of transverse magnetic field presented by Damseh et al. (2008). Mamun et al. (2007) studied combined effect of conduction and viscous dissipation on MHD free convection flow along a vertical flat plate. Parveen and Chowdhury (2009) considered stability analysis of the laminar boundary layer flow.

Recently Alim et al (2012) studied the heat generation effects on MHD natural convection flow along a vertical wavy surface with variable thermal conductivity. Very recently Kabir et al (2013) also investigated the effect of viscous dissipation on MHD natural convection flow along a vertical wavy surface.

Further, the problems of natural convective heat and mass transfer flows under the influence of a magnetic field, which is subject matter of MHD, have been paid attention of number of researchers because of possible applications in many branches of science, engineering and geophysical process. Considering this numerous application of MHD, Alam et al (1997) have been studied the problem of convection from a wavy vertical surface in presence transverse magnetic field. Tashtoush and Al-Odat (2004) investigated magnetic field effect on heat fluid flow over a wavy surface with a valuable heat flux. The problem of free convection from a vertical wavy surface embedded in a uniform porous media in presence of an external magnetic field and internal heat generation/absorption effects was formulated by Haddy et al (2006). Ahmed (2008) investigated MHD free convection flow along heated vertical wavy surface with heat generation. The viscosity and thermal conductivity of the fluid have been assumed to be constant in most of the above studied. However it is known that these physical properties may be changed significantly with temperature. Parveen and Alim (2011) investigated Joul heating dependent on temperature. Alim et al (2012) investigated the heat generation effects on MHD natural convection flow along a vertical wavy surface with variable thermal conductivity. From the above investigation it is found that variation of viscosity with temperature in presence of magnetic field is an interesting macroscopic physical phenomenon in fluid dynamics In most of the above investigations the effect of temperature dependent physical properties on MHD natural convection flow along wavy surface with heat generation have been ignored.

The main objective of the present work is to analyze the effect of temperature dependent physical properties on MHD natural convection flow along a uniformly heated vertical wavy surface with heat generation. There are five parameters of interest in the present problem, namely, the thermal conductivity parameter γ, heat generation parameter Q, magnetic parameter M, the amplitude of the waviness α of the surface and Prandtl number Pr. The numerical solutions regarding the velocity and temperature fields will be presented for different selected values of the established dimensionless parameters. The influences of these various parameters on the velocity and temperature fields will be exhibited in the present analysis. Also numerical values of local shearing stress and the rate of heat transfer will be calculated in terms of the skin-friction coefficient C_{fx} and Nusselt number Nu_x respectively for a wide range of the axial distance x starting from the leading edge for different values of the relevant parameters mentioned above and these are shown in tabular form. It may be expected that the temperature-dependent physical properties and the waviness of the surface have a great influence on the velocity and temperature fields, so that their effects should be taken into account with other useful parameters associated.

Thus this thesis is composed of Six Chapters. An introduction and basic concept of boundary layer theory, natural convection flows, and temperature dependent physical properties are presented in CHAPTER I. Earlier researches i.e. literature review related to our present problem are presented in CHAPTER II. Methodology of the problem, basic governing equation, physical model of the problem, formulation of the problem, and dimensional analysis with simplifying assumptions are given in CHAPTER III. The general Finite Difference procedure has given in CHAPTER IV. CHAPTER V is concerned with the detailed of results discussions, Figures and Tables. In CHAPTER VI the conclusions gained from this work have been discussed.

CHAPTER III

Temperature Dependent Thermal Conductivity on MHD Natural Convection Flow along a Vertical Wavy Surface with Heat Generation

3.1 Methodology of the Problem

The finite-difference methods are numerical methods for approximating the solutions of differential equations using finite difference equations to approximate derivatives. We use the Keller Box method to obtain the solutions of the coupled momentum and energy equations for external flows. The numerical method and computer program are presented in such a way reader can easily apply them to other problems including internal flows.

3.2 Derivatives

Finite-difference methods approximate the solutions to differential equations by replacing derivative expressions with approximately equivalent difference quotients. That is, the first derivative of a function f is defined by the definition:

$$f'(x_i) = \lim_{\Delta x_i \to 0} \frac{f(x_i + \Delta x_i) - f(x_i)}{\Delta x_i}$$

Then the approximation for that derivative is

$$f'(x_i) \approx \frac{f(x_i + \Delta x_i) - f(x_i)}{\Delta x_i}$$

for some small value of Δx_i, in fact, this is the forward difference equation for the first derivative. Using this and similar formulae to replace derivative expressions in differential equations, we can approximate their solutions without the need for calculus.

3.3 Governing Equations

Magnetohydrodynamic equations are the ordinary electromagnetic and hydrodynamic equations modified to take account of the interaction between the motion of the fluid and electromagnetic field. Formulation of electromagnetic theory in mathematical form is known as Maxwell's equations. Maxwell's basic equations show the relation of basic field quantities and their production. But it is assumed that all velocities are small in comparison with the speed of light. Before writing down the MHD equations it is essential to know about the ordinary electromagnetic equations and hydromagnetic equations, which are as follows (Cramer and Pai (1974)).

Charge Continuity: $$\nabla.\vec{D} = \rho_e \tag{3.1}$$

Current Continuity: $$\nabla.\vec{J} = -\frac{\partial \rho_e}{\partial t} \tag{3.2}$$

Magnetic field continuity: $$\nabla.\vec{B} = 0 \tag{3.3}$$

Ampere's Law: $$\nabla \wedge \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{3.4}$$

Faraday's Law: $$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.5}$$

Constitutive equations for D and B: $$\vec{D} = \varepsilon' \vec{E} \text{ and } \vec{B} = \mu_e \vec{H} \tag{3.6}$$

Total current density flow: $$\vec{J} = \sigma_0\left(\vec{E} + \vec{q} \wedge \vec{B}\right) + \rho_e \vec{q} \tag{3.7}$$

The above equations (3.1) to (3.7) are Maxwell's equations where $\vec{D}$ is the electron displacement, ρ_e is the charge density, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, $\vec{B}_0$ is the magnetic field strength, $\vec{J}$ is the current density, $\partial\vec{D}/\partial t$ is the displacement current, ε' is the electric permeability of the medium, μ_e is the magnetic permeability of the medium, $\vec{q}$ is the velocity vector and σ_0 is the electric conductivity.

The electromagnetic equations as shown above are not usually applied in their present form and require interpretation and several assumptions to provide the set to be used in MHD. In MHD a fluid is considered that is grossly neutral. The charge density ρ_e in

Maxwell's equations must then be interpreted, as an excess charge density, which is generally not large. If it is disregard the excess charge density then it must disregard the displacement current. In most problems the displacement current, the excess charge density and the current due to convection of the excess charge are small. Taking into this effect the electromagnetic equations can be reduced to the following form:

$$\nabla.\vec{D} = 0 \tag{3.8}$$

$$\nabla.\vec{J} = 0 \tag{3.9}$$

$$\nabla.\vec{B} = 0 \tag{3.10}$$

$$\nabla \wedge \vec{H} = \vec{J} \tag{3.11}$$

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.12}$$

$$\vec{D} = \varepsilon^{/} \vec{E} \text{ and } \vec{B} = \mu_e \overline{B}_0 \tag{3.13}$$

$$\vec{J} = \sigma_0\left(\vec{E} + \vec{q} \wedge \vec{B}\right) \tag{3.14}$$

We shall now suitably represent below the equations of fluid dynamics to take account of the electromagnetic phenomena.

The continuity equation

The MHD continuity equation for viscous incompressible electrically conducting fluid remains same as that of usual continuity equation

$$\nabla \cdot \vec{q} = 0 \tag{3.15}$$

The Navier-Stokes equation

The motion of the conducting fluid across the magnetic field generates electric currents, which change the magnetic field and the action of the magnetic field on these current give rises to mechanical forces, which modify the flow of the fluid. Thus, the fundamental equation of the magneto-fluid combines the equations of the motion from fluid mechanics with Maxwell's equations from electrodynamics.

Then the Navier-stokes equation for a viscous incompressible fluid in the time independent form may be written in the following form:

$$\rho(\vec{q}\cdot\nabla)\vec{q} = -\nabla P + \mu\nabla^2\vec{q} + \vec{F} + \vec{J}\times\vec{B} \tag{3.16}$$

where ρ is the fluid density, μ is the viscosity and P is the pressure, $\vec{q} = (u, v)$, u and v are the velocity components along the x and y directions respectively, $\vec{F}$ is the body force per unit volume which is defined as $-\rho g$, the terms $\vec{J}$ and $\vec{B}$ are respectively the current density and magnetic induction vectors and the term $\vec{J}\times\vec{B}$ is the force on the fluid per unit volume produced by the interaction of the current and magnetic field in the absence of excess charges. Here $\vec{B} = \mu_e \overline{B}_0$, μ_e being the magnetic permeability of the fluid, $\overline{B}_0$ is the uniformly distributed transverse magnetic field strength.The first term on the right hand side of equation (3.16) is the pressure gradient, second term is the viscosity, third term is the body force per unit volume and last term is the electromagnetic force due to motion of the fluid.

The energy equation

The energy equation for a viscous incompressible fluid is obtained by adding the electromagnetic energy term into the classical gas dynamic energy equation. This equation can be written as

$$\rho C_P(\vec{q}\cdot\nabla)T = \nabla\cdot(k\nabla T) + (\vec{J}\times\vec{q}) \tag{3.17}$$

where, $\vec{q} = (u, v)$, u and v are the velocity components along the x and y directions respectively, k is the thermal conductivity, C_P is the specific heat with constant pressure. T is the temperature of the fluid in the boundary layer. The left side of equation (3.17) represents the net energy transfer due to mass transfer, the first term on the right hand side represents conductive heat transfer and second term is Joule heating term due to the resistance of the fluid to the flow of current. Also ∇ is the vector differential operator and is defined for two dimensional case as

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$$

where $\vec{i}$ and $\vec{j}$ are the unit vectors along x and y directions respectively. When the external electric field is zero and the induced electric field is negligible, the current density is related to the velocity by Ohm's law as follows

$$\vec{J} = \sigma_0\left(\vec{q}\times\vec{B}\right) \tag{3.18}$$

where $\left(\vec{q}\times\vec{B}\right)$ is electrical fluid vector and σ_0 denotes the electric conductivity of the fluid. Under the condition that the magnetic Reynolds number is small, the induced magnetic field is negligible compared with applied field. This condition is well satisfied in terrestrial applications, especially so in (low velocity) free convection flows. So it can be written as

$$\vec{B} = B_0\vec{j} \tag{3.19}$$

Bringing together equations (3.18) and (3.19) the force per unit volume $\vec{J}\times\vec{B}$ acting along the x-axis takes the following form

$$\vec{J}\times\vec{B} = \sigma_0 B_0 u\hat{k} \tag{3.20}$$

3.4 Physical Model of the Problem

In this study the steady of two dimensional laminar free convection boundary layer flow of a viscous incompressible and electrically conducting fluid along a vertical wavy surface in presence of uniform transverse magnetic field of strength $\underline{B}_0$ with temperature dependent physical properties like viscosity and thermal conductivity is considered. It is assumed that the wavy surface is electrically insulated and is maintained at a uniform temperature T_w. Far away the wavy plate, the fluid is stationary and is kept at a temperature T_∞. The surface temperature T_w is greater than the ambient temperature T_∞ that is $T_w > T_\infty$. A uniform magnetic field of strength is imposed along the y-axis i.e. normal direction to the surface and x -axis is taken along the surface.

The flow configuration of the wavy surface and the two-dimensional Cartesian coordinate system are shown in figure 2.1.

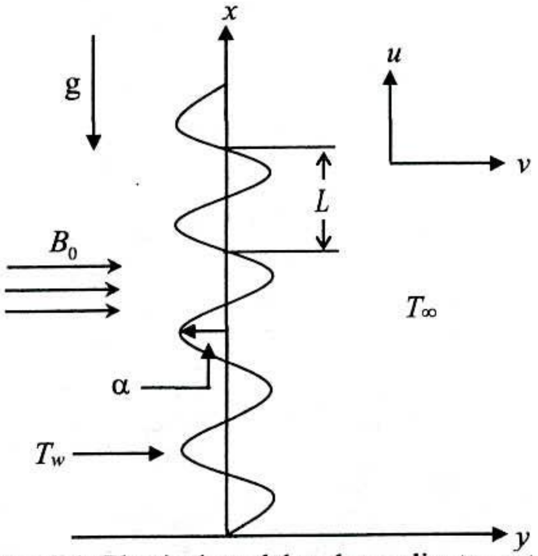

Figure 3.1: Physical model and coordinate system

The boundary layer analysis outlined below allows $\bar{\sigma}(X)$ being arbitrary, but our detailed numerical work assumed that the surface exhibits sinusoidal deformations. The wavy surface may be described by

$$Y_w = \bar{\sigma}(X) = \alpha \sin\left(\frac{n\pi X}{L}\right) \tag{3.21}$$

where α is the amplitude and L is the wave length associated with the wavy surface.

3.5 Formulation of the Problem

The conservation equations for the flow characterized with steady, laminar and two-dimensional boundary layer; under the usual Boussinesq approximation, the continuity, momentum and energy equations can be written as:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{3.22}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \nu\nabla^2\bar{u} + g\beta(T - T_\infty) - \frac{\sigma_0 B_0 \bar{u}}{\rho} \tag{3.23}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{y}} + \nu\nabla^2\bar{v} \tag{3.24}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \frac{k}{\rho C_p}\nabla^2 T + \frac{Q_0(T-T_\infty)}{\rho c_p} \tag{3.25}$$

where $(\bar{x},\bar{y})$ are the dimensional coordinates along and normal to the tangent of the surface and $(\bar{u},\bar{v})$ are the velocity components parallel to $(\bar{x},\bar{y})$, $\nabla^2(=\partial^2/\partial\bar{x}^2+\partial^2/\partial\bar{y}^2)$ is the Laplacian operator, g is the acceleration due to gravity, $\bar{p}$ is the dimensional pressure of the fluid, ρ is the density, B_0 is the strength of magnetic field, σ_0 is the electrical conduction, k is the thermal conductivity of the fluid in the boundary layer region depending on the fluid temperature, β is the coefficient of thermal expansion, $\nu(=\mu/\rho)$ is the kinematic viscosity, μ is the dynamic viscosity, C_p is the specific heat due to constant pressure and Q_0 is the source parameter.

The boundary conditions relevant to the above problem are

$$\begin{aligned} &\bar{u}=0,\ \bar{v}=0,\ T=T_w \quad at\ \bar{y}=\bar{y}_w=\bar{\sigma}(\bar{x}) \\ &\bar{u}=0,\quad T=T_\infty,\quad \bar{p}=p_\infty \quad as\quad \bar{y}\to\infty \end{aligned} \tag{3.26}$$

where T_w is the surface temperature, T_∞ is the ambient temperature of the fluid and p_∞ is the pressure of fluid outside the boundary layer.

There are very few forms of thermal conductivity variation available in the literature. Among them we have considered that one which is appropriate for liquid introduced by Hossain et al. (2001) as follows:

$$k=k_\infty\left[1+\gamma^*(T-T_\infty)\right] \tag{3.27}$$

where k_∞ is the thermal conductivity of the ambient fluid and $\gamma^*=\frac{1}{k_f}\left(\frac{\partial k}{\partial T}\right)_f$ is a constant evaluated at the film temperature of the flow $T_f=1/2(T_w+T_\infty)$.

Using Prandtl's transposition theorem to transform the irregular wavy surface into a flat surface as extended by Yao (1983) and boundary-layer approximation, the following dimensionless variables were introduced for non-dimensionalizing the governing equations:

$$\bar{x} = Lx, \quad \bar{y} = LGr^{-\frac{1}{4}}y + \bar{\sigma}, \; \bar{p} = Gr\frac{\rho v^2}{L^2}p$$

$$\bar{u} = \frac{\mu}{\rho L}Gr^{\frac{1}{2}}u, \quad \bar{v} = \frac{\mu}{\rho L}Gr^{\frac{1}{4}}v + Gr^{\frac{1}{2}}\sigma_x u \tag{3.28}$$

$$\sigma_x = \frac{d\bar{\sigma}}{d\bar{x}} = \frac{d\sigma}{dx}, Gr = \frac{g\beta(T_w - T_\infty)}{v^2}L^3, \theta = \frac{T - T_\infty}{T_w - T_\infty}$$

where θ is the non-dimensional temperature function and (u, v) are the dimensionless velocity components parallel to (x, y).

$$u = \frac{\rho L}{\mu_\infty}Gr^{-1/2}\bar{u}$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{x}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{x}} = \frac{\rho L}{\mu_\infty}Gr^{-1/2}\frac{\partial \bar{u}}{\partial \bar{x}} \qquad \left[\because \frac{\partial y}{\partial \bar{x}} = -\frac{\sigma_x}{L}Gr^{1/4}\right]$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\mu_\infty}{\rho L^2}Gr^{1/2}\left[\frac{\partial u}{\partial x} - Gr^{1/4}\sigma_x\frac{\partial u}{\partial y}\right] \tag{3.28i}$$

$$v = \frac{\rho L}{\mu_\infty}Gr^{-1/4}\left(\bar{v} - \sigma_x\bar{u}\right)$$

$$\frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{y}} = \frac{\rho L}{\mu_\infty}Gr^{-1/4}\left[\frac{\partial \bar{v}}{\partial \bar{y}} - \sigma_x\frac{\partial \bar{u}}{\partial \bar{y}}\right] \qquad \left[\because \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\mu_\infty}{\rho L^2}Gr^{3/4}\frac{\partial u}{\partial y}\right]$$

$$\frac{\partial \bar{v}}{\partial \bar{y}} = \frac{\mu_\infty}{\rho L^2}Gr^{1/2}\left[\frac{\partial v}{\partial y} + Gr^{1/4}\sigma_x\frac{\partial u}{\partial y}\right] \qquad \left[\because \frac{\partial y}{\partial \bar{y}} = \frac{1}{L}Gr^{1/4}\right] \tag{3.28ii}$$

$$\bar{v} = \frac{\mu_\infty}{\rho L}\left(\sigma_x Gr^{1/2}u + Gr^{1/4}v\right)$$

$$\frac{\partial \bar{v}}{\partial \bar{x}} = \frac{\mu_\infty}{\rho L}Gr^{1/2}\left\{\frac{1}{L}\sigma_{xx}u + \sigma_x\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{x}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{x}}\right)\right\} + \frac{\mu_\infty}{\rho L}Gr^{1/4}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{x}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{x}}\right) \tag{3.28iii}$$

$$= \frac{\mu_\infty}{\rho L^2}\left[Gr^{1/4}\frac{\partial v}{\partial x} - Gr^{1/2}\sigma_x\frac{\partial v}{\partial y} + Gr^{1/2}\sigma_{xx}u + Gr^{1/2}\sigma_x\frac{\partial u}{\partial x} - Gr^{3/4}\sigma_x^2\frac{\partial u}{\partial y}\right]$$

$$\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\mu_\infty}{\rho L^2}Gr^{3/4}\frac{\partial u}{\partial y} \tag{3.28iv}$$

$$\frac{\partial \overline{p}}{\partial \overline{x}} = \frac{\rho v^2}{L^3} Gr \left(\frac{\partial p}{\partial x} - Gr^{1/4} \sigma_x \frac{\partial p}{\partial y} \right) \tag{3.28v}$$

$$\frac{\partial \overline{p}}{\partial \overline{y}} = \frac{\rho v^2}{L^3} Gr^{5/4} \frac{\partial p}{\partial y} \tag{3.28vi}$$

$$\frac{\partial^2 \overline{u}}{\partial \overline{x}^2} = \frac{v}{L^2} Gr^{1/2} \left[\begin{array}{l} \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) \frac{\partial x}{\partial \overline{x}} + \frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) \frac{\partial y}{\partial \overline{x}} \\ -Gr^{1/4} \{ \frac{1}{L} \sigma_{xx} \frac{\partial u}{\partial y} + \sigma_x [\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) \frac{\partial x}{\partial \overline{x}} + \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \frac{\partial y}{\partial \overline{x}}]\} \end{array} \right] \tag{3.28vii}$$

$$= \frac{v}{L^3} Gr^{1/2} [\frac{\partial^2 u}{\partial x^2} - 2\sigma_x Gr^{1/4} \frac{\partial^2 u}{\partial x \partial y} - Gr^{1/4} \sigma_{xx} \frac{\partial u}{\partial y} + Gr^{1/2} \sigma_x^2 \frac{\partial^2 u}{\partial y^2}] \tag{3.28viii}$$

$$\frac{\partial^2 \overline{u}}{\partial \overline{y}^2} = \frac{v}{L^2} Gr^{3/4} \left[\frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \frac{\partial y}{\partial \overline{y}} + \frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) \frac{\partial x}{\partial \overline{y}} \right]$$

$$= \frac{v}{L^3} Gr \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 \overline{v}}{\partial \overline{x}^2} = \left[\frac{\partial}{\partial x}(\frac{\partial \overline{v}}{\partial \overline{x}}) \frac{\partial x}{\partial \overline{x}} + \frac{\partial}{\partial y}(\frac{\partial \overline{v}}{\partial \overline{x}}) \frac{\partial y}{\partial \overline{x}} \right]$$

$$= \frac{v}{L^3} \left[\begin{array}{l} Gr^{1/4} \frac{\partial^2 v}{\partial x^2} - Gr^{1/2} \sigma_{xx} \frac{\partial v}{\partial y} - 2Gr^{1/2} \sigma_x \frac{\partial^2 v}{\partial x \partial y} + Gr^{1/2} \sigma_{xxx} u \\ \quad + 2Gr^{1/2} \sigma_{xx} \frac{\partial u}{\partial x} + Gr^{1/2} \sigma_x \frac{\partial^2 u}{\partial x^2} - 3Gr^{3/4} \sigma_x \sigma_{xx} \frac{\partial u}{\partial y} \\ \quad - 2Gr^{3/4} \sigma_x^2 \frac{\partial^2 u}{\partial x \partial y} + Gr^{3/4} \sigma_x^2 \frac{\partial^2 v}{\partial y^2} + Gr \sigma_x^3 \frac{\partial^2 u}{\partial y^2} \end{array} \right] \tag{3.28ix}$$

$$\frac{\partial^2 \overline{v}}{\partial \overline{y}^2} = \left[\frac{\partial}{\partial x}(\frac{\partial \overline{v}}{\partial \overline{y}}) \frac{\partial x}{\partial \overline{y}} + \frac{\partial}{\partial y}(\frac{\partial \overline{v}}{\partial \overline{y}}) \frac{\partial y}{\partial \overline{y}} \right]$$

$$= \frac{v}{L^3} [Gr^{3/4} \frac{\partial^2 v}{\partial y^2} + Gr \sigma_x \frac{\partial^2 u}{\partial y^2}] \tag{3.28x}$$

$$\nu\nabla^2\overline{u} = \nu\left(\frac{\partial^2\overline{u}}{\partial^2\overline{x}} + \frac{\partial^2\overline{u}}{\partial^2\overline{y}}\right)$$

$$= \frac{\nu}{L^3}\left[\left(\frac{\partial^2}{\partial x^2} - 2\sigma_x Gr^{\frac{1}{4}}\frac{\partial^2}{\partial x\partial y} - \sigma_{xx}Gr^{\frac{1}{2}}\frac{\partial}{\partial y} + \sigma_x{}^2 Gr^{\frac{1}{2}}\frac{\partial^2}{\partial y^2}\right) + Gr^{\frac{1}{2}}\frac{\partial^2}{\partial y^2}\right]\frac{\nu}{L}Gr^{\frac{1}{2}}u$$

(3.28 xi)

$$\nu\nabla^2\overline{v} = \frac{\nu}{L^2}\left[\frac{\partial^2}{\partial x^2} - 2xGr^{\frac{1}{2}}\frac{\partial^2}{\partial x\partial y} - \sigma_{xx}Gr^{\frac{1}{2}}\frac{\partial}{\partial y} + \sigma_x{}^2 Gr^{\frac{1}{2}}\frac{\partial^2}{\partial y^2} + Gr^{\frac{1}{2}}\frac{\partial^2}{\partial y^2}\right]\left(\frac{\nu}{L}Gr^{\frac{1}{4}}v + \frac{\nu}{L}\sigma_x{}^2 u\right)$$

(3.28xii)

$$g\beta(T - T_\infty) = g\beta(T_w - T_\infty)\theta = \frac{\nu^2}{L^3}Gr\theta \qquad (3.28xiii)$$

$$T = T_\infty + (T_w - T_\infty)\theta$$

$$\frac{\partial T}{\partial\overline{x}} = (T_w - T_\infty)\left(\frac{\partial\theta}{\partial x}\frac{\partial x}{\partial\overline{x}} + \frac{\partial\theta}{\partial y}\frac{\partial y}{\partial\overline{x}}\right) \qquad [\because T_w - T_\infty = \Delta T] \qquad (2.28xiv)$$

$$= \frac{\Delta T}{L}\left(\frac{\partial\theta}{\partial x} - Gr^{1/4}\sigma_x\frac{\partial\theta}{\partial y}\right)$$

$$\frac{\partial T}{\partial\overline{y}} = \Delta T\left(\frac{\partial\theta}{\partial x}\frac{\partial x}{\partial\overline{y}} + \frac{\partial\theta}{\partial y}\frac{\partial y}{\partial\overline{y}}\right)$$

Or, $\frac{\partial T}{\partial\overline{y}} = \frac{\Delta T}{L}Gr^{1/4}\frac{\partial\theta}{\partial y}$ (3.28xv)

$$\frac{\partial^2 T}{\partial\overline{x}^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial\overline{x}}\right)\frac{\partial x}{\partial\overline{x}} + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial\overline{x}}\right)\frac{\partial y}{\partial\overline{x}}$$

$$= \frac{\Delta T}{L^2}\left[\frac{\partial^2\theta}{\partial x^2} - Gr^{1/4}\sigma_{xx}\frac{\partial\theta}{\partial y} - 2Gr^{1/4}\sigma_x\frac{\partial^2\theta}{\partial x\partial y} + Gr^{1/2}\sigma_x^2\frac{\partial^2\theta}{\partial y^2}\right] \qquad (3.28xvi)$$

$$\frac{\partial^2 T}{\partial\overline{y}^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial\overline{y}}\right)\frac{\partial x}{\partial\overline{y}} + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial\overline{y}}\right)\frac{\partial y}{\partial\overline{y}}$$

Or, $\frac{\partial^2 T}{\partial\overline{y}^2} = \frac{\Delta T}{L^2}Gr^{1/2}\frac{\partial^2\theta}{\partial y^2}$ (3.28xvii)

Introducing the above dimensionless dependent and independent variables given in equations (3.28) – (3.28xvii), into equation (3.22) – (3.25) the following dimensionless form of the governing equations are obtained after ignoring terms of smaller orders of magnitude in Gr, the Grashof number defined in (3.28).

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3.29}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+Gr^{1/4}\sigma_x\frac{\partial p}{\partial y}+\left(1+\sigma_x^2\right)\frac{\partial^2 u}{\partial y^2}-Mu+\theta \tag{3.30}$$

$$\sigma_x\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-Gr^{1/4}\frac{\partial p}{\partial y}+\sigma_x\left(1+\sigma_x^2\right)\frac{\partial^2 u}{\partial y^2}-\sigma_{xx}u^2 \tag{3.31}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\Pr}\left(1+\sigma_x{}^2\right)\frac{\partial^2\theta}{\partial y^2}+Q\theta \tag{3.32}$$

where $\Pr=\dfrac{C_p\mu}{k_\infty}$ is the Prandtl number, $Q=\dfrac{Q_0L^2}{\mu c_p Gr^{1/2}}$ is the heat generation parameter, $\gamma=\gamma^*(T_w-T_\infty)$ is the thermal conductivity variation parameter and $M=\dfrac{\sigma_0 B_0 L^2}{\mu Gr^{1/2}}$ is the magnetic parameter.

It can easily be seen that the convection induced by the wavy surface is described by equations. (3.29) – (3.32). We further notice that, equation. (3.31) indicates that the pressure gradient along the y-direction is of $O(Gr^{-1/4})$, which implies that lowest order pressure gradient along x -direction can be determined from the inviscid flow solution. For the present problem this pressure gradient ($\partial p/\partial x=0$) is zero. Equation. (3.31) further shows that $Gr^{1/4}\partial p/\partial y$ is of $O(1)$ and is determined by the left-hand side of this equation. Thus, the elimination of $\partial p/\partial y$ from equations. (3.30) and (3.31) leads to the following equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(1+\sigma_x^{\ 2}\right)\frac{\partial^2 u}{\partial y^2} - \frac{\sigma_x \sigma_{xx}}{1+\sigma_x^{\ 2}}u^2 + \frac{1}{1+\sigma_x^{\ 2}}\theta - \frac{M}{1+\sigma_x^{\ 2}}u \tag{3.33}$$

The corresponding boundary conditions for the present problem then turn into

$$\left.\begin{array}{lll} u = v = 0, & \theta = 1 & at \quad y = 0 \\ u = \theta = 0, & p = 0 & as \quad y \to \infty \end{array}\right\}. \tag{3.34}$$

Now we introduce the following transformations to reduce the governing equations to a convenient form:

$$\psi = x^{3/4} f(x,\eta), \quad \eta = yx^{-1/4}, \quad \theta = \theta(x,\eta) \tag{3.35}$$

where $f(x, \eta)$ is the dimensionless stream function, η is the pseudo similarity variable and ψ is the stream function that satisfies the equation (3.29) and is defined by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{3.36}$$

$$u = x^{3/4}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}\right) = x^{1/2} f' \tag{3.36i}$$

$$\begin{aligned} v &= -\frac{3}{4}x^{-1/4} f - x^{3/4}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \\ &= \frac{1}{4}x^{-1/4}\eta f' - x^{3/4}\frac{\partial f}{\partial x} - \frac{3}{4}x^{-1/4} f \qquad \left[\because \frac{\partial \eta}{\partial x} = -\frac{1}{4x}\eta\right] \end{aligned} \tag{3.36bii}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}x^{-1/2} f' + x^{1/2}\left(\frac{\partial f'}{\partial x} - \frac{1}{4x}\eta f''\right) \tag{3.36iii}$$

$$\frac{\partial u}{\partial y} = x^{1/4} f'' \qquad \left[\because \frac{\partial \eta}{\partial y} = x^{-1/4}\right]. \tag{3.36iv}$$

$$\frac{\partial^2 u}{\partial y^2} = x^{1/4}\left(\frac{\partial f''}{\partial \eta}\frac{\partial \eta}{\partial y}\right) = f''' \tag{3.36v}$$

$$\begin{aligned} \frac{\partial \theta}{\partial x} &= \left(\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \\ &= \frac{\partial \theta}{\partial x} - \frac{1}{4x}\eta\theta' \end{aligned} \tag{3.36vi}$$

$$\frac{\partial \theta}{\partial y} = x^{-1/4}\theta' \tag{3.36vii}$$

$$\frac{\partial^2 \theta}{\partial y^2} = x^{-1/2}\theta'' \tag{3.36viii}$$

Introducing the transformations given in equation. (3.35) and calculations given above equations (3.36) – (3.36viii) into equations. (3.33) and (3.32) the following system of non linear equations is obtained:

$$\left(1+\sigma_x^{\ 2}\right)f''' + \frac{3}{4}ff'' - \left(\frac{1}{2} + \frac{x\sigma_x\sigma_{xx}}{1+\sigma_x^2}\right)f'^2 + \frac{1}{1+\sigma_x^{\ 2}}\theta - \frac{Mx^{1/2}}{1+\sigma_x^2}f' = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) \tag{3.37}$$

$$\frac{1}{\Pr}\left(1+\sigma_x^{\ 2}\right)\theta'' + \frac{3}{4}f\theta' + x^{\frac{1}{2}}Q\theta = x\left(f'\frac{\partial \theta}{\partial x} - \theta'\frac{\partial f}{\partial x}\right) \tag{3.38}$$

The boundary conditions (3.34) now take the following form:

$$\left.\begin{aligned} &f(x,o) = f'(x,o) = 0, \quad \theta(x,o) = 1 \\ &f'(x,\infty) = 0, \qquad \theta(x,\infty) = 0 \end{aligned}\right\} \tag{3.39}$$

In the above equations prime denote the differentiation with respect to η.

Solutions of the local non-similar partial differential equations (3.37) and (3.38) together with the boundary condition (3.39) are obtained numerically by using implicit finite difference method with Keller-Box Scheme.

In practical applications, the physical quantities of principle interest are the shearing stress τ_w in terms of the skin-friction coefficients C_{fx} and the rate of heat transfer in terms of Nusselt number Nu_x which can be written as:

$$C_{fx} = \frac{2\tau_w}{\rho U^2} \text{ and } Nu_x = \frac{q_w x}{k_\infty (T_w - T_\infty)} \tag{3.40}$$

where $\tau_w = (\mu \bar{n}.\nabla \bar{u})_{y=0}$ and $q_w = -k(\bar{n}.\nabla T)_{y=0}$ (3.41)

where

$$q_w = -k(\bar{n}.\nabla T)_{y=o}$$
$$= -k\frac{\bar{i}h_x + \bar{j}h_y}{\sqrt{h_x^2 + h_y^2}}.\left(\bar{i}\frac{\partial T}{\partial \bar{x}} + \bar{j}\frac{\partial T}{\partial \bar{y}}\right)_{y=0} \tag{3.41i}$$

where $h(\bar{x},\bar{y}) = \bar{y} - \sigma(\bar{x})$

Now,

$$\frac{\partial h}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} + \frac{\partial h}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x} = \frac{\partial \bar{y}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} - \frac{\partial \sigma(\bar{x})}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} \quad \left[\frac{\partial \bar{y}}{\partial x} = \frac{\partial \bar{y}}{\partial y}\frac{\partial y}{\partial x} = 0\right]$$

$$\Rightarrow Lh_x = -L\sigma_x$$

$$\Rightarrow h_x = -\sigma_x \tag{i}$$

$$\frac{\partial h}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial y} + \frac{\partial h}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial y} = \frac{\partial \bar{y}}{\partial y} - \frac{\partial \sigma(\bar{x})}{\partial y} \qquad \left[\frac{\partial \bar{x}}{\partial y} = \frac{\partial \bar{x}}{\partial x}\frac{\partial x}{\partial y} = 0\right]$$

$$\Rightarrow LGr^{-1/4}h_y = LGr^{-1/4}$$

$$\Rightarrow h_y = 1 \tag{ii}$$

$$\bar{n} = \frac{\bar{i}h_x + \bar{j}h_y}{\sqrt{h_x^2 + h_y^2}} = \frac{-\bar{i}\sigma_x + \bar{j}}{\sqrt{\sigma_x^2 + 1}} \tag{iii}$$

$$(\bar{n}.\nabla T) = \frac{\bar{i}h_x + jh_y}{\sqrt{h_x^2 + h_y^2}}.\left(\bar{i}\frac{\partial T}{\partial \bar{x}} + \bar{j}\frac{\partial T}{\partial \bar{y}}\right)$$

$$= \frac{\Delta T}{L\sqrt{\sigma_x^2 + 1}}\left[-\sigma_x\frac{\partial \theta}{\partial x} + Gr^{1/4}(\sigma_x^2 + 1)\frac{\partial \theta}{\partial y}\right]$$

$$\therefore (\bar{n}.\nabla T)_{y=0} = \frac{\Delta T}{L}x^{-1/4}Gr^{1/4}\sqrt{\sigma_x^2 + 1}\,\theta'(x,0) \qquad \because (\frac{\partial \theta}{\partial x})_{y=0} = 0, (\frac{\partial \theta}{\partial y})_{y=0} = x^{-1/4}\theta'(x,0)$$

$$\therefore q_w = -k\frac{\Delta T}{L}x^{-1/4}Gr^{1/4}\sqrt{\sigma_x^2 + 1}\,\theta'(x,0) \tag{3.41ii}$$

where $\Delta T = (T_w - T_\infty)$

Now,

$$\tau_w = \left(\mu \bar{n}.\nabla \bar{u} \right)_{y=0}$$

$$= \mu_\infty (1+\gamma) \frac{\bar{i} f_x + \bar{j} f_y}{\sqrt{f_x^2 + f_y^2}} . \left(\bar{i} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{j} \frac{\partial \bar{u}}{\partial \bar{y}} \right)_{y=0}$$

$$= \frac{\mu_\infty}{\rho L^2} Gr^{3/4} (1+\gamma) x^{1/4} \sqrt{1+\sigma_x^2} \quad f''(x,o) \tag{3.41iii}$$

Here $\bar{n} = \frac{\bar{i} h_x + \bar{j} h_y}{\sqrt{h_x^2 + h_y^2}}$. is the unit normal to the surface. Using the transformations (3.35) and calculations (3.41i) to (3.41iv) into equation (3.40), the local Nusselt number, Nu_x and the local skin friction coefficient, C_{fx} take the following forms:

$$C_{fx} (Gr/x)^{1/4} / 2 = (1+\gamma) \sqrt{1+\sigma_x^{\ 2}} \quad f''(x,o) \tag{3.42}$$

$$Nu_x (Gr/x)^{-1/4} = -\sqrt{1+\sigma_x^{\ 2}} \theta'(x,o) \tag{3.43}$$

Finally, it should be mentioned that for the computational purpose the period of oscillations in the waviness of this surface has been considered to be π.

3.6 Numerical Procedure

The transformed boundary layer equations solved numerically with the help of implicit finite difference method together with the Keller-Box scheme (1978) and used by Hossain et al. (1996, 1997, 1999, 2000, 2001). To begin with, the partial differential equations are first converted into a system of first order differential equations. Then these equations are expressed in finite difference forms by approximating the functions and their derivatives in terms of the center differences. Denoting the mesh points in the x and η-plane by x_i and η_j where i = 1, 2 ,......., M and j = 1, 2,......,N, central difference approximations are made, such that those equations involving x explicitly are centered at $(x_{i-1/2}, \eta_{j-1/2})$ and the remainder at $(x_i, \eta_{j-1/2})$, where $\eta_{j-1/2} = 1/2(\eta_j + \eta_{j-1})$ etc. The above central difference approximations reduce the system of first order differential equations to a set of non-linear difference equations for the unknown at x_i in terms of their values at x_{i-1}. The resulting set of non-linear difference equations are solved by using the Newton's quasi-linearization method. The Jacobian matrix has a block-tridiagonal structure and the difference equations are efficiently solved using a block-matrix version of the Thomas algorithm. In the program test, a finer axial step size is tried and find to give acceptable accuracy. A uniform grid of 201 points is used in x- direction with $\Delta x = 0.05$, while a non-uniform grid of 76 points lying between $\eta = 0.0$ and 10.017 is chosen. Grid points are concentrated towards the heated surface in order to improve resolution and the accuracy of the computed values of the surface shear stress and rate of heat transfer. During the program test, the convergent criteria for the relative errors between two iterations are less 10^{-5}. It means that iterative procedure is stopped when the maximum change between successive iterates is less then 10^{-5}.

3.7 Implicit Finite Difference Method (IFDM)

To apply the aforementioned method, equations (3.37) and (3.38) and the boundary condition (3.39) are first converted into the following system of first order equations. For this purpose we introduce new dependent variables $u(\xi,\eta)$, $v(\xi,\eta)$, $p(\xi,\eta)$ and $g(\xi,\eta)$ so that the transformed momentum and energy equations can be written as:

$$f' = u \tag{3.44}$$

$$u' = v \tag{3.45}$$

$$g' = \theta' = p \tag{3.46}$$

$$p_1 v' + p_2 f v - p_3 u^2 + p_4 g - p_6 u = \xi\left(u\frac{\partial u}{\partial \xi} - v\frac{\partial f}{\partial \xi}\right) \tag{3.47}$$

$$\frac{1}{\Pr} P_1 p' + P_2 f \; p + \xi^{\frac{1}{2}} P_5 g = \xi \; \left(u \; \frac{\partial g}{\partial \xi} - p \; \frac{\partial f}{\partial \xi}\right) \tag{3.48}$$

where $x = \xi$, $\theta = g$ and

$$P_1 = \left(1+\sigma_x^2\right),\; P_2 = \frac{3}{4},\; P_3 = \frac{1}{2} + \frac{x\sigma_x\sigma_{xx}}{1+\sigma_x^2},\; P_4 = \frac{1}{1+\sigma_x^2},\; P_6 = \frac{Mx^{1/2}}{1+\sigma_x^2},$$

and $Q = P_5$

and the boundary conditions (3.39) are

$$f(\xi,0)=0,\ u(\xi,0)=0, g(\xi,0)=1$$
$$u(\xi,\infty)=0,\ g\ (\xi,\infty)=0 \tag{3.49}$$

Now consider the net rectangle on the (ξ,η) plane shown in the Figure 3.2 and denote the net points by

$$\xi^0=0,\ \xi^n=\xi^{n-1}+k_n,\ n=1,2,\cdots\cdots\cdots\cdots,N$$
$$\eta_0=0,\ \eta_j=\eta_{j-1}+h_{j,}\ \ j=1,2,\cdots\cdots\cdots\cdots,J \tag{3.50}$$

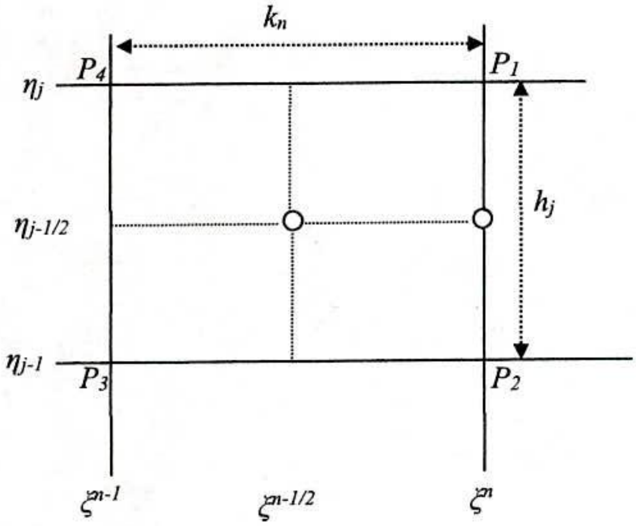

Figure 3.2: Net rectangle of difference approximations for the Box scheme.

Here n and j are just sequence of numbers on the (ξ,η) plane, k_n and h_j are the variable mesh widths. Approximate the quantities f, u, v and p at the points (ξ^n,η_j) of the net by f_j^n, u_j^n, v_j^n, p_j^n which call net function. It is also employed that the notation P_j^n for the quantities midway between net points shown in figure 3.2 and for any net function as

$$\xi^{n-1/2}=\frac{1}{2}(\xi^n+\xi^{n-1}) \tag{3.51}$$

$$\eta_{j-1/2}=\frac{1}{2}(\eta_j+\eta_{j-1}) \tag{3.52}$$

$$g_j^{n-1/2} = \frac{1}{2}(g_j^n + g_j^{n-1}) \tag{3.53}$$

$$g_{j-1/2}^n = \frac{1}{2}(g_j^n + g_{j-1}^n) \tag{3.54}$$

The finite difference approximations according to box method to the three first order ordinary differential equations (3.44) – (3.46) are written for the midpoint $(\xi^n, \eta_{j-1/2})$ of the segment P_1P_2 shown in the figure 3.2 and the finite difference approximations to the two first order differential equations (3.47) and (3.48) are written for the midpoint $(\xi^{n-1/2}, \eta_{j-1/2})$ of the rectangle $P_1P_2P_3P_4$. This procedure yields

$$\frac{f_j^n - f_{j-1}^n}{h_j} = u_{j-1/2}^n = \frac{u_j^n + u_{j-1/2}^n}{2} \tag{3.55}$$

$$\frac{u_j^n - u_{j-1}^n}{h_j} = v_{j-1/2}^n = \frac{v_{j-1}^n + v_j^n}{2} \tag{3.56}$$

$$\frac{g_j^n - g_{j-1}^n}{h_j} = p_{j-1/2}^n = \frac{p_{j-1}^n + p_j^n}{2} \tag{3.57}$$

$$\begin{aligned}&\frac{1}{2}(P_1)_{j-1/2}^n\left(\frac{v_j^n - v_{j-1}^n}{h_j}\right) + \frac{1}{2}(P_1)_{j-1/2}^n\left(\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + (P_2 fv)_{j-1/2}^{n-1/2} - (P_3 u^2)_{j-1/2}^{n-1/2}\\&+(P_4 g)_{j-1/2}^{n-1/2} - (P_6 u)_{j-1/2}^{n-1/2} = \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n} - v_{j-1/2}^{n-1/2}\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right)\end{aligned} \tag{3.58}$$

$$\begin{aligned}&\frac{1}{2\Pr}\{(P_1)_{j-1/2}^n\}\left(\frac{P_j^n - P_{j-1}^n}{h_j}\right) + \frac{1}{2\Pr}\{(P_1)_{j-1/2}^n\}\left(\frac{P_j^{n-1} - P_{j-1}^{n-1}}{h_j}\right) + (P_2 fp)_{j-1/2}^{n-1/2}\\&+(P_5 g)_{j-1/2}^{n-1/2} = \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{g_{j-1/2}^n - g_{j-1/2}^{n-1}}{k_n} - P_{j-1/2}^{n-1/2}\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right)\end{aligned} \tag{3.59}$$

Now the equation (3.58) can be written as

$$\frac{1}{2}(P_1)^n_{j-1/2}\left(\frac{v^n_j - v^n_{j-1}}{h_j}\right) + \frac{1}{2}(P_1)^{n-1}_{j-1/2}\left(\frac{v^{n-1}_j - v^{n-1}_{j-1}}{h_j}\right) + \frac{1}{2}\left\{(P_2 fv)^n_{j-1/2} + (P_2 fv)^{n-1}_{j-1/2}\right\}$$

$$-\frac{1}{2}\left\{(P_3 u^2)^n_{j-1/2} + (P_3 u^2)^{n-1}_{j-1/2}\right\} + \frac{1}{2}\left\{\left(P_4 g^n_{j-1/2}\right) + \left(P_4 g^{n-1}_{j-1/2}\right)\right\} - \frac{1}{2}\left\{\left(P_6 u^n_{j-1/2}\right) + \left(P_6 u^{n-1}_{j-1/2}\right)\right\}$$

$$= \frac{1}{2k_n}\xi^{n-1/2}_{j-1/2}\left(u^n_{j-1/2} + u^{n-1}_{j-1/2}\right)\left(u^n_{j-1/2} - u^{n-1}_{j-1/2}\right)$$

$$-\frac{1}{2k_n}\xi^{n-1/2}_{j-1/2}\left(v^n_{j-1/2} + v^{n-1}_{j-1/2}\right)\left(f^n_{j-1/2} - f^{n-1}_{j-1/2}\right)$$

Or, $(P_1)^n_{j-1/2}\left(\frac{v^n_j - v^n_{j-1}}{h_j}\right) + (P_1)^{n-1}_{j-1/2}\left(\frac{v^{n-1}_j - v^{n-1}_{j-1}}{h_j}\right) + (P_2)^n_{j-1/2}\,(fv)^n_{j-1/2} + (P_2)^{n-1}_{j-1/2}\,(fv)^{n-1}_{j-1/2}$

$-(P_3)^n_{j-1/2}\,(u^2)^n_{j-1/2} - (P_3)^{n-1}_{j-1/2}\,(u^2)^{n-1}_{j-1/2} + (P_4)^n_{j-1/2}\,g^n_{j-1/2} + (P_4)^{n-1}_{j-1/2}\,g^{n-1}_{j-1/2}$

$-(P_6)^n_{j-1/2}(u)^n_{j-1/2} - (P_6)^{n-1}_{j-1/2}(u)^{n-1}_{j-1/2}$

$= \alpha_n\left\{(u^2)^n_{j-1/2} - (u^2)^{n-1}_{j-1/2} - (fv)^n_{j-1/2} + f^{n-1}_{j-1/2}v^n_{j-1/2} - f^n_{j-1/2}v^{n-1}_{j-1/2} + (fv)^{n-1}_{j-1/2}\right\}$

Or, $(P_1)^n_{j-1/2}\left(\frac{v^n_j - v^n_{j-1}}{h_j}\right) + \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}(fv)^n_{j-1/2} - \left\{(P_3)^n_{j-1/2} + \alpha_n\right\}(u^2)^n_{j-1/2}$

$+(P_4)^n_{j-1/2}\,g^n_{j-1/2} - (P_5)^n_{j-1/2}(u)^n_{j-1/2} + (P_6)^n_{j-1/2} + \alpha_n\left(f^n_{j-1/2}v^{n-1}_{j-1/2} - v^n_{j-1/2}f^{n-1}_{j-1/2}\right)$

$= \alpha_n\left[\left\{-(u^2)^{n-1}_{j-1/2} + (fv)^{n-1}_{j-1/2}\right\}\right] - (P_2)^{n-1}_{j-1/2}(fv)^{n-1}_{j-1/2}$

$+(P_3)^{n-1}_{j-1/2}(u^2)^{n-1}_{j-1/2} - (P_4)^{n-1}_{j-1/2}g^{n-1}_{j-1/2} + (P_5)^{n-1}_{j-1/2}(u)^{n-1}_{j-1/2} - (P_6)^{n-1}_{j-1/2}(pv)^{n-1}_{j-1/2}$

$-(P_1 T)^{n-1}_{j-1/2}\left(\frac{v^{n-1}_j - v^{n-1}_{j-1}}{h_j}\right)$

i.e., $R^{n-1}_{j-1/2} = \alpha_n\left\{(fv)^{n-1}_{j-1/2} - (u^2)^{n-1}_{j-1/2}\right\} - L^{n-1}_{j-1/2}$ (say) (3.60)

where $\alpha_n = \frac{1}{k_n}\xi^{n-1/2}_{j-1/2}$ (3.61)

$$\begin{aligned} R^{n-1}_{j-1/2} &= (P_1)^n_{j-1/2}\,h_j^{-1}\left(v^n_j - v^n_{j-1}\right) + \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}(fv)^n_{j-1/2} \\ &- \left\{(P_3)^n_{j-1/2} + \alpha_n\right\}(u^2)^n_{j-1/2} + (P_4)^n_{j-1/2}g^n_{j-1/2} - (P_5)^n_{j-1/2}(u)^n_{j-1/2} \\ &+ (P_6)^n_{j-1/2}(pv)^n_{j-1/2} + \alpha_n\left(f^n_{j-1/2}\,v^{n-1}_{j-1/2} - v^n_{j-1/2}\,f^{n-1}_{j-1/2}\right) \end{aligned} \qquad (3.62)$$

$$\text{and } L_{j-1/2}^{n-1} = (P_1)_{j-1/2}^{n-1} h_j^{-1}\left(v_j^{n-1} - v_{j-1}^{n-1}\right) + (P_2)_{j-1/2}^{n-1}(fv)_{j-1/2}^{n-1} - (P_3)_{j-1/2}^{n-1}(u^2)_{j-1/2}^{n-1} + (P_4)_{j-1/2}^{n-1} g_{j-1/2}^{n-1} - (P_5)_{j-1/2}^{n-1}(u)_{j-1/2}^{n-1} + (P_6)_{j-1/2}^{n-1}(pv)_{j-1/2}^{n-1} \tag{3.63}$$

Again from the equation (3.59) then

$$\frac{1}{\Pr}\left[\left\{(P_1)_{j-1/2}^{n}\right\}\left(\frac{p_j^n - p_{j-1}^n}{h_j}\right) + \left\{(P_1)_{j-1/2}^{n-1}\right\}\left(\frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right)\right] + \left\{(P_2 fp)_{j-\frac{1}{2}}^{n} + (P_2 fp)_{j-\frac{1}{2}}^{n-1}\right\}$$
$$+ (P_5 g)_{j-1/2}^{n} + (P_5 g)_{j-1/2}^{n-1}$$
$$= \alpha_n\left[\left(u_{j-\frac{1}{2}}^{n} + u_{j-\frac{1}{2}}^{n-1}\right)\left(g_{j-\frac{1}{2}}^{n} - g_{j-\frac{1}{2}}^{n-1}\right) - \left(p_{j-\frac{1}{2}}^{n} + p_{j-\frac{1}{2}}^{n-1}\right)\left(f_{j-\frac{1}{2}}^{n} - f_{j-\frac{1}{2}}^{n-1}\right)\right]$$

$$\text{Or, } \frac{1}{\Pr}\left\{(P_1)_{j-1/2}^{n}\right\}\left(\frac{p_j^n - p_{j-1}^n}{h_j}\right) + \frac{1}{\Pr}\left\{(P_1)_{j-1/2}^{n-1}\right\}\left(\frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) + (P_2)_{j-\frac{1}{2}}^{n}(fp)_{j-1/2}^{n}$$
$$+ (P_2)_{j-\frac{1}{2}}^{n-1}(fp)_{j-1/2}^{n-1} + (P_5)_{j-1/2}^{n}(g)_{j-1/2}^{n} + (P_5)_{j-1/2}^{n-1}(g)_{j-1/2}^{n-1}$$
$$= \alpha_n\left[\begin{matrix}(ug)_{j-1/2}^{n} - u_{j-1/2}^{n} g_{j-1/2}^{n-1} + g_{j-1/2}^{n} u_{j-1/2}^{n-1} - (ug)_{j-1/2}^{n-1} - (fp)_{j-1/2}^{n} \\ + p_{j-1/2}^{n} f_{j-1/2}^{n-1} - f_{j-1/2}^{n} p_{j-1/2}^{n-1} + (fp)_{j-1/2}^{n-1}\end{matrix}\right]$$
$$\text{Or, } \frac{1}{\Pr}\left\{(P_1)_{j-1/2}^{n}\right\}\left(\frac{p_j^n - p_{j-1}^n}{h_j}\right) + \left\{(P_2)_{j-\frac{1}{2}}^{n} + \alpha_n\right\}(fp)_{j-1/2}^{n} - \alpha_n (ug)_{j-1/2}^{n}$$
$$+ \alpha_n\left[u_{j-1/2}^{n} g_{j-1/2}^{n-1} - g_{j-1/2}^{n} u_{j-1/2}^{n-1} - p_{j-1/2}^{n} f_{j-1/2}^{n-1} + f_{j-1/2}^{n} p_{j-1/2}^{n-1}\right] + (P_5)_{j-1/2}^{n}(g)_{j-1/2}^{n}$$
$$= -\frac{1}{\Pr}\left\{(P_1)_{j-1/2}^{n-1}\right\}\left(\frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) - (P_2)_{j-1/2}^{n-1}(fp)_{j-1/2}^{n-1} + \alpha_n\left[(fp)_{j-1/2}^{n-1} - (ug)_{j-1/2}^{n-1}\right] - (P_5)_{j-1/2}^{n-1}(g)_{j-1/2}^{n-1}$$

$$\text{i.e., } \quad T_{j-1/2}^{n-1} = -M_{j-1/2}^{n-1} + \alpha_n\left\{(fp)_{j-1/2}^{n-1} - (ug)_{j-1/2}^{n-1}\right\}\text{(say)} \tag{3.64}$$

$$\text{where, } \quad \alpha_n = \frac{1}{k_n}\xi_{j-1/2}^{n-1/2} \tag{3.65}$$

$$M_{j-\frac{1}{2}}^{n-1} = \frac{h_j^{-1}}{\Pr}(P_1)_{j-\frac{1}{2}}^{n-1}\left\{p_j^{n-1} - p_{j-1}^{n-1}\right\} + (P_2)_{j-\frac{1}{2}}^{n-1}(fp)_{j-\frac{1}{2}}^{n-1} + (P_5)_{j-1/2}^{n-1}(g)_{j-1/2}^{n-1} \tag{3.66}$$

$$T_{j-1/2}^{n-1} = \frac{1}{\Pr}(P_1)_{j-1/2}^{n} h_j^{-1}\left(p_j^n - p_{j-1}^n\right) + \left\{(P_2)_{j-1/2}^{n} + \alpha_n\right\}(fp)_{j-1/2}^{n} - \alpha_n (ug)_{j-1/2}^{n}$$
$$+ \alpha_n\left(u_{j-1/2}^{n} g_{j-1/2}^{n-1} - g_{j-1/2}^{n} u_{j-1/2}^{n-1}\right) - \alpha_n\left(p_{j-1/2}^{n} f_{j-1/2}^{n-1} - f_{j-1/2}^{n} p_{j-1/2}^{n-1}\right) + (P_5)_{j-1/2}^{n}(g)_{j-1/2}^{n} \tag{3.67}$$

The boundary condition becomes

$$f_0^n = 0, \quad u_0^n = 0, g_0^n = 1$$
$$u_j^n = 0, \quad g_j^n = 0 \tag{3.68}$$

CHAPTER IV

General Procedure

4.1 The Finite Difference Procedure

It is assumed that $f_j^{n-1}, u_j^{n-1}, v_j^{n-1}, g_j^{n-1}, p_j^{n-1}$, for $0 \le j \le J$ are known. Then equations (3.51) to (3.68) form a system of (5*J*+5) non linear equations for the solutions of the (5*J*+5) unknowns $\left(f_j^n, u_j^n, v_j^n, g_j^n, p_j^n\right), j = 0, 1, 2, 3, \ldots\ldots J$. These non-linear systems of algebraic equations are to be linearized by Newton's Quassy linearization method. The iterates $\left(f_j^i, u_j^i, v_j^i, g_j^i, p_j^i\right)$, $i = 0, 1, 2, 3 \ldots\ldots N$ are defined with initial values equal those at the previous *x*-station (which is usually the best initial available). For the higher iterates the following forms can be written

$$f_j^{(i+1)} = f_j^i + \delta f_j^i \tag{4.1}$$

$$u_j^{(i+1)} = u_j^i + \delta u_j^i \tag{4.2}$$

$$v_j^{(i+1)} = v_j^i + \delta v_j^i \tag{4.3}$$

$$g_j^{(i+1)} = g_j^i + \delta g_j^i \tag{4.4}$$

$$p_j^{(i+1)} = p_j^i + \delta p_j^i \tag{4.5}$$

Now by substituting the right hand sides of the above equations in place of f_j^n, u_j^n, v_j^n and g_j^n in equations (3.55)-(3.57) and in equations (3.60) and (3.64) dropping the terms that are quadratic in $\delta f_j^i, \delta u_j^i, \delta v_j^i$ and δp_j^i, then take the following linear system of algebraic form

Or, $\dfrac{f_j^{(i)} - f_{j-1}^{(i)}}{h_j} + \dfrac{\delta f_j^{(i)} - \delta f_{j-1}^{(i)}}{h_j} = u_{j-\frac{1}{2}}^{(i)} + \delta u_{j-\frac{1}{2}}^{(i)} = \dfrac{1}{2}\left\{u_j^{(i)} + \delta u_j^{(i)} + u_{j-1}^{(i)} + \delta u_{j-1}^{(i)}\right\}$

Or, $f_j^{(i)} + \delta f_j^{(i)} - f_{j-1}^{(i)} - \delta f_{j-1}^{(i)} = \dfrac{h_j}{2}\left\{u_j^{(i)} + \delta u_j^{(i)} + u_{j-1}^{(i)} + \delta u_{j-1}^{(i)}\right\}$

$$\delta f_j^{(i)} - \delta f_{j-1}^{(i)} - \frac{h_j}{2}\left(\delta u_j^{(i)} + \delta u_{j-1}^{(i)}\right) = (r_1)_j \text{ (say)} \tag{4.6}$$

where, $(r_1)_j = f_{j-1}^{(i)} - f_j^{(i)} + \dfrac{h_j}{2} u_{j-1}^{(i)}$

Similarly,

$$\delta u_j^{(i)} - \delta u_{j-1}^{(i)} - \frac{h_j}{2}\left(\delta v_j^{(i)} + \delta v_{j-1}^{(i)}\right) = (r_4)_j \tag{4.7}$$

$$\delta g_j^{(i)} - \delta g_{j-1}^{(i)} - \frac{h_j}{2}\left(\delta p_j^{(i)} + \delta p_{j-1}^{(i)}\right) = (r_5)_j \tag{4.8}$$

where, $(r_4)_j = u_{j-1}^{(i)} - u_j^{(i)} + \dfrac{h_j}{2} v_{j-1}^{(i)}$

and

$$(r_5)_j = g_{j-1}^{(i)} - g_j^{(i)} + \frac{h_j}{2} p_{j-1}^{(i)}$$

Then equation (3.62) becomes,

$$\begin{aligned}
&(P_1)_{j-1/2}^{n} h_j^{-1}\left(v_j^{(i)} + \delta v_j^{(i)} - v_{j-1}^{(i)} - \delta v_{j-1}^{(i)}\right) + \left\{(P_2)_{j-1/2}^{n} + \alpha_n\right\}\left\{(fv)_{j-1/2}^{(i)} + \delta(fv)_{j-1/2}^{(i)}\right\} \\
&- \left\{(P_3)_{j-1/2}^{n} + \alpha_n\right\}\left\{(u^2)_{j-1/2}^{(i)} + \delta(u^2)_{j-1/2}^{(i)}\right\} + (P_4)_{j-1/2}^{n}\left\{g_{j-1/2}^{(i)} + \delta g_{j-1/2}^{(i)}\right\} \\
&- (P_5)_{j-1/2}^{n}\left\{(u)_{j-1/2}^{(i)} + \delta(u)_{j-1/2}^{(i)}\right\} + (P_6)_{j-1/2}^{n}\left\{(pv)_{j-1/2}^{(i)} + \delta(pv)_{j-1/2}^{(i)}\right\} \\
&+ \alpha_n\left\{f_{j-1/2}^{(i)} + \delta f_{j-1/2}^{(i)}\right\} v_{j-1/2}^{n-1} - \alpha_n\left(v_{j-1/2}^{(i)} + \delta v_{j-1/2}^{(i)}\right) f_{j-1/2}^{n-1} = R_{j-1/2}^{n-1}
\end{aligned}$$

$$\text{Or, } (P_1)_{j-1/2}^n h_j^{-1}\left(v_j^{(i)}+\delta v_j^{(i)}-v_{j-1}^{(i)}-\delta v_{j-1}^{(i)}\right)$$
$$+\left\{(P_2)_{j-1/2}^n+\alpha_n\right\}\left\{(fv)_{j-1/2}^{(i)}+\frac{1}{2}\left(f_j^{(i)}\delta v_j^{(i)}+v_j^{(i)}\delta f_j^{(i)}+f_{j-1}^{(i)}\delta v_{j-1}^{(i)}+v_{j-1}^{(i)}\delta f_{j-1}^{(i)}\right)\right\}$$
$$-\left\{(P_3)_{j-1/2}^n+\alpha_n\right\}\left\{(u^2)_{j-1/2}^{(i)}+u_j^{(i)}\delta u_j^{(i)}+u_{j-1}^{(i)}\delta u_{j-1}^{(i)}\right\}$$
$$+(P_4)_{j-1/2}^n\left\{g_{j-1/2}^{(i)}+\frac{1}{2}\left(\delta g_j^{(i)}+\delta g_{j-1}^{(i)}\right)\right\}-(P_5)_{j-1/2}^n\left\{(u)_{j-1/2}^{(i)}+\frac{1}{2}\left(\delta(u)_j^{(i)}+\delta(u)_{j-1}^{(i)}\right)\right\}$$
$$+(P_6)_{j-1/2}^n\left\{(pv)_{j-1/2}^{(i)}+\frac{1}{2}\left(p_j^{(i)}\delta v_j^{(i)}+v_j^{(i)}\delta p_j^{(i)}+p_{j-1}^{(i)}\delta v_{j-1}^{(i)}+v_{j-1}^{(i)}\delta p_{j-1}^{(i)}\right)\right\}$$
$$+\alpha_n\left\{f_{j-1/2}^{(i)}+\frac{1}{2}\left(\delta f_j^{(i)}+\delta f_{j-1}^{(i)}\right)\right\}v_{j-1/2}^{n-1}-\alpha_n\left(v_{j-1/2}^{(i)}+\frac{1}{2}\left(\delta v_j^{(i)}+\delta v_{j-1}^{(i)}\right)\right)f_{j-1/2}^{n-1}=R_{j-1/2}^{n-1}$$

$$\text{Or, } (s_1)_j\,\delta v_j^{(i)}+(s_2)_j\,\delta v_{j-1}^{(i)}+(s_3)_j\,\delta f_j^{(i)}+(s_4)_j\,\delta f_{j-1}^{(i)}+(s_5)_j\,\delta u_j^{(i)}$$
$$+(s_6)_j\,\delta u_{j-1}^{(i)}+(s_7)_j\,\delta g_j^{(i)}+(s_8)_j\,\delta g_{j-1}^{(i)}+(s_9)_j\,\delta p_j^{(i)}+(s_{10})_j\,\delta p_{j-1}^{(i)}=(r_2)_j \qquad \text{(say)}(4.9)$$

where

$$(r_2)_j=R_{j-1/2}^{n-1}-(P_1)_{j-1/2}^n h_j^{-1}\left(v_j^{(i)}-v_{j-1}^{(i)}\right)-\left\{(P_2)_{j-1/2}^n+\alpha_n\right\}(fv)_{j-1/2}^{(i)}$$
$$+\left\{(P_3)_{j-1/2}^n+\alpha_n\right\}(u^2)_{j-1/2}^{(i)}-(P_4)_{j-1/2}^n g_{j-1/2}^{(i)}+(P_5)_{j-1/2}^n u_{j-1/2}^{(i)} \qquad (4.10)$$
$$-(P_6)_{j-1/2}^n(pv)_{j-1/2}^{(i)}-\alpha_n\left(f_{j-1/2}^{(i)}v_{j-1/2}^{n-1}-v_{j-1/2}^{(i)}f_{j-1/2}^{n-1}\right)$$

$$(s_1)_j=h_j^{-1}(P_1)_{j-1/2}^n+\frac{(P_2)_{j-1/2}^n+\alpha_n}{2}f_j^{(i)}+\frac{1}{2}(P_6)_{j-1/2}^n p_j^{(i)}-\frac{\alpha_n}{2}f_{j-1/2}^{n-1} \qquad (4.11)$$

$$(s_2)_j=-h_j^{-1}(P_1)_{j-1/2}^n+\frac{(P_2)_{j-1/2}^n+\alpha_n}{2}f_{j-1}^{(i)}+\frac{1}{2}(P_6)_{j-1/2}^n p_{j-1}^{(i)}-\frac{\alpha_n}{2}f_{j-1/2}^{n-1} \qquad (4.12)$$

$$(s_3)_j=\frac{(P_2)_{j-1/2}^n+\alpha_n}{2}v_j^{(i)}+\frac{\alpha_n}{2}v_{j-1/2}^{n-1} \qquad (4.13)$$

$$(s_4)_j=\frac{(P_2)_{j-1/2}^n+\alpha_n}{2}v_{j-1}^{(i)}+\frac{\alpha_n}{2}v_{j-1/2}^{n-1} \qquad (4.14)$$

$$(s_5)_j=-\left\{(P_3)_{j-1/2}^n+\alpha_n\right\}u_j^{(i)}-\frac{(P_5)_{j-1/2}^n}{2} \qquad (4.15)$$

$$(s_6)_j = -\left\{(P_3)^n_{j-1/2} + \alpha_n\right\}u^{(i)}_{j-1} - \frac{(P_5)^n_{j-1/2}}{2} \tag{4.16}$$

$$(s_7)_j = \frac{(P_4)^n_{j-1/2}}{2} \tag{4.17}$$

$$(s_8)_j = \frac{(P_4)^n_{j-1/2}}{2} \tag{4.18}$$

$$(s_9)_j = \frac{(P_6)^n_{j-1/2}}{2} v^{(i)}_j \tag{4.19}$$

$$(s_{10})_j = \frac{(P_6)^n_{j-1/2}}{2} v^{(i)}_{j-1} \tag{4.20}$$

Similarly by using the equations (4.1) to (4.5), then the equation (3.67) can be written as

$$\begin{aligned}
&\frac{1}{\Pr} h_j^{-1} (P_1)^n_{j-1/2}\left(p^{(i)}_j + \delta p^{(i)}_j - p^{(i)}_{j-1} - \delta p^{(i)}_{j-1}\right) + \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}\left\{(fp)^{(i)}_{j-1/2} + \delta(fp)^{(i)}_{j-1/2}\right\} \\
&- \alpha_n\left\{(ug)^{(i)}_{j-1/2} + \delta(ug)^{(i)}_{j-1/2}\right\} + \alpha_n\left\{\left(u^{(i)}_{j-1/2} + \delta u^{(i)}_{j-1/2}\right)g^{n-1}_{j-1/2} - u^{n-1}_{j-1/2}\left(g^{(i)}_{j-1/2} + \delta g^{(i)}_{j-1/2}\right)\right\} \\
&- \alpha_n\left\{\left(p^{(i)}_{j-1/2} + \delta p^{(i)}_{j-1/2}\right)f^{n-1}_{j-1/2} - p^{n-1}_{j-1/2}\left(f^{(i)}_{j-1/2} + \delta f^{(i)}_{j-1/2}\right)\right\} + (P_5)^n_{j-1/2}\left\{g^{(i)}_{j-1/2} + \delta g^{(i)}_{j-1/2}\right\} \\
&= T^{n-1}_{j-1/2}
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\Pr} h_j^{-1} (P_1)^n_{j-1/2}\left(p^{(i)}_j + \delta p^{(i)}_j - p^{(i)}_{j-1} - \delta p^{(i)}_{j-1}\right) \\
&+ \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}\left\{(fp)^{(i)}_{j-1/2} + \frac{1}{2}\left\{\delta(fp)^{(i)}_j + \delta(fp)^{(i)}_{j-1}\right\}\right\} \\
&- \alpha_n\left\{(ug)^{(i)}_{j-1/2} + \frac{1}{2}\left\{\delta(ug)^{(i)}_j + \delta(ug)^{(i)}_{j-1}\right\}\right\} + \alpha_n\left\{u^{(i)}_{j-1/2} + \frac{1}{2}\left(\delta u^{(i)}_j + \delta u^{(i)}_{j-1}\right)\right\}g^{n-1}_{j-1/2} \\
&- \alpha_n\left\{g^{(i)}_{j-1/2} + \frac{1}{2}\left(\delta g^{(i)}_j + \delta g^{(i)}_{j-1}\right)\right\}u^{n-1}_{j-1/2} + (P_5)^n_{j-1/2}\left\{\delta g^{(i)}_j + \delta(g)^{(i)}_{j-1}\right\} \\
&- \alpha_n\left\{p^{(i)}_{j-1/2} + \frac{1}{2}\left(\delta p^{(i)}_j + \delta p^{(i)}_{j-1}\right)\right\}f^{n-1}_{j-1/2} + \alpha_n\left\{f^{(i)}_{j-1/2} + \frac{1}{2}\left(\delta f^{(i)}_j + \delta f^{(i)}_{j-1}\right)\right\}p^{n-1}_{j-1/2} = T^{n-1}_{j-1/2}
\end{aligned}$$

i.e.,

$$(t_1)_j\,\delta p_j^{(i)} + (t_2)_j\,\delta p_{j-1}^{(i)} + (t_3)_j\,\delta f_j^{(i)} + (t_4)_j\,\delta f_{j-1}^{(i)} + (t_5)_j\,\delta u_j^{(i)} + (t_6)_j\,\delta u_{j-1}^{(i)} + (t_7)_j\,\delta g_j^{(i)} + (t_8)_j\,\delta g_{j-1}^{(i)} + (t_9)_j\,\delta v_j^{(i)} + (t_{10})_j\,\delta v_{j-1}^{(i)} = (r_3)_j\ (say) \tag{4.21}$$

where

$$(r_3)_j = T_{j-1/2}^{n-1} - \frac{1}{\Pr} h_j^{-1} (P_1)_{j-1/2}^n \left(p_j^{(i)} - p_{j-1}^{(i)}\right) - \left\{(P_2)_{j-1/2}^n + \alpha_n\right\}(fp)_{j-1/2}^{(i)} + \alpha_n (ug)_{j-1/2}^{(i)} + \alpha_n \left(g_{j-1/2}^{(i)} u_{j-1/2}^{n-1} - g_{j-1/2}^{n-1} u_{j-1/2}^{(i)}\right) + \alpha_n \left(p_{j-1/2}^{(i)} f_{j-1/2}^{n-1} - p_{j-1/2}^{n-1} f_{j-1/2}^{(i)}\right) - (P_5)_{j-1/2}^n g_{j-1/2}^{(i)} \tag{4.22}$$

$$(t_1)_j = \frac{1}{\Pr} h_j^{-1} (P_1)_{j-1/2}^n + \frac{(P_2)_{j-1/2}^n + \alpha_n}{2} f_j^{(i)} - \frac{\alpha_n}{2} f_{j-1/2}^{n-1} \tag{4.23}$$

$$(t_2)_j = -\frac{1}{\Pr} h_j^{-1} (P_1)_{j-1/2}^n + \frac{(P_2)_{j-1/2}^n + \alpha_n}{2} f_{j-1}^{(i)} - \frac{\alpha_n}{2} f_{j-1/2}^{n-1} \tag{4.24}$$

$$(t_3)_j = \frac{(P_2)_{j-1/2}^n + \alpha_n}{2} p_j^{(i)} + \frac{\alpha_n}{2} p_{j-1/2}^{n-1} \tag{4.25}$$

$$(t_4)_j = \frac{(P_2)_{j-1/2}^n + \alpha_n}{2} p_{j-1}^{(i)} + \frac{\alpha_n}{2} p_{j-1/2}^{n-1} \tag{4.26}$$

$$(t_5)_j = -\frac{\alpha_n}{2} g_j^{(i)} + \frac{\alpha_n}{2} g_{j-1/2}^{n-1} \tag{4.27}$$

$$(t_6)_j = -\frac{\alpha_n}{2} g_{j-1}^{(i)} + \frac{\alpha_n}{2} g_{j-1/2}^{n-1} \tag{4.28}$$

$$(t_7)_j = -\frac{\alpha_n}{2}u_j^{(i)} - \frac{\alpha_n}{2}u_{j-1/2}^{n-1} + \frac{(P_5)_{j-1/2}^n}{2} \tag{4.29}$$

$$(t_8)_j = -\frac{\alpha_n}{2}u_{j-1}^{(i)} - \frac{\alpha_n}{2}u_{j-1/2}^{n-1} + \frac{(P_5)_{j-1/2}^n}{2} \tag{4.30}$$

$$(t_9)_j = 0 \tag{4.31}$$

$$(t_{10})_j = 0 \tag{4.32}$$

The boundary conditions (3.68) becomes

$$\begin{aligned} &\delta f_0^n = 0, \quad \delta u_0^n = 0, \ \delta g_0^n = 0 \\ &\delta u_j^n = 0, \quad \delta g_j^n = 0 \end{aligned} \tag{4.33}$$

which just express the requirement for the boundary conditions to remain during the iteration process. Now the system of linear equations (4.6)-(4.9) and (4.21) together with the boundary conditions (4.33) can be written in matrix or vector form, where the coefficient matrix has a block tri-diagonal structure. The whole procedure, namely reduction to first order followed by central difference approximations, Newton's Quasi-linearization method and the block tri-diagonal representation Thomas algorithm, is well known as the Keller-box method.

CHAPTER V

Results and Discussions

The effect of temperature dependent physical properties on MHD natural convection flow along a vertical wavy surface with heat generation has been investigated. Using the aforementioned numerical procedure, the numerical results obtained using the governing equations (3.37) and (3.38) with the boundary conditions (3.39) are displayed through graphs and tables below. There are five parameters of interest in the present problem, namely, the thermal conductivity parameter γ, heat generation parameter Q, magnetic parameter M, the amplitude to length ratio of the wavy surface α of the surface and Prandtl number Pr. The numerical solutions regarding the velocity and temperature distributions are presented for different selected values of the established dimensionless parameters. The influences of these various parameters on the velocity and temperature fields are presented in Figure 5.1 through Figure 5.20 and some of the numerical results regarding coefficients skin friction and heat transfer are given in tabular form in Table 5.1 and Table 5.2.

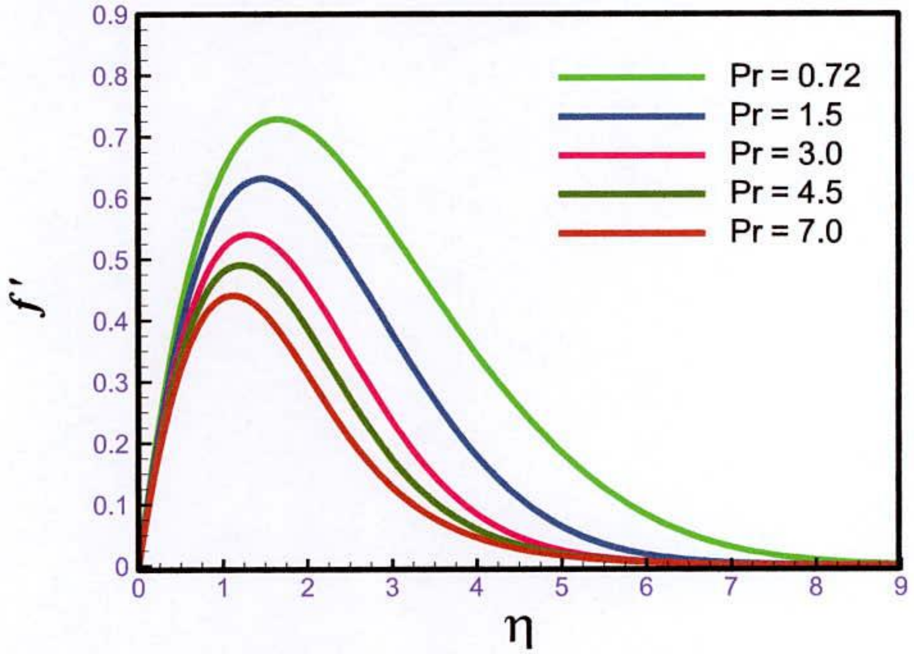

Figure 5.1: Velocity profiles f' against dimensionless distance η for different values of Prandtl number Pr while $\alpha = 0.2$, $M = 0.5$, $\gamma = 3.0$ and $Q = 0.5$.

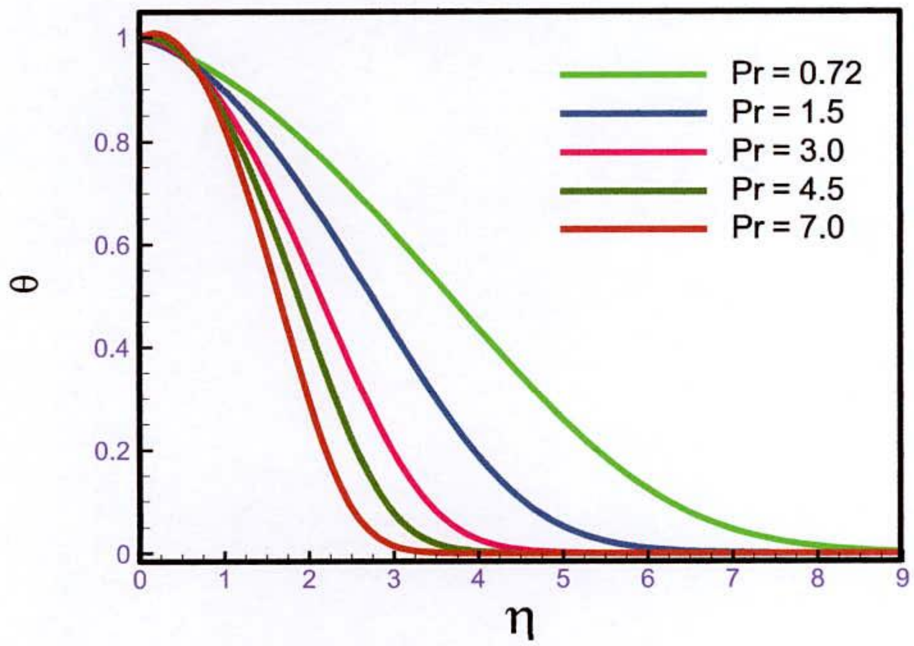

Figure 5.2: Temperature distribution θ against dimensionless distance η for different values of Prandtl number Pr while $\alpha = 0.2$, $M = 0.5$, $\gamma = 3.0$ and $Q = 0.5$.

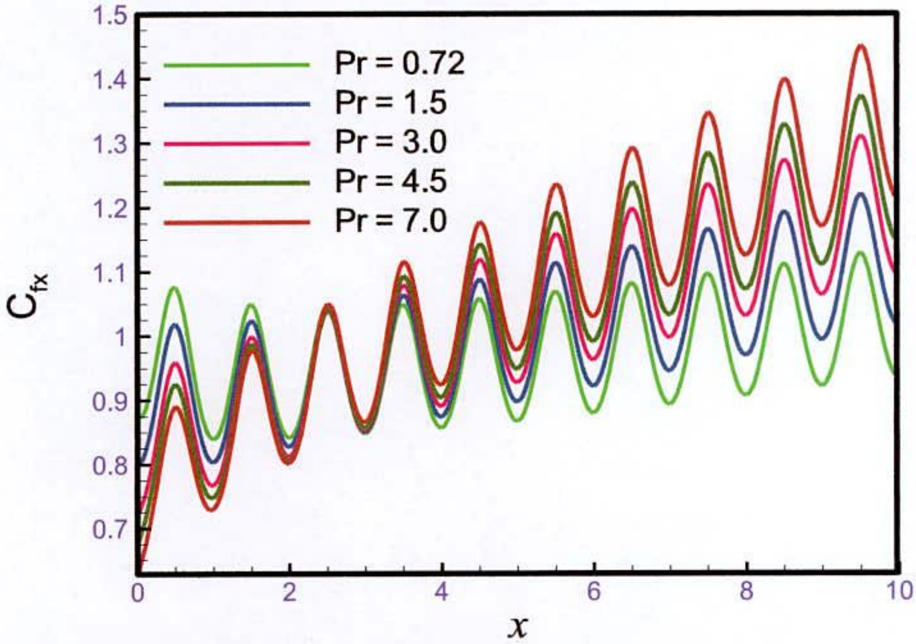

Figure 5.3: Variation of skin fiction coefficient C_{fx} against dimensionless distance x for different values of Prandtl number Pr with $M = 0.5$, $\alpha = 0.2$, $\gamma = 3.0$ and $Q = 0.5$.

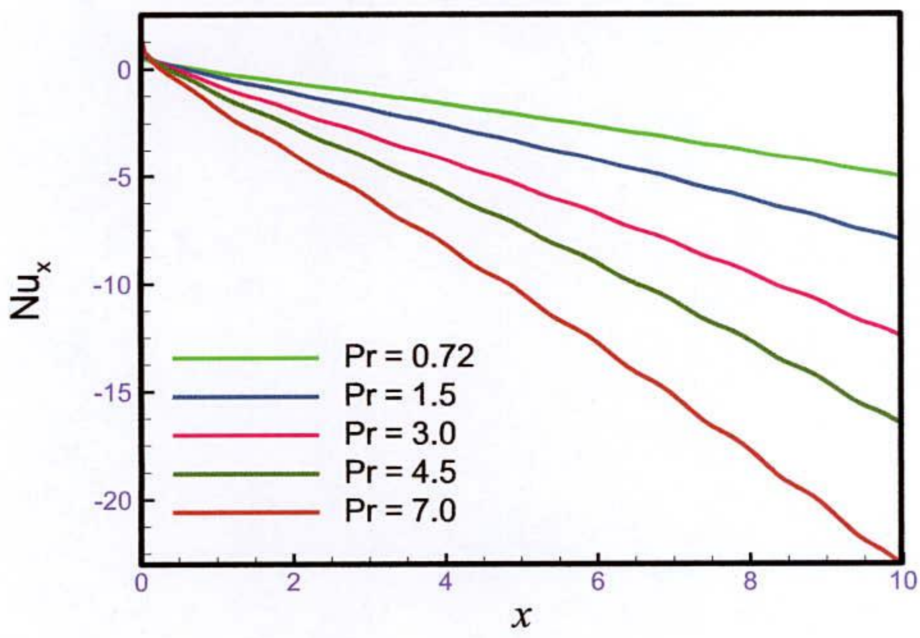

Figure 5.4: Variation of rate of heat transfer Nu_x against dimensionless distance x for different values of Prandtl number Pr with $M = 0.5$, $\alpha = 0.2$, $\gamma=3.0$ and $Q = 0.5$.

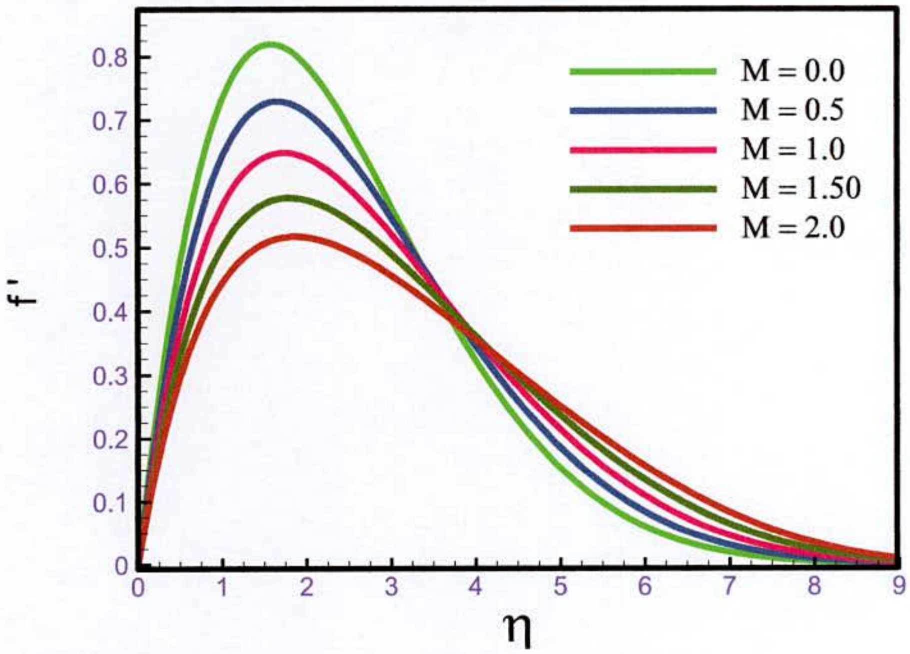

Figure 5.5: Velocity profiles f' against dimensionless distance η for different values of magnetic parameter M with Pr = 0.72, $\alpha = 0.2$, $\gamma=3.0$ and $Q = 0.5$.

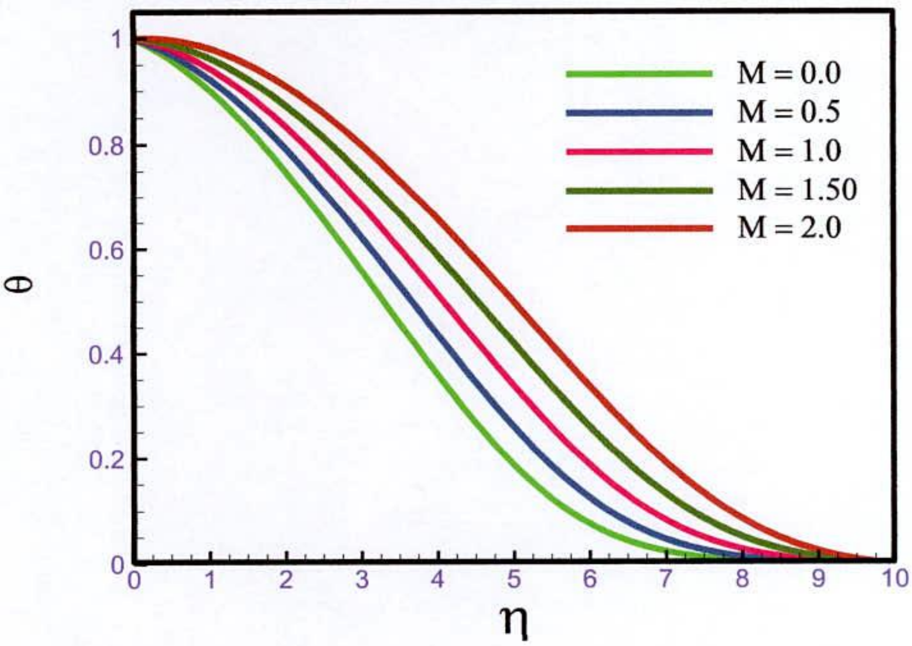

Figure 5.6: Temperature distribution θ against dimensionless distance η for different values of magnetic parameter M with Pr = 0.72, $\alpha = 0.2$, $\gamma= 3.0$ and $Q = 0.5$.

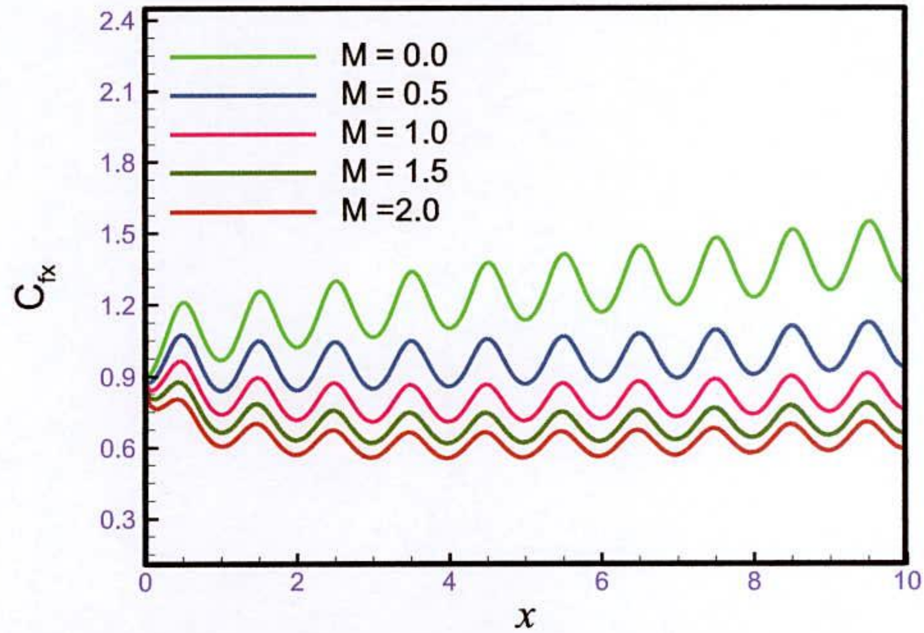

Figure 5.7 Variation of skin fiction coefficient C_{fx} against dimensionless distance x for different values of magnetic parameter M with Pr = 0.72, $\alpha = 0.2$, $\gamma = 3.0$ and $Q = 0.1$.

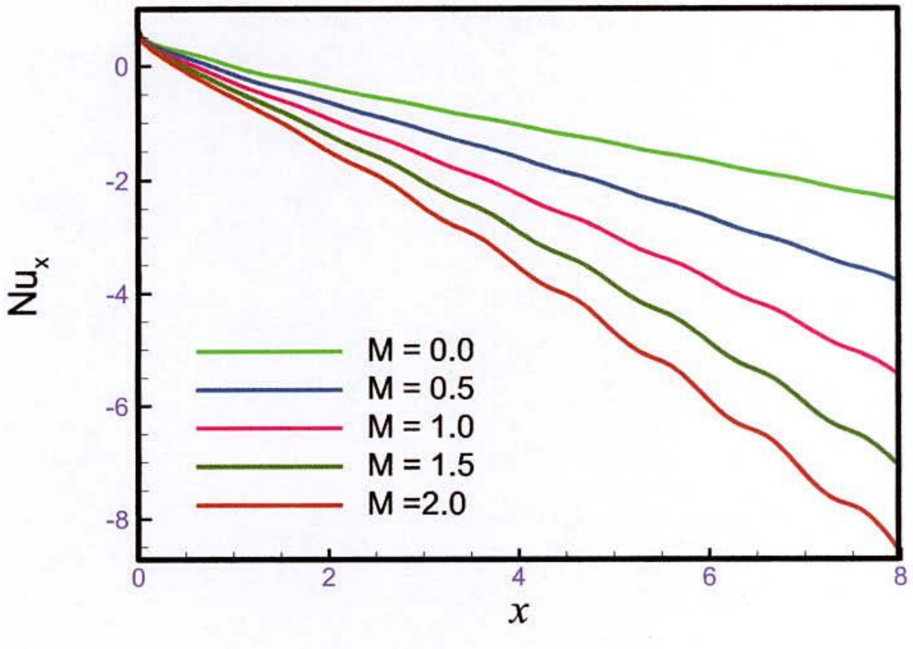

Figure 5.8 Variation of rate of heat transfer Nu_x against dimensionless distance x for different values of magnetic parameter M with Pr = 0.72, $\alpha = 0.2$, $\gamma = 3.0$ and $Q = 0.1$.

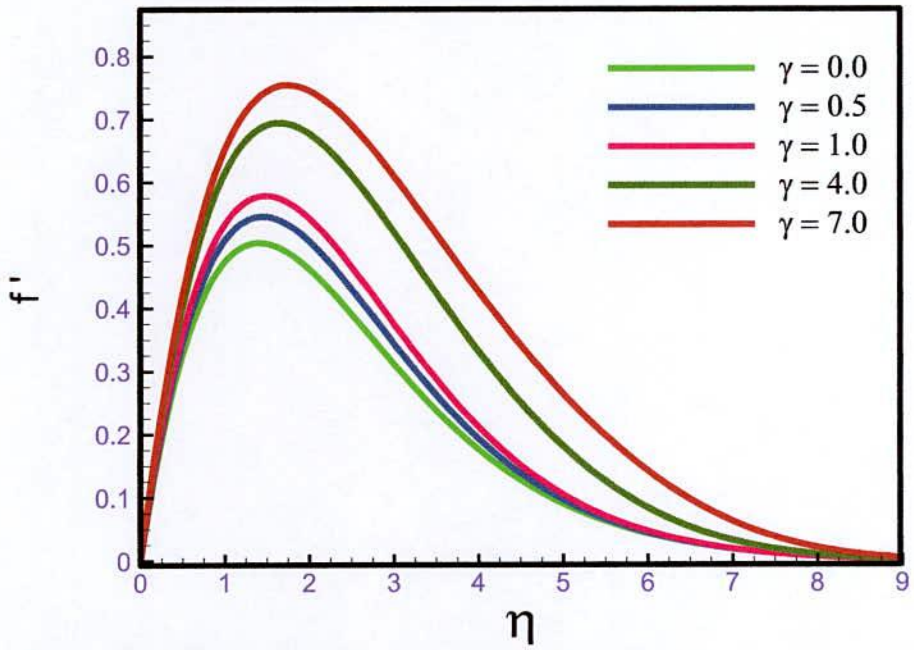

Figure 5.9: Velocity profiles f' against dimensionless distance η for different values of thermal conductivity parameter γ with Pr = 0.72, $\alpha = 0.2$, $M = 0.5$ and $Q = 0.1$.

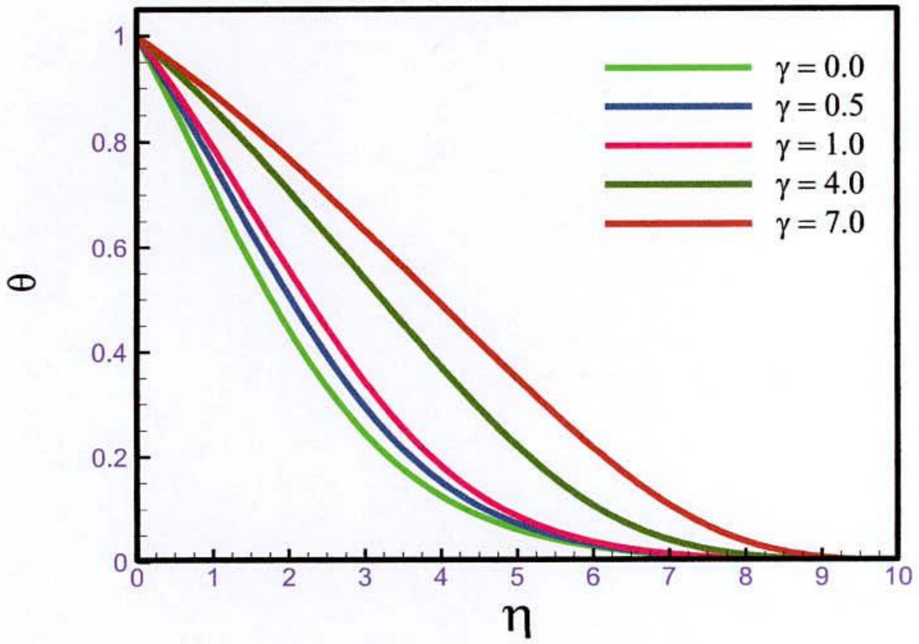

Figure 5.10: Temperature distribution θ against dimensionless distance η for different values of thermal conductivity parameter γ with Pr = 0.72, $\alpha = 0.2$, $M = 0.5$ and $Q = 0.1$.

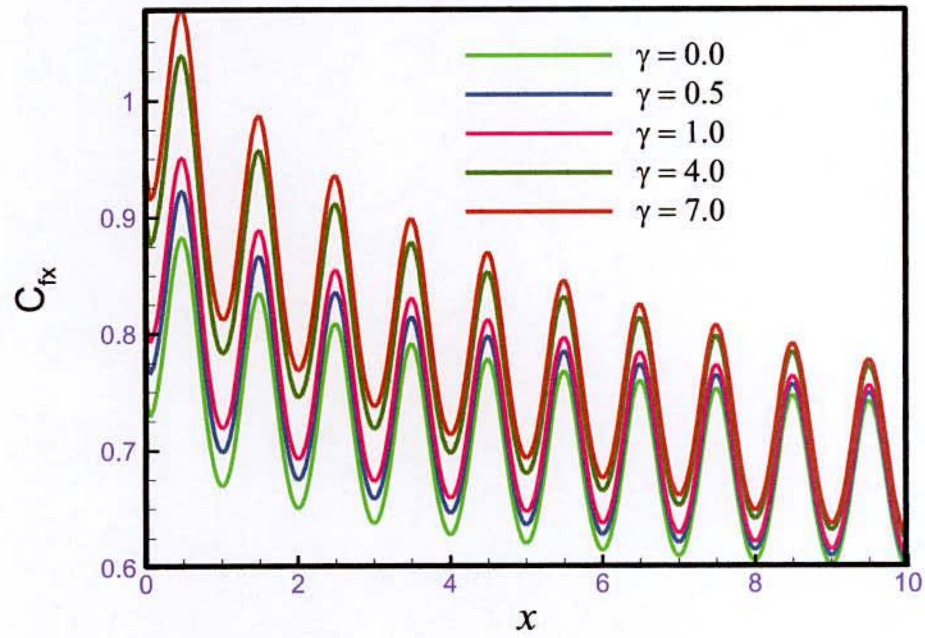

Figure 5.11 Variation of skin fiction coefficient C_{fx} against dimensionless distance x for different values of thermal conductivity parameter γ with $\Pr = 0.72$, $\alpha = 0.2$, $M = 0.5$ and $Q = 0.1$.

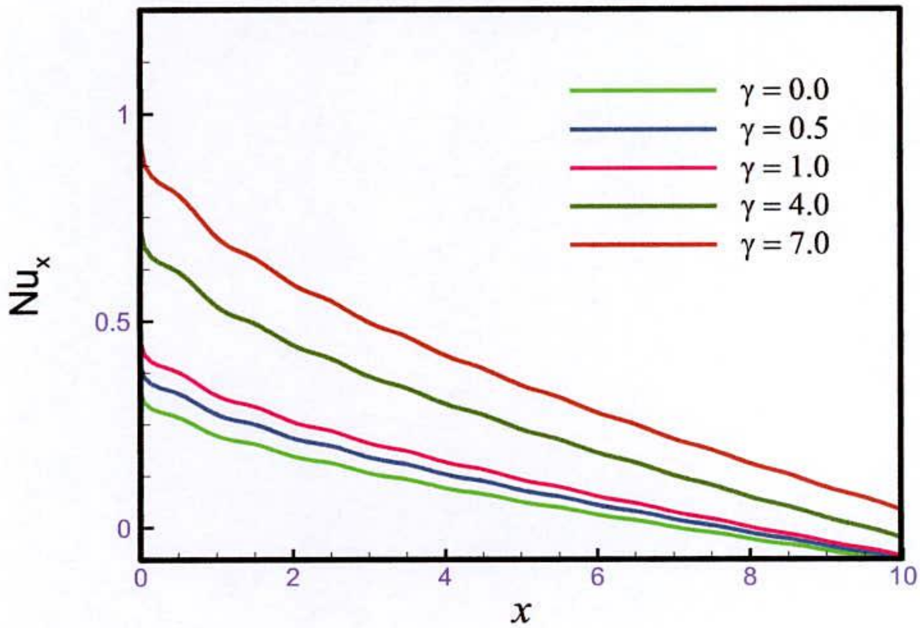

Figure 5.12 Variation of rate of heat transfer Nu_x against dimensionless distance x for different values of thermal conductivity parameter γ with $\Pr = 0.72$, $\alpha = 0.2$, $M = 0.5$ and $Q = 0.1$.

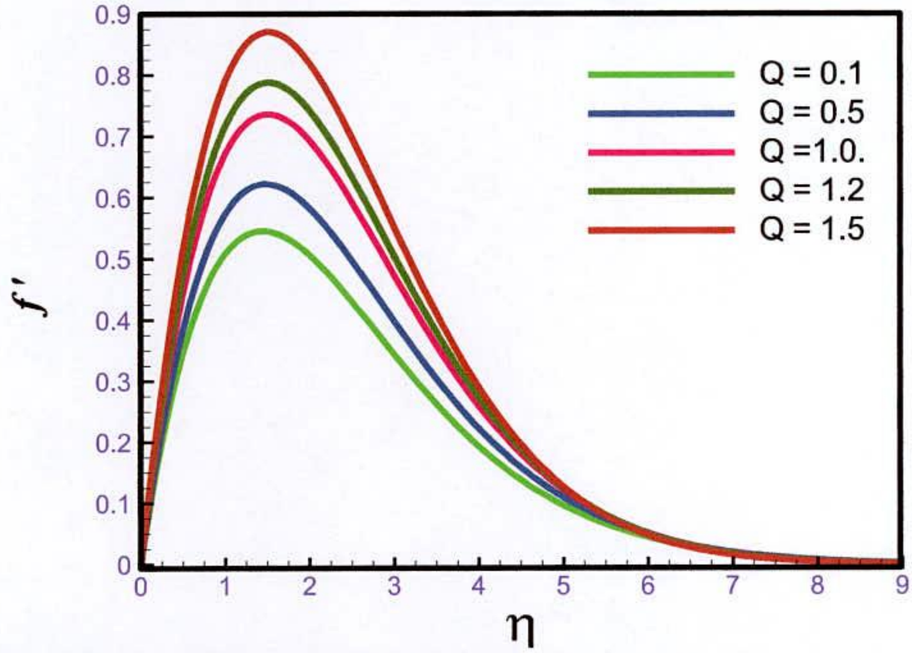

Figure 5.13: Velocity profiles f' against dimensionless distance η for different values of heat generation parameter Q with Pr = 0.72, $\alpha = 0.2$, $M = 0.5$ and $\gamma = 0.5$.

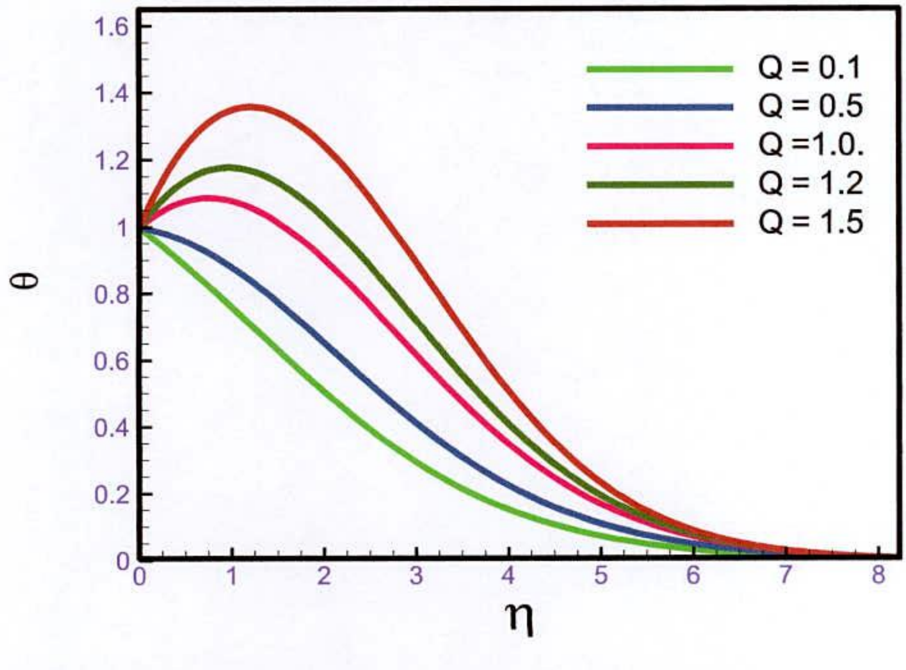

Figure 5.14: Temperature distribution θ against dimensionless distance η for different values of heat generation parameter Q with $Pr = 0.72$, $\alpha = 0.2$, $M = 0.5$ and $\gamma = 0.5$.

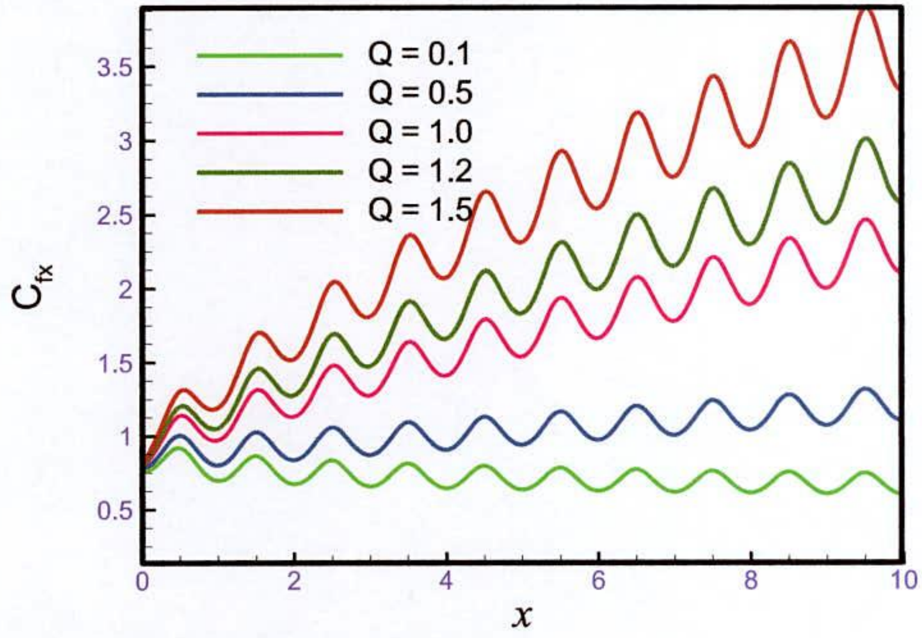

Figure 5.15: Variation of skin fiction coefficient C_{fx} against dimensionless distance x for different values of heat generation parameter Q with $M = 0.5$, Pr = 0.72, $\gamma = 0.0$ and $\alpha = 0.2$.

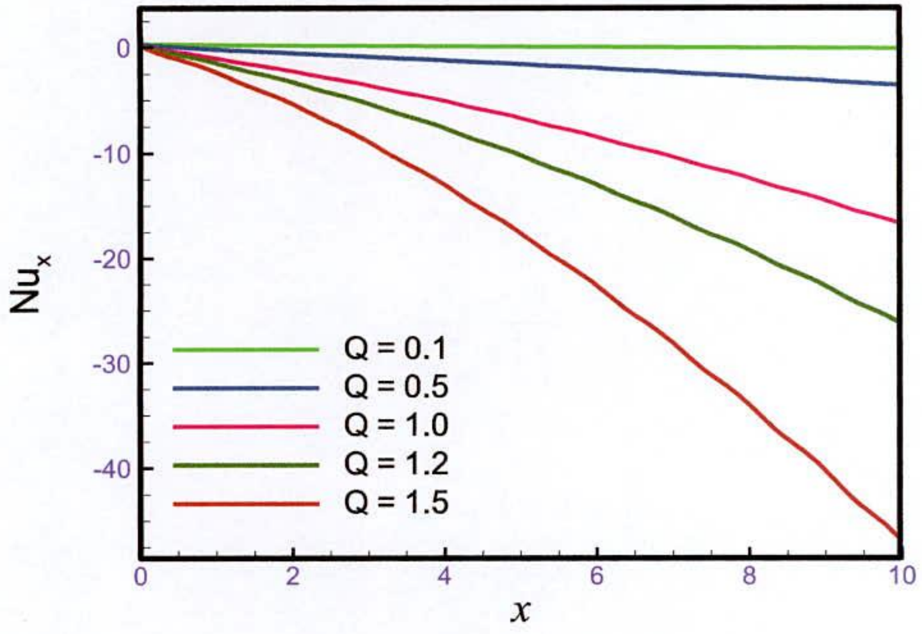

Figure 5.16: variation of rate of heat transfer Nu_x against dimensionless distance x for different values of heat generation parameter Q with $M = 0.5$, $Pr = 0.72$, $\gamma = 0.0$ and $\alpha = 0.2$.

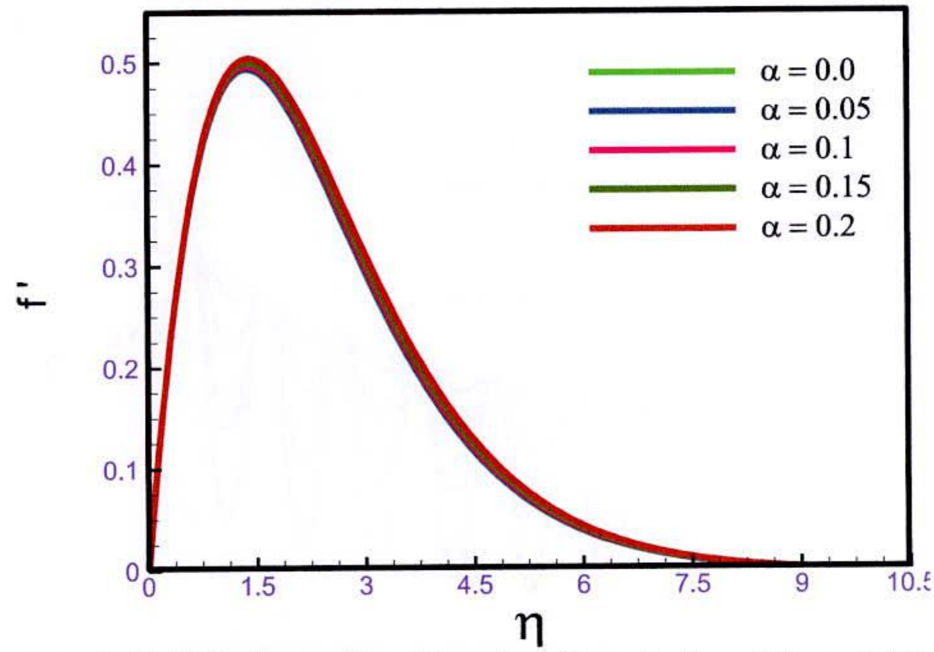

Figure 5.17: Velocity profiles f' against dimensionless distance η for different values of amplitude to the length ratio of wavy surface α with $\Pr = 0.72$, $Q = 0.5$, $M = 0.5$ and $\gamma = 0.0$.

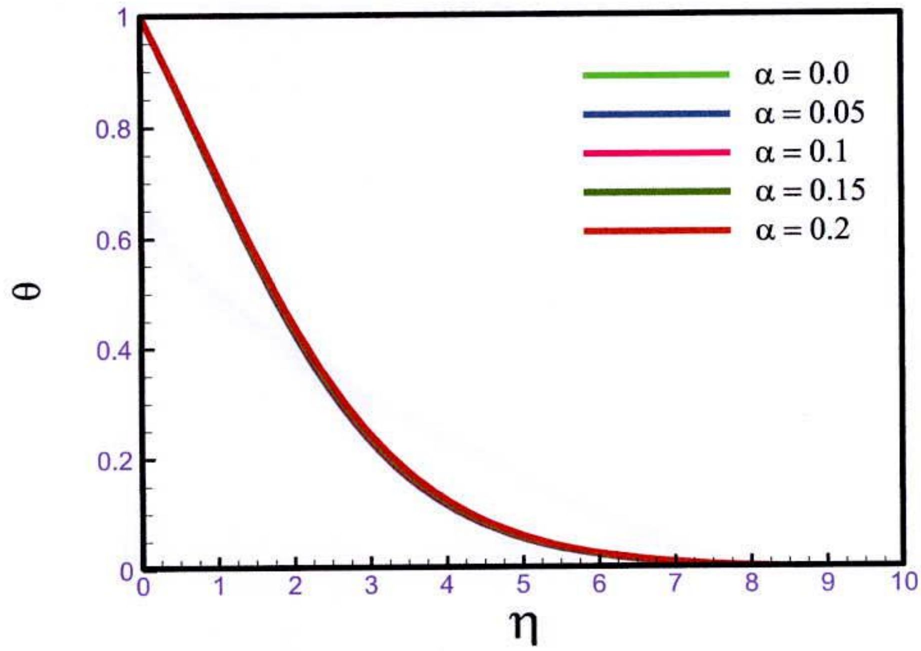

Figure 5.18: Temperature distribution θ against dimensionless distance η for different values of amplitude to the length ratio of wavy surface α with $\Pr = 0.72$, $Q = 0.5$, $M = 0.5$ and $\gamma = 0.0$

Numerical results for velocity and temperature fields and the shearing stress in terms of skin friction coefficients C_{fx} and the rate of heat transfer in terms of the Nusselt number Nu_x are obtained for different values Prandtl number Pr = 0.72, 1.5, 3.0, 4.5 and 7.0 (Pr = 0.72 which corresponds to the air at 20^0C, Pr = 3.0 and 7.0 which correspond to water at 60^0C and 20^0C respectively), the magnetic parameter $M = 0.0$ (non magnetic field), 0.5, 1.0, 1.5 and 2.0, the thermal conductivity parameter $\gamma = 0.0$, 0.5, 1.0, 4.0 and 7.0, the heat generation parameter $Q = 0.1$, 0.5, 1.0, 1.2 and 1.5 and the amplitude to length ratio of the wavy surface ranging from $\alpha = 0.0$ (flat plate) to 0.2 are displayed in Figure 5.1 to Figure 5.20.

Figure 5.1 and Figure 5.2 deal with the effect of Prandtl number (Pr = 0.72, 1.5, 3.0, 4.5, 7.0) with other fixed controlling parameters, such as, magnetic parameter $M = 0.5$, thermal conductivity parameter $\gamma = 3.0$, heat generation parameter $Q = 0.5$ and amplitude to length ratio of the wavy surface $\alpha = 0.2$, on the velocity $f'(x,\eta)$ and the temperature $\theta(x,\eta)$ fields respectively. Prandtl number is the ratio of viscous force and thermal force. Increasing values of Pr, increases viscosity and decreases thermal action of the fluid. If viscosity increases, then fluid does not move freely. Because of this fact, it is observed from Figure 5.1 that the velocity of the fluid decreases quickly against η for increasing values of Prandtl number. The maximum values of velocity are 0.72978 (for Pr = 0.72) and 0.44170 (for Pr = 7.0) and maximum values of temperature are 1.00000 (for Pr = 0.72) and 1.00928 (for Pr = 7.0). Here velocity decreases by 39.47% when Pr increases from 0.72 to 7.0. From figure 5.2, we see that very close to the surface, temperature increases with increasing Pr but away from it temperature decreases quickly with the increase in Pr. This is due to the formation of the thermal boundary layer close to the surface. In Figures 5.3 and 5.4, the surface shear stress in terms of the local skin friction coefficient C_{fx} and the rate of heat transfer in terms of Nusselt number Nu_x are depicted graphically for different values of Prandtl number (Pr = 0.72, 1.5, 3.0, 4.5 and 7.0) with other fixed controlling parameters $M = 0.5$, $\gamma = 3.0$, $Q = 0.5$ and $\alpha = 0.2$. From Figure 5.3, it is observed that as the Prandtl number increases the skin friction coefficient decreases monotonically up to the position $x = 2.50$ and then crosses the side and increases with increasing values of the Prandtl number Pr. From Figure 5.4, a reverse situation is observed for the rate of heat transfer. It is evident that increasing values of Prandtl number Pr, speed up the decay of the temperature field away from the heated surface with a consequent increase in the rate of heat transfer. The maximum values of local skin friction coefficient are 1.12901, 1.45073 and the rate of heat transfer in terms of the local Nusselt number are 0.64631and and 1.44945 for Pr = 0.72 and 7.0 respectively. It is seen that the local skin friction coefficient increases by 28.49% and the rate of heat transfer decreases as Pr increases from 0.72 to 7.0. Figure 5.5 shows the effect of magnetic parameter M on the velocity field. It is observed from the figure that magnetic field acting along the direction normal to the surface and for the fixed controlling parameters Pr = 0.72, $\gamma = 3.0$, $Q = 0.5$ and $\alpha = 0.2$, the fluid velocity decreases with the increase of the magnetic parameter. Here it also observed that the position of peak velocity moves toward the interface with increasing M and crosses the side at about $\eta = 3.50$. As electrically conducting fluid in presence of magnetic field generates electrical current, the magnetic field is changed and the fluid motion is moderated. The interaction of the magnetic field and moving electric charge carried by the flowing fluid induces a force, which tends to oppose the fluid motion. Figure 5.6 observes the effect of magnetic parameter M on the temperature field. It is observed that at the surface the temperature is maximum, it decreases away from the surface and finally leads to zero asymptotically against η. From figure it is seen that the temperature increases significantly as M increases. It is evident that the induced magnetic fields decrease the temperature gradient at the wall and increase the temperature in the flow region due to the interaction. The influence of magnetic field, on the local skin friction coefficient and local rate of heat transfer are illustrated in Figure 5.7 and Figure 5.8 respectively for fixed values of other controlling parameters Pr = 0.72, $\gamma = 3.0$, $Q = 0.5$ and $\alpha = 0.2$. Figure 5.7 shows the effect of magnetic parameter on the local rate of skin friction coefficient for fixed values of controlling parameters. Since electrically conducting fluid in presence of magnetic field generates electrical current, as a result, the velocity gradient $f''(x, 0)$ increases with the effect of magnetic field. It is observed that the local rate of skin friction coefficient decreases at different position of x due to increasing values of the magnetic parameter M. The maximum values of local skin friction coefficient C_{fx} is 1.54970 and 0.64631 for $M = 0.0$ and 2.0. It is shown that the skin friction coefficient decreases by 58.29% when M increases from 0.0 to 2.0. On the other hand, Figure 5.8 shows the effect of magnetic parameter on the local rate of heat transfer for fixed values of controlling parameters Pr = 0.72, $\gamma = 3.0$, $Q = 0.5$ and $\alpha = 0.2$. As electrically conducting fluid in presence of magnetic field generates electrical current, as a result, the rate heat transfer in terms of the local Nusselt number Nu_x decreases due to increasing values of the magnetic parameter M at different position of x. However, the local rate of heat transfer remain unchanged in its initial position as M increases from 0.0 to 2.0. Therefore, the magnetic field acts against the direction of fluid flow and reduce both the skin friction and the rate of heat transfer. Figure 5.9 and Figure 5.10 deal with the effect of thermal conductivity parameter (γ = 0.0, 0.5, 1.0, 4.0, 7.0) with other fixed controlling parameters $\Pr = 0.72$, $M = 0.5$, $Q = 0.1$ and $\alpha = 0.2$, on the velocity $f'(x,\eta)$ and the temperature $\theta(x,\eta)$ against η, respectively. It is observed from Figure 5.9 that velocity increases with the increasing values of thermal conductivity parameter γ and in Figure 5.10 it is noted that the temperature increases significantly in the upstream with increasing γ and converse to zero asymptotically for large η.

The effect of temperature dependent thermal conductivity (γ = 0.0, 0.5, 1.0, 4.0, 7.0) on the surface shear stress in terms of the local skin friction coefficient C_{fx} and the rate of heat transfer in terms of the local Nusselt number Nu_x are depicted graphically in Figures 5.11 and 5.12 respectively, against the axial distance of x fixing all other parameters $\Pr = 0.72$, $M = 0.5$, $Q = 0.5$ and $\alpha = 0.2$. Figure 5.11 indicates that increasing values of the thermal conductivity sharply increased the surface shear stress in terms of the frictional force in the direction of x and a increase in the values of γ, the rate of heat transfer along the wavy surface is observed increasing as is seen from Figure 5.12. Moreover, the maximum values of local skin friction coefficient C_{fx} are 0.88315, 0.92273, 0.95112, 1.03825, 1.07745 for γ = 0.0, 0.5, 1.0, 4.0, and 7.0 respectively which occurs at $x = 0.5$ and it is seen that the local skin friction coefficient C_{fx} increases by 22.00% as the value of thermal conductivity parameter grows up from 0.0 to 7.0. Furthermore, maximum values of local the rate of heat transfer are 0.32836 for $\gamma = 0.0$ and 0.93787 for $\gamma = 7.0$, respectively each of which occurs at different position of x. The rate of heat transfer increases by 185.62% as the value of thermal conductivity parameter enhances from 0.0 to 7.0. Here it is concluded that the higher the value of thermal conductivity parameter, the skin friction coefficient and the corresponding rate of heat transfer are also higher.

Figure 5.13 and Figure 5.14 deal with the effect of heat generation parameter (Q = 0.1, 0.5, 1.0, 1.2, 1.5.), with other fixed controlling parameters $\Pr = 0.72$, $M = 0.5$, $\gamma = 0.5$ and $\alpha = 0.2$, on the velocity $f'(x,\eta)$ and the temperature $\theta(x,\eta)$ against η. It is observed from

Figure 5.13 that with the increasing values of heat generation parameter Q, velocity increases and in Figure 5.14 the same situation it noted, that is, increasing values of heat generation parameter Q also increased the temperature.

Figures 5.15 and 5.16 show respectively the effect of heat generation parameter ($Q = 0.1$, 0.5, 1.0, 1.2, 1.5.) on the local skin friction coefficient C_{fx} and the local heat transfer rate in the direction of x keeping all other parameters fixed (Pr = 0.72, $M = 0.5$, $\gamma = 0.5$ and $\alpha = 0.2$). Figure 5.15 and Figure 5.16 indicate that with the increasing values of the heat generation parameter Q, the skin friction in terms of C_{fx} increases but the rate of heat transfer in terms of Nu_x decreases very slowly. However, the maximum values of the skin friction coefficient and the heat transfer rate are 0.92273 and 0.39386 for $Q = 0.1$ which occurs at the surface and 3.89676, 0.39388 for $Q = 1.5$ which occurs at the axial position of $x = 0.50$. It is seen that the skin friction coefficient increases up to 322.30% and the heat transfer rate is unchanged in its initial position as Q increases from 0.1 to 1.5.

Figure 5.17 demonstrates the velocity variation for variation of the amplitude to length ratio of the wavy surface ($\alpha = 0.0, 0.05, 0.1, 0.15, 0.2$) with Pr = 0.72, $M = 0.5$, $\gamma = 0.0$ and $Q = 0.5$ and the corresponding temperature $\theta(x,\eta)$ distribution is shown in Figure 5.18. From Figure 5.17, it is revealed that the velocity $f'(x,\eta)$ increases slowly against η with the increase of α. From Figure 5.18 we see that no significant effect on the temperature $\theta(x,\eta)$ for increasing values of the amplitude to length ratio of the wavy surface. Figures 5.19 and 5.20 show that the increase in the value of the amplitude to length ratio of wavy surface ($\alpha = 0.0, 0.05, 0.1, 0.15, 0.2$) leads to decrease the value of the skin friction coefficient C_{fx} and the rate of heat transfer in terms Nu_x while Pr = 0.72, $M = 0.5$, $\gamma = 0.0$ and $Q = 0.5$. Frictional force depends on the smoothness of the surface, temperature and nature of fluid. Surface becomes more roughened for increasing values of amplitude to length ratio of wavy surface. and velocity decreases at the local points. However, the maximum values of the skin friction coefficient and the heat transfer rate are 0.95584 and 0.35680 for $\alpha = 0.0$ which occurs at the surface and 0.88351, 0.32836 for $\alpha = 0.2$ which occurs at the axial position of $x = 0.50$. It is seen that the skin friction coefficient and the heat transfer rate decrease by 7.60% and 7.97% respectively as α increases from 0.0 to 0.2.

Numerical values proportional to the coefficients of skin friction C_{fx} and the rate of heat transfer in terms of the Nusselt number Nu_x, are obtained for different values of the Prandtl number Pr (= 0.72, 1.5, 3.0, 4.5 and 7.0) with fixed values of other controlling parameters ($M = 0.5$, $\gamma = 3.0$, $Q = 0.5$ and $\alpha = 0.2$) are given in Table 5.1 and 5.2, respectively. It is observed that with the increase in Pr, the coefficient of skin friction decreases as well as the rate of heat transfer decreases except at $x = 0$.

Table 5.1: Comparison of the values proportional to the coefficient of skin friction C_{fx} against x for the variation of Prandtl's number Pr with other fixed controlling values $M = 0.5$, $\gamma = 3.0$, $Q = 0.5$ and $\alpha = 0.2$.

x	**Pr = 0.72**	**Pr =1.5**	**Pr=3.0**	**Pr=4.5**	**Pr = 7.0**
0.00	0.87961	0.80045	0.72108	0.67452	0.62447
0.50	1.07772	1.01989	0.96042	0.92636	0.89151
1.00	0.83974	0.80387	0.76800	0.74869	0.73059
1.50	1.05084	1.02512	1.00010	0.98816	0.97937
2.00	0.84175	0.82765	0.81266	0.80592	0.80228
2.50	1.04651	1.04260	1.03961	1.04202	1.05070
3.00	0.84784	0.85067	0.85296	0.85709	0.86595
3.50	1.05040	1.06471	1.07989	1.09372	1.11596
4.00	0.85682	0.87411	0.89093	0.90432	0.92406
4.50	1.05874	1.08894	1.11996	1.14374	1.17762
5.00	0.86774	0.89785	0.92743	0.94889	0.97816
5.50	1.06989	1.11436	1.15952	1.19221	1.23647
6.00	0.88003	0.92175	0.96279	0.99148	1.02925
6.50	1.08297	1.14048	1.19846	1.23923	1.29292
7.00	0.89331	0.94570	0.99719	1.03247	1.07795
7.50	1.09741	1.16700	1.23675	1.28494	1.34729
8.00	0.90730	0.96960	1.03074	1.07210	1.12467
8.50	1.11285	1.19371	1.27440	1.32946	1.39983
9.00	092181	0.99340	1.06352	1.11053	1.16969
9.50	1.12901	1.22050	1.31140	1.37289	1.45073
10.0	0.93669	1.01705	1.09558	1.14789	1.21322

Table 5.2: Comparison of the values proportional to the rate of heat transfer in terms of Nusselt number Nu_x against x for the variation of Prandtl number Pr with other fixed controlling values $M = 0.5$, $\gamma = 3.0$, Q = 0.5 and $\alpha = 0.2$.

x	**Pr = 0.72**	**Pr =1.5**	**Pr = 3.0**	**Pr = 4.5**	**Pr = 7.0**
0.00	0.64613	0.85781	1.09851	1.25825	1.44945
0.50	0.15918	0.11968	-0.02458	-0.20232	-0.52440
1.00	-0.13729	-0.33399	-0.72459	-1.12278	-1.78679
1.50	-0.39220	-0.72103	-1.31364	-1.89509	-2.85176
2.00	-0.63028	-1.08991	-1.88430	-2.64817	-3.89442
2.50	-0.87466	-1.46881	-2.47523	-3.43507	-4.99720
3.00	-1.11085	-1.83261	-3.03048	-4.16099	-5.99285
3.50	-1.35945	-2.22358	-3.64447	-4.98339	-7.15541
4.00	-1.60137	-2.59610	-4.20903	-5.71837	-8.15692
4.50	-1.86123	-3.00689	-4.85481	-6.58312	-9.38065
5.00	-2.11145	-3.39199	-5.43651	-7.33833	-10.40513
5.50	-2.38555	-3.82659	-6.11822	-8.24963	-11.69265
6.00	-2.64555	-4.22679	-6.72049	-9.02917	-12.74725
6.50	-2.93512	-4.68660	-7.44011	-9.98947	-14.10083
7.00	-3.20511	-5.10337	-8.06515	-10.79574	-15.18740
7.50	-3.51108	-5.58867	-8.82269	-11.80466	-16.60571
8.00	-3.79222	-6.02286	-9.47137	-12.63927	-17.72564
8.50	-4.11414	-6.53338	-10.26602	-13.69495	-19.20506
9.00	-4.40758	-6.98549	-10.93911	-14.55832	-20.35892
9.50	-4.74457	-7.52137	-11.77129	-15.66192	-21.90264
10.0	-5.04957	-7.99156	-12.46893	-16.55427	-23.09071

CHAPTER VI

Conclusions and Recommendations

The effect of temperature dependent physical properties on MHD natural convection flow of viscous incompressible fluid along a uniformly heated vertical wavy surface with heat generation has been investigated in this study. The numerical procedure based on the implicit finite difference method, known as Keller-Box scheme is adopted to solve the nonlinear system of partial differential equations. The numerical results of velocity and temperature fields, the surface shear stress in terms of skin friction coefficient and the rate of heat transfer in terms of local Nusselt number have been presented graphically for some selected values of appeared parameters consisting of thermal conductivity variation parameter γ □, heat generation parameter Q, magnetic parameter M, amplitude to the length ratio of wavy surface α and Prandtl number Pr. Some numerical results of the skin friction coefficient and the rate of heat transfer also have been presented in tabular forms.

On the basis of the figures, it is concluded that:

- the velocity of the fluid decreases quickly against η for increasing values of Prandtl number.
- temperature increases with increasing Pr very close to the surface, but away from it temperature decreases quickly with the increase in Pr.
- the skin friction coefficient decreases monotonically with the increases of Prandtl number within $x = 2.50$ and then crosses the side and increases with increasing values of the Prandtl number Pr.
- a reverse situation is observed for the rate of heat transfer.
- the fluid velocity decreases with the increase of the magnetic parameter M.
- the temperature increases significantly as M increases.
- the local rate of skin friction coefficient decreases at different position of x due to increasing values of the magnetic parameter M.

- the rate heat transfer in terms of the local Nusselt number Nu_x decreases due to increasing values of the magnetic parameter M at different position of x.
- velocity increases with the increasing values of thermal conductivity parameter γ.
- the temperature increases significantly in the upstream with increasing γ and converse to zero asymptotically for large η.
- increasing values of the thermal conductivity sharply increased the surface shear stress in terms of the frictional force in the direction of x.
- with a increase in the values of γ, the rate of heat transfer along the wavy surface is observed increasing.
- velocity increases with the increasing values of heat generation parameter Q.
- increasing values of heat generation parameter Q increased the temperature of the fluid.
- with the increasing values of the heat generation parameter Q, the skin friction in terms of C_{fx} increases but the rate of heat transfer in terms of Nu_x decreases very slowly.
- the velocity increases slowly against η with the increase of the amplitude to length ratio of the wavy surface α.
- there is no significant effect of α on the temperature field.
- the increase in the value of α leads to decrease the value of the skin friction coefficient C_{fx} and the rate of heat transfer in terms Nu_x.

On the basis of the tables, it is concluded that:

- with the increase in Pr, the coefficient of skin friction decreases.
- the rate of heat transfer decreases except at $x = 0$.

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