On Some Analytical Approximate Solutions of Fourth Order Damped Oscillatory and Near Critically Damped Nonlinear Systems

by

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Dedicated to My

Beloved Parents

And

Affectionate Son

Declaration

This is to certify that the thesis work entitled "On Some Analytical Approximate Solutions of Fourth Order Damped Oscillatory and Near Critically Damped Nonlinear Systems" has been carried out by Md. Habibur Rahman in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above research work or any part of the work has not been submitted anywhere for the award of any degree or diploma.

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The author

Abstract

At first Krylov and Bogoliubov presented a perturbation method known as "the asymptotic averaging method" in the theory of nonlinear oscillations. Primarily, the method was proposed only to get the periodic solutions of second order autonomous systems with small nonlinearities. Later, the method has been extended by Bogoliubov and Mitropolskii. At present the method is used to obtain the solutions of second and higher order nonlinear equations for damped oscillatory, over damped, near critically damped, critically damped, more critically damped systems under some special conditions. The unified Krylov-Bogoliubov-Mitropolskii (KBM) method is used to find approximate solutions of fourth order nonlinear systems with large damping. In this thesis, the KBM method has been modified and elaborated to find out the solutions of fourth order damped oscillatory and near critically damped non-oscillatory nonlinear systems by imposing some restrictions on the eigen-values. For verification of the results obtained by the modified KBM method, we have compared them with those obtained by the fourth order Runge-Kutta method and a nice matching is observed.

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CHAPTER 1

INTRODUCTION

Differential equation is a mathematical tool, which has its applications in many branches of knowledge of mankind. Numerous physical, mechanical, chemical, biological, biochemical, and many other relations appear mathematically in the form of differential equations that are linear or nonlinear, autonomous or non-autonomous. Generally, in many physical phenomena, such as spring-mass systems, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, pendulums, the motion of the rotating mass around another body, etc. the differential equations occur. Also in ecology and economics the differential equations are vastly used. Basically, many differential equations involving physical phenomena are nonlinear. Differential equations, which are linear are comparatively easy to solve and nonlinear are laborious and in some cases it is impossible to solve them analytically. In such situations, mathematicians convert the nonlinear equations into linear equations by imposing some conditions. The method of small oscillations is a well-known example of the linearization. But, such a linearization is not always possible and when it is not, then the original nonlinear equation itself must be used. With the discovery of numerous phenomena of self-excitation of circuits containing nonlinear equations of electricity, like, electron tubes, gaseous discharge, etc. and in many cases of nonlinear mechanical vibrations of special types, the method of small oscillations becomes inadequate for their analytical treatment. The knowledge of the nonlinear equations is generally confined to a variety of rather special cases. There exists an important difference between the phenomena, which oscillate in steady state and the phenomena governed by linear differential equations with constant coefficients. For example, oscillations of a pendulum with small amplitudes, in that the amplitude of the ultimate stable oscillations seems to be entirely independent of the initial conditions, where as in oscillations governed by a linear differential equation with constant coefficients, it depends upon the initial conditions. Originally the Krylov-Bogoliubov-Mitropolskii (KBM) method was developed for the systems only to obtain the periodic solutions of second order nonlinear differential equations. Now the method is used to obtain oscillatory as well as damped, critically damped, over damped, near critically damped, more critically damped oscillatory and non-oscillatory solutions of second, third, fourth etc. order nonlinear differential equations by imposing some specific conditions to make the solutions uniform. To solve nonlinear differential equations there exist some methods. Among the methods, the method of perturbations, *i.e.*, asymptotic expansions in terms of a small parameter, are foremost. Perturbation methods have recently received much attention as methods for accurately and quickly computing numerical solutions of dynamic stochastic economic equilibrium models, both single-agent or rational-expectations models and multi-agent or game-theoretic models. A perturbation method is based on the following aspects: The equations to be solved are sufficiently "smooth" or sufficiently differentiable a number of times in the required regions of variables and parameters. At first Van der pol paid attention to the new oscillations i.e., self-excitations and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing the differential equation in the sense of the method of small oscillations, one simply eliminates the possibility of investigating such problems. Thus it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. The method of Krylov and Bogoliubov is an asymptotic method in the sense that $\varepsilon \to 0$. An asymptotic series itself may not be convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as ε tends to zero. It may be noted that the term asymptotic is frequently used in the theory of oscillations in the sense that $\varepsilon \to \infty$. But in this case the mathematical method is quite different. It is an important approach to the study of such nonlinear oscillations is the small parameter expansion. Two widely spreaded methods in this theory are mainly used in the literature; one is averaging asymptotic method of KBM and the other is multi-time scale method. Among the methods used to study nonlinear systems with a small nonlinearity, the KBM method is particularly convenient and is the extensively used technique to obtain the approximate solutions. The method of KBM starts with the solution of linear equation (sometimes called the generating solution of the linear equation), assuming that in the nonlinear case, the amplitude and phase in the solution of the linear differential equation are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the solution of a second order equation. It is customary in the KBM method that correction terms (i.e., the terms with small parameter) in the solutions do not contain secular terms. These assumptions are mainly valid for second and third order equations. But for the fourth order differential equation the correction terms sometimes contain secular terms, although the solution is generated by the classical KBM asymptotic method. Consequently, the traditional solutions fail to explain the proper situation of the systems. To remove the presence of secular terms and obtain the desired results, we need to impose some special conditions. The main target of this thesis is to find out these limitations and determine the proper solutions under some special conditions. The method has its use mainly in engineering and technology, notably in mechanics, electrical circuit theory and also used in population dynamics, chemistry, control theory, plasma physics, etc.. It may be noted that most of the representers have tried to find the solutions of second and third order nonlinear systems. Although some investigators have obtained the solutions of fourth order nonlinear differential equations, which have not been studied extensively.

In this thesis, we have chosen a fourth order nonlinear autonomous differential equations, that describes damped oscillatory and near critically damped non-oscillatory systems with small nonlinearities, to solve by the modified KBM method and the quality of the solution is being tested.

We are going to propose a perturbation technique to solve a fourth order nonlinear differential equation of the form

$$\frac{d^4x}{dt^4}+c_1\frac{d^3x}{dt^3}+c_2\frac{d^2x}{dt^2}+c_3\frac{dx}{dt}+c_4x=-\varepsilon f(x),$$

where ε is a very small positive parameter ; c_1, c_2, c_3, c_4 are arbitrary constants and f is a given nonlinear function.

In Chapter 2, the review of literature is presented. In 2.1 information regarding damped oscillatory nonlinear system is presented and in 2.2 that regarding near critically damped nonlinear system is presented. In Chapter 3, the methodology is being discussed. In 3.1 the solution procedure of a fourth order damped oscillatory system is discussed where as in 3.2 the solution procedure of a fourth order near critically damped non-oscillatory nonlinear system is discussed. In Chapter 4, the results and discussions is presented. In 4.1 solution of fourth order damped oscillatory nonlinear system with an example and its discussion are presented. In 4.2 the solution of fourth order near critically damped non-oscillatory nonlinear system with an example and its discussion are presented. Finally, in Chapter 5, the conclusion is presented.

CHAPTER 2

LITERATURE REVIEW

2.1 Damped oscillatory nonlinear systems

Nonlinear differential equations show peculiar characters. But mathematical formulations of many physical problems often result in differential equations, which are nonlinear. In many situations, linear differential equation is substituted for a nonlinear differential equation, which approximates the former equation closely enough to give expected result. In many cases such linearization is not possible, and, when it is not, the original nonlinear differential equation must be tackled directly. During last several decades in the 20^{th} century, some Russian scientists like Mandelstam and Papalexi [55], Krylov and Bogoliubov [50], Bogoliubov and Mitropolskii [33] unitedly investigated the nonlinear dynamics. To solve nonlinear differential equations there exist some methods. Among the methods, the method of perturbations, i.e., an asymptotic expansion in terms of small parameter is foremost.

Firstly, Krylov and Bogoliubov [50] considered equations of the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}, t, \varepsilon\right), \qquad (2.1.1)$$

where ε is a small positive parameter and f is a power series in ε, whose coefficients are polynomials in $x, \frac{dx}{dt}$, $\sin t$ and $\cos t$ and their proposed solution procedure is known as Krylov-Bogoliubov (KB) method. In general, f does not contain either ε or t. To describe the behavior of nonlinear oscillations by the solutions obtained by perturbation method, Lindstedt [54], Gylden [48], Liapounoff [52], Poincare [68] discussed only periodic solutions, transient were not considered. Most probably, Poisson initiated approximate solutions of nonlinear differential equations around 1830 and the technique was established by Liouville. The KBM method started with the solution of the linear equation, assuming that in the nonlinear systems, the amplitude and phase in the solution of the linear equation are time dependent functions rather than constants. This procedure introduced an additional condition on the first derivative of the assumed solution for determining the solution. Some meritful works were done and elaborative uses have been made by Stoker [75], McLachlan [56], Minorsky [58], Nayfeh [65], Bellman [32]. Duffing [47] investigated many significant results about the periodic solutions of equation of the form

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega^2x=-\varepsilon x^3 \tag{2.1.2}$$

Sometimes different types nonlinear phenomena occur, when the amplitude of the dependent variable of a dynamical system is less than or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation having these phenomena of the form

$$\frac{d^2x}{dt^2}-\varepsilon(1-x^2)\frac{dx}{dt}+x=0 \tag{2.1.3}$$

This equation is known as Van der Pol [76] equation. Kruskal [49] has extended the KB method to solve the fully nonlinear differential equation of the form

$$\frac{d^2x}{dt^2}=F(x,\frac{dx}{dt},\varepsilon) \tag{2.1.4a}$$

Cap [46] has studied nonlinear systems of the form

$$\frac{d^2x}{dt^2}+\omega^2 f(x)=\varepsilon F(x,\frac{dx}{dt}) \tag{2.1.4b}$$

Since, generally, f dose not contain either ε or t, thus the equation (2.1.1) becomes

$$\frac{d^2x}{dt^2}+\omega^2x=\varepsilon f(x,\frac{dx}{dt}) \tag{2.1.5}$$

As pointed certain that, in the treatment of nonlinear oscillations by perturbation method, only periodic solutions were discussed, transients were not considered by different investigators, where as Krylov and Bogoliubov first discussed transient response.

If $\varepsilon=0$, the equation (2.1.5) reduces to linear equation and its solution is

$$x=a\cos(\omega t+\theta), \tag{2.1.6}$$

where a and θ are arbitrary constants to be determined using initial conditions.

If $\varepsilon \neq 0$, but is very small, then Krylov and Bogoliubov assumed that the solution of (2.1.5) is still given by (2.1.6) together with the derivative of the form

$$\frac{dx}{dt} = -a\omega \sin(\omega t + \theta), \tag{2.1.7}$$

where a and θ are functions of t, rather than being constants.

Thus the solution of (2.1.5) of the form

$$x = a(t)\cos(\omega t + \theta(t)) \tag{2.1.8}$$

and the derivative equation (2.1.7) becomes

$$\frac{dx}{dt} = -\,a(t)\omega \sin(\omega t + \theta(t)) \tag{2.1.9}$$

Differentiating the assumed solution (2.1.8) with respect to t, we obtain

$$\frac{dx}{dt} = \frac{da}{dt}\cos\psi - a\omega\sin\psi - a\frac{d\theta}{dt}\sin\psi, \tag{2.1.10}$$

where $\psi = \omega t + \theta(t)$

Using the equations (2.1.7) and (2.1.10), we get

$$\frac{da}{dt}\cos\psi = a\,\frac{d\theta}{dt}\sin\psi \tag{2.1.11}$$

Again differentiating (2.1.9) with respect to t, we have

$$\frac{d^2x}{dt^2} = -\frac{da}{dt}\,\omega\sin\psi - a\,\omega^2\cos\psi - a\omega\frac{d\theta}{dt}\cos\psi \tag{2.1.12}$$

Putting the value of $\dfrac{d^2x}{dt^2}$ from (2.1.12) into the equation (2.1.5) and using equations (2.1.8) and (2.1.9), we obtain

$$\frac{da}{dt}\,\omega\sin\psi + a\omega\frac{d\theta}{dt}\cos\psi = -\varepsilon\, f\,(a\cos\psi,\ -a\omega\sin\psi) \tag{2.1.13}$$

Solving (2.1.11) and (2.1.13), we have

$$\frac{da}{dt} = -\frac{\varepsilon}{\omega}\sin\psi \; f\,(a\cos\psi,\, -a\omega\sin\psi) \tag{2.1.14}$$

$$\frac{d\theta}{dt} = -\frac{\varepsilon}{a\omega}\cos\psi \; f\,(a\cos\psi,\, -a\omega\sin\psi) \tag{2.1.15}$$

Thus it is observed that, a basic differential equation (2.1.5) of the second order in the unknown x, reduces to two first order differential equations (2.1.14) and (2.1.15) in the unknowns a and θ.

Moreover, $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ are proportional to ε; a and θ are slowly varying functions of the time period $T = \frac{2\pi}{\omega}$. It is noted that these first-order equations are now written in terms of the amplitude a and phase θ as dependent variables. Therefore, the right sides of equations (2.1.14) and (2.1.15) show that both $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ are periodic functions of time T. For the fact, the right-hand terms of these equations contain a small parameter ε and also contain both a and θ, which are slowly varying functions of the time with period $T = \frac{2\pi}{\omega}$. We can transform the equations (2.1.14) and (2.1.15) into more convenient form. Now, expanding $\sin\psi\; f(a\cos\psi, -a\omega\sin\psi)$ and $\cos\psi\; f(a\cos\psi, -a\omega\sin\psi)$ in Fourier series in the total phase ψ, the first approximate solution of (2.1.5), by averaging (2.1.14) and (2.1.15) with period $T = \frac{2\pi}{\omega}$, is

$$\begin{aligned}\frac{da}{dt} &= -\frac{\varepsilon}{2\pi\,\omega}\int_0^{2\pi}\sin\psi\, f(a\cos\psi, -a\omega\sin\psi)\, d\psi,\\ \frac{d\theta}{dt} &= -\frac{\varepsilon}{2\pi\,\omega a}\int_0^{2\pi}\cos\psi\, f(a\cos\psi, -a\omega\sin\psi)\, d\psi,\end{aligned} \tag{2.1.16}$$

where a and θ are independent of time under the integrals.

Krylov and Bogoliubov [50] called their method asymptotic in the sense that $\varepsilon \to 0$. An asymptotic series itself is not convergent, but for a fixed number of terms the approximate solution tends to the exact solution as ε tends to zero. Later, this technique has been extended mathematically by Bogoliubov and Mitropolskii [33], and extended to non-stationary vibrations by Mitropolskii [59]. They assumed the solution of the nonlinear differential equation (2.1.5) of the form

$$x = a\cos\psi + \varepsilon\, u_1\,(a,\psi) + \varepsilon^2\, u_2\,(a,\psi) + \dots\dots\dots + \varepsilon^n\, u_n\,(a,\psi) + O(\varepsilon^{n+1}), \quad (2.1.17)$$

where u_k, ($k = 1, 2, \dots, n$) are periodic functions of ψ with a period 2π, and the quantities a and ψ are functions of time t, defined by

$$\frac{da}{dt} = \varepsilon\, A_1(a) + \varepsilon^2\, A_2\,(a) + \dots\dots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}),$$

$$\frac{d\psi}{dt} = \omega + \varepsilon\, B_1(a) + \varepsilon^2 B_2\,(a) + \dots\dots + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}) \qquad (2.1.18)$$

The functions u_k, A_k and B_k, ($k = 1, 2, \dots, n$) are to be chosen in such a way that the equation (2.1.17), after replacing a and ψ by the functions defined in equation (2.1.18), is a solution of (2.1.5). Since there are no restrictions in choosing functions A_k and B_k, it generates the arbitrariness in the definitions of the functions u_k (Bogoliubov and Mitropolskii [33]). To remove this arbitrariness, the following additional conditions are imposed

$$\int_0^{2\pi} u_k\,(b,\psi)\cos\psi\, d\psi = 0,$$
$$\int_0^{2\pi} u_k\,(a,\psi)\sin\psi\, d\psi = 0. \qquad (2.1.19)$$

Absences of secular terms in all successive approximations are guaranteed by these conditions. Differentiating (2.1.17) two times with respect to t, substituting the values of x, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ into (2.1.5), using these relations in (2.1.18) and equating the coefficients of ε^k, ($k = 1, 2, \cdots, n$) result a recursive system

$$\omega^2 \left(\frac{\partial^2 u_k}{\partial \psi^2} + u_k \right) = f^{(k-1)}(a, \psi) + 2\omega\left(aB_k \cos\psi + A_k \sin\psi\right), \tag{2.1.20}$$

where $f^{(0)}(a,\psi) = f\,(a\cos\psi, -a\omega\sin\psi)$ and

$$\begin{aligned} f^{(1)}(a,\psi) &= u_1\, f_x(a\cos\psi, -a\omega\sin\psi) + (A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}) \\ &\times f_{\frac{dx}{dt}}(\cos\psi, -a\omega\sin\psi) + (aB_1^2 - A_1\frac{dA_1}{da})\cos\psi \\ &+ (2A_1B_1 - aA_1\frac{dB_1}{da})\sin\psi - 2\omega\,(A_1\frac{\partial^2 u_1}{\partial a\,\partial\psi} + B_1\frac{\partial^2 u_1}{\partial\psi^2}) \end{aligned} \tag{2.1.21}$$

Here $f^{(k-1)}$ is a periodic function of ψ with period 2π which depends also on the amplitude a.

Therefore, $f^{(k-1)}$ and u_k can be expanded in a Fourier series as

$$\begin{aligned} f^{(k-1)}(a,\psi) &= g_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}\left(g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi\right), \\ u_k(a,\psi) &= v_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}\left(v_n^{(k-1)}(a)\cos n\psi + w_n^{(k-1)}(a)\sin n\psi\right), \end{aligned} \tag{2.1.22}$$

where

$$\begin{aligned} g_0^{(k-1)} &= \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\,d\psi \\ g_n^{(k-1)} &= \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\cos n\psi\,d\psi \\ h_n^{(k-1)} &= \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\sin n\psi\,d\psi, \quad n \ge 1 \end{aligned} \tag{2.1.23}$$

Here, $v_1^{(k-1)} = w_1^{(k-1)} = 0$ for all values of k, because both integrals of (2.1.19) vanish.

Substituting these values into the equation (2.1.20), we obtain

$$\omega^2 v_0^{(k-1)}(a) + \sum_{n=2}^{\alpha} \omega^2 (1-n^2)[v_n^{(k-1)}(a)\cos n\psi + w_n^{(k-1)}(a)\sin n\psi]$$
$$= g_0^{(k-1)}(a) + (g_1^{(k-1)}(a) + 2\omega a B_k)\cos n\psi + (h_1^{(k-1)}(a) + 2\omega A_k)\sin\psi \tag{2.1.24}$$
$$+ \sum_{n=2}^{\alpha} [g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi]$$

Now equating the coefficients of the harmonics of the same order, yield

$$g_1^{(k-1)}(a) + 2\omega a B_k = 0, \qquad h_1^{(k-1)}(a) + 2\omega A_k = 0, \quad v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2},$$

$$v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2 (1-n^2)}, \quad w_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2 (1-n^2)}, \; n \ge 1 \tag{2.1.25}$$

These are the sufficient conditions to obtain the desired order of approximation. For the first order approximation, we have

$$A_1 = -\frac{h_1^{(0)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos t\psi, -aw\sin\psi)\sin\psi \, d\psi,$$
$$B_1 = -\frac{g_1^{(0)}}{2a\omega} = -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos t\psi, -aw\sin\psi)\cos\psi \, d\psi. \tag{2.1.26}$$

Thus, the variational equations in (2.1.18) become

$$\frac{da}{dt} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi \, d\psi,$$
$$\frac{d\psi}{dt} = \omega - \frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi \, d\psi. \tag{2.1.27}$$

It is seen that, the equations of (2.1.27) are similar to the equations in (2.1.16). Thus, the first order solution obtained by Bogoliubov and Mitropotskii [33] is identical to the original solution obtained by Krylov and Mitropotskii [50]. Secondly, higher order solutions can be found easily. The correction term u_1 is obtained by (2.1.22) on using (2.1.25) as

$$u_1 = \frac{g_0^{(0)}(a)}{\omega^2} + \sum_{n=2}^{\alpha} \frac{g_n^{(0)}(a)\cos n\psi + h_n^{(0)}(a)\sin n\psi}{\omega^2 (1-n^2)} \tag{2.1.28}$$

The solution (2.1.17) together with u_1 is known as the first order improved solution in which a and ψ are obtained from (2.1.27). If the values of the functions A_1 and B_1 are substituted from (2.1.26) into the second relation of (2.1.21), the function $f^{(1)}$ and in the similar way, the functions A_2, B_2 and u_2 can be found. Therefore, the determination of the higher order approximation is complete. The KB method is very similar to that of Van der Pol and related to it. Van der Pol applied the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$, on the other hand Krylov-Bogoliubov applied the same method to the basic solution $x = a\cos(\omega t + \theta)$ of the same equation. Thus in the KB method the varied constants are a and θ, while in the Van der Pol's method the constants are a and b. The method of Krylov-Bogoliubov seems more interesting from the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation.

Volosov [77] and Museenkov [64] also obtained higher order effects. The solution of the equation (2.1.4a) is based on recurrent relations and is given as the power series of the small parameter. Cap [46] solved the equation (2.1.4b) by using elliptical functions in the sense of Krylov and Bogoliubov. The method of Krylov-Bogoliubov (KB) has been extended by Popov [69] to damped nonlinear systems represented by

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = \varepsilon f\left(\frac{dx}{dt}, x\right), \tag{2.1.29}$$

where $-2k\dfrac{dx}{dt}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the Popov's method [69] in the physical systems, involved damping force, Mendelson [57] and Bojadziev [42] retrieved Popov's results. In case of damped nonlinear systems the first equation of (2.1.18) has been replaced by

$$\frac{da}{dt} = -ka + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}) \tag{2.1.18a}$$

Murty and Deekshatulu [61] developed a simple analytical method to obtain the time response of second order nonlinear over-damped systems with small nonlinearity represented by the equation (2.1.29), based on the Krylov-Bogoliubov method of variation of parameters. Alam [21] extended the KBM method to find solutions of over-damped nonlinear systems, when one root of the auxiliary equation becomes much smaller than the other root. According to the KBM method, Murty *et al.* [62] found a hyperbolic type asymptotic solution of an over-damped system represented by the nonlinear differential equation (2.1.29), *i. e.*, in the case $k > \omega$. They used hyperbolic functions, $\cosh\varphi$ and $\sinh\varphi$ instead of their circular counterpart, which was used by Krylov, Bogoliubov, Mitropolskii, Popov and Mendelson. In case of oscillatory or damped oscillatory process $\cosh\varphi$ may be used arbitrarily for all kinds of initial conditions. But in case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions (Bojadziev and Edwards [43], Murty *et al.* [62], Murty [63]). Murty [63] has presented a unified KBM method for solving the nonlinear systems represented by the equation (2.1.29), which cover the undamped, damped and overdamped cases. Bojadziev and Edwards [43] investigated solutions of oscillatory and non-oscillatory systems represented by (2.1.29) when k and ω are slowly varying functions of time t. Arya and Bojadziev [30, 31] examined damped oscillatory systems and time-dependent oscillating systems with slowly varying parameters and delay. Sattar [73] has developed an asymptotic method to solve a second order critically damped nonlinear system represented by (2.1.29). He has found the asymptotic solution of the system (2.1.29) of the form,

$$x = a(1+\psi) + \varepsilon\, u_1(a,\psi) + \cdots + \varepsilon^n u_n(a,\psi) + O\left(\varepsilon^{n+1}\right), \qquad (2.1.30)$$

where a is defined by the equation (2.1.18a) and ψ is defined by

$$\frac{d\psi}{dt} = 1 + \varepsilon\, C_1(a) + \varepsilon^2\, C_2\,(a) + \ldots\ldots + \varepsilon^n\, C_n\,(a) + O(\varepsilon^{n+1}) \qquad (2.1.18b)$$

Osiniskii [66], extended the KBM method to a third order nonlinear differential equation,

$$\frac{d^3x}{dt^3} + c_1\frac{d^2x}{dt^2} + c_2\frac{dx}{dt} + c_3 x = \varepsilon f\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right), \qquad (2.1.31)$$

where ε is a small positive parameter and f is a nonlinear function. He assumed the asymptotic solution of (2.1.31) in the form

$$x = a + b\cos\psi + \varepsilon\, u_1(a,b,\psi) + \cdots + \varepsilon^n\, u_n(a,b,\psi) + o\left(\varepsilon^{n+1}\right), \tag{2.1.32}$$

where each u_k $(k = 1,2,\dots,n)$ is a periodic function of ψ with period 2π and a, b and ψ are functions of time t , given by

$$\begin{aligned}
\frac{da}{dt} &= -\lambda a + \varepsilon\, A_1(a) + \varepsilon^2\, A_2\,(a) + \dots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}),\\
\frac{db}{dt} &= -\mu b + \varepsilon\, B_1(b) + \varepsilon^2\, B_2\,(b) + \dots + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}),\\
\frac{d\psi}{dt} &= \omega + \varepsilon\, C_1(b) + \varepsilon^2\, C_2\,(b) + \dots + \varepsilon^n\, C_n(b) + O(\varepsilon^{n+1}),
\end{aligned} \tag{2.1.33}$$

where $-\lambda,\ -\mu \pm \omega$ are the eigen-values of the equation (2.1.31) when $\varepsilon = 0$.

By using the KBM method, Bojadziev [34] has investigated solutions of nonlinear damped oscillatory systems with small time lag. Bojadziev [39] has also found solutions of damped forced nonlinear vibrations with small time delay. Bojadziev [40], Bojadziev and Chan [41] applied the KBM method to solve the problems of population dynamics. Bojadziev [42] used the KBM method to investigate solutions of nonlinear systems arised from biological and biochemical fields. Lin and Khan [53] have also used the KBM method to some biological problems. Proskurjakov [70] and Bojadziev *et al.* [35] have investigated periodic solutions of nonlinear systems by the KBM and Poincare method, and compared the two solutions. Bojadziev and Lardner [36, 37] have investigated monofrequent oscillations in mechanical systems including the case of internal resonance, governed by hyperbolic differential equations with small nonlinearities. Bojadziev and Lardner [38] have also investigated solution for a certain hyperbolic partial differential equation with small nonlinearity and large time delay included into both unperturbed and perturbed parts of the equation. Rauch [71] has studied oscillations of a third order nonlinear autonomous system. Bojadziev [44] and Bojadziev and Hung [45] developed a technique by using the method of KBM to investigate a weakly nonlinear mechanical system with strong damping. Osiniskii [67] has also extended the KBM method to a third order nonlinear partial differential equation with initial friction and relaxation. Mulholland [60] studied nonlinear oscillations governed by a third order differential equation. Lardner and Bojadziev [51] investigated nonlinear damped oscillations governed by a third order partial differential equation. They introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b. Bojadziev [44] and Bojadziev and Hung [45] used at least two trial solutions to investigate time dependent differential systems; one is for resonant case and the other is for the non-resonant case. But Alam [26] used only one set of variational equations, arbitrarily for both resonant and non-resonant cases. Alam *et al.* [28] presented a general form of the KBM method for solving nonlinear partial differential equations. Raymond and Cabak [72] examined the effects of internal resonance on impulsive forced nonlinear systems with two-degree-of-freedom. Later, Alam [15, 17] has extended the method to n-th order nonlinear systems. Alam [18, 24] has also extended the KBM method for certain non-oscillatory nonlinear systems when the eigen-values of the unperturbed equation are real and non-positive. Alam [11] has presented a new perturbation method based on the KBM method to find approximate solutions of second order nonlinear systems with large damping. Alam *et al.* [13] investigated perturbation solution of a second order time-dependent nonlinear system based on the modified Krylov-Bogoliubov method. Sattar [74] has extended the KBM asymptotic method for three-dimensional over-damped nonlinear systems. Alam *et al.* [12] extended the KBM method to certain non-oscillatory nonlinear systems with varying coefficients. Later, Alam [23] has unified the KBM method for solving n-th order nonlinear differential equation with varying coefficients. Alam and Sattar [10] studied time dependent third order oscillating systems with damping based on an extension of the asymptotic method of Krylov-Bogoliubov-Mitropolskii. Alam [21] and Alam *et al.* [27] have developed a simple method to obtain the time response of some order over-damped nonlinear systems together with slowly varying coefficients under some special conditions. Later, Alam [17] and Alam and Hossain [22] have extended the method presented in [21] to obtain the time response of n-th order $(n \geq 2)$, over-damped systems. Alam [19, 20] has also developed a method for obtaining non-oscillatory solution of third order nonlinear systems. Alam and Sattar [8] presented a unified KBM method for solving third order nonlinear systems. Alam [14] has also presented a unified KBM method, which is not the formal form of the original KBM method, for solving n-th order nonlinear systems. The solution contains some unusual variables, yet this solution is very important. Alam [25] has also presented a modified and compact form of the Krylov-Bogoliubov-Mitropolskii unified method for solving a n-th order nonlinear differential equation. The formula presented in [25] is compact, systematic and practical, and easier than that of [14]. Alam [26] developed a general formula based on the extended KBM method, for obtaining asymptotic solution of an n-th order time dependent quasi linear differential equation with damping. Akbar *et al.* [1] presented an asymptotic method based on the KBM method to solve the fourth order over-damped nonlinear systems. Later, Akbar *et al.* [2] extended the method present in [1] for the fourth order damped oscillatory systems. Akbar *et al.* [3] also developed a simple technique for obtaining certain over-damped solution of an n-th order nonlinear differential equation. Akber *et al.* [4] presented the KBM unified method for solving n-th order nonlinear systems under some special conditions including the case of internal resonance. Akbar *et al.* [6] also developed perturbation theory for fourth order nonlinear systems with large damping.

2.2 Near critically damped nonlinear systems

The Krylov-Bogoliubov-Mitropolskii (KBM) method [33, 50], was basically developed to find periodic solutions of second order nonlinear differential equations with small nonlinearities,

$$\frac{d^2x}{dt^2}+\omega^2 x=-\varepsilon f(x,\frac{dx}{dt}), \tag{2.2.1}$$

where ε is very small positive parameter but not equal to zero.

First, Popov [69] has extended the KB method. The KBM [33, 50] method is particularly easy to understand and extensively used to obtain approximate solution of weakly nonlinear systems. 0For the physical importance, Mendelson [57] reproduced Popov's results [69]. Murty *et al.* [62] and Alam [9] extended the method to nonlinear over-damped systems. However, both over-damped solutions [62, 9] are not up to the mark for certain damping effects especially near to the critically damped. Alam [9] has developed a new perturbation technique to find approximate analytical solution of second order both over-damped and critically damped nonlinear systems. First, Alam and Sattar [7] developed a method to solve third order critically damped autonomous nonlinear systems. Alam [16] redeveloped the method presented in [7] to find approximate solutions of critically damped nonlinear systems in the presence of different damping forces by considering different sets of variational equations. Later, he unified the KBM method for solving critically damped nonlinear systems [29]. Alam [20] studied a third order critically damped nonlinear system whose unequal eigen-values are in integral multiple. Alam [20] has also extended the method to a third order over-damped system when two of the eigen-values are almost equal (*i.e.*, the system is near to the criticallly damped) and the rest is small. Recently, Alam [24] has presented an asymptotic method for certain third order non-oscillatory nonlinear system, which gives desired results when the damping force is near to the critical damping force.

In this thesis, we want to develop the KBM method to solve fourth order damped oscillatory and near critically damped non-oscillatory nonlinear systems with small nonlinearities.

CHAPTER 3

METHODOLOGY

In this chapter, we have discussed the methodology to solve fourth order nonlinear differential equation. On the basis of the eigen-values of the unperturbed form of the equation the solution may be oscillatory or non-oscillatory. In 3.1 we have considered the oscillatory system and in 3.2 we have considered the non-oscillatory system. In the oscillatory case the system considered is damped, where as, in the non-oscillatory case the system is near critically damped.

3.1 Approximate Solutions of Fourth Order Damped Oscillatory Nonlinear Systems

Let us consider a weakly nonlinear damped oscillatory system, which is governed by the differential equation

$$\frac{d^4x}{dt^4}+c_1\frac{d^3x}{dt^3}+c_2\frac{d^2x}{dt^2}+c_3\frac{dx}{dt}+c_4x=\varepsilon f(x), \tag{3.1.1}$$

where ε is a small positive quantity, f is the given nonlinear function and c_1, c_2, c_3, c_4 are arbitrary constants defined in terms of the eigen values $-\lambda_i$ (i=1, 2, 3, 4) of the unperturbed form of (3.1.1) as $c_1=\sum_{i=1}^{4}\lambda_i$, $c_2=\sum_{\substack{i,j=1\\ i\neq j}}^{4}\lambda_i\lambda_j$, $c_3=\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4}\lambda_i\lambda_j\lambda_k$

and $c_4=\prod_{i=1}^{4}\lambda_i$.

Suppose, for the damped oscillations, the two eigen-values say $-\lambda_1,-\lambda_2$ are real and the other two $-\lambda_3,-\lambda_4$ are complex, so they are conjugate to each other.

If $\varepsilon=0$, then the unperturbed solution of the equation (3.1.1) is,

$$x(t,0)=\sum_{i=1}^{4}a_{i,0}\,e^{-\lambda_i t}, \tag{3.1.2}$$

where $a_{i,0}$ (i = 1, 2, 3, 4) are arbitrary constants.

If $\varepsilon \neq 0$, then we consider the solution of the equation (3.1.1) of the form

$$x(t,\varepsilon)=\sum_{i=1}^{4} a_i e^{-\lambda_i t}+\varepsilon u_1(a_1,a_2,a_3,a_4,t)+\varepsilon^2 u_2(a_1,a_2,a_3,a_4,t)+\varepsilon^3\cdots, \quad (3.1.3)$$

where each a_i (i = 1, 2, 3, 4) satisfies the differential equations

$$\frac{da_i(t)}{dt}=\varepsilon A_i(a_1,a_2,a_3,a_4,t)+\varepsilon^2 B_i(a_1,a_2,a_3,a_4,t)+\varepsilon^3\cdots \quad (3.1.4)$$

We will remain confined within some first few terms, 1, 2, 3, ... , m in the series expansion (3.1.3) and (3.1.4) and calculate the functions u_1, u_2, u_3,.... and A_i, B_i,$\cdots$ (i=1, 2, 3, 4), so that , a_i appearing in (3.1.3) and (3.1.4) will satisfy the given differential equation (3.1.1) with correctness of ε^{m+1}. Basically, the solution can be obtained up to the correctness of any order of approximation. However, owing to the rapidly growing algebraic complexity for the formulae, the solution is, in general, confined to a lower order, usually the first [62]. In order to determine these functions, it is assumed that the functions u_1, u_2, u_3,.... do not contain the fundamental terms, which are included in the series expansion (3.1.3) of order ε^0. Thus, the first approximated solution may be taken as

$$x(t,\varepsilon)=\sum_{i=1}^{4} a_i e^{-\lambda_i t}+\varepsilon u_1, \quad (3.1.5)$$

where each a_i (i = 1, 2, 3, 4) satisfies the following form of the differential equations

$$\frac{da_i(t)}{dt}=\varepsilon A_i(a_1,a_2,a_3,a_4,t) \quad (3.1.6)$$

Differentiating (3.1.3) four times with respect to t, then substituting x and the derivatives, $\frac{d^4x}{dt^4}, \frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}$ in the original equation (3.1.1), using the relations in (3.1.4) and equating the coefficients of ε, we will obtain equation involving u_1 and A_i (i = 1, 2, 3, 4)

with $f^{(0)}=f(x_o)$ on the right ,where $x_0=\sum_{i=1}^{4} a_i(t)e^{-\lambda_i t}$.

In general, the function $f^{(o)}$ can be expanded in a Taylor series as

$$f^{(0)} = \sum_{m_1=-\infty, m_2=-\infty, m_3=-\infty, m_4=-\infty}^{\infty,\infty,\infty,\infty} F_{m_1, m_2, m_3, m_4} a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(-m_1\lambda_1 - m_2\lambda_2 - m_3\lambda_3 - m_4\lambda_4)t}.$$

According to our assumptions, u_1 does not contain the fundamental terms, the obtained equation can be separated into five equations for unknown functions, A_1, A_2, A_3, A_4 and u_1.

Putting the value of $f^{(0)}$ and separating in terms of $e^{-\lambda_i t}$ (i =1,2,3,4), we will get separate equations involving A_1, A_2, A_3, A_4 and u_1 respectively.

The particular solutions of these equations give the results of the functions A_1, A_2, A_3, A_4 and u_1. The values of A_1, A_2, A_3, A_4 will be used on (3.1.6) to determine a_i (i = 1, 2, 3, 4) and the final values to be substituted on (3.1.5). Thus, the determination of the first approximate solution is completed. The above process can be applied to the higher order approximations.

3.2 Asymptotic Solutions of Fourth Order Near Critically Damped Non-Oscillatory Nonlinear Systems

Let us consider the following fourth order weakly nonlinear ordinary differential equation

$$\frac{d^4x}{dt^4} + e_1 \frac{d^3x}{dt^3} + e_{2e} \frac{d^2x}{dt^2} + e_3 \frac{dx}{dt} + e_4 x = -\varepsilon f(x), \tag{3.2.1}$$

where ε is a positive small parameter; f is the given nonlinear function and e_1, e_2, e_3, c_4

are constants, defined in terms of the eigen values $-\lambda_i$(i=1, 2, 3, 4) of the unperturbed form of (3.2.1) as $e_1 = \sum_{i=1}^{4}\lambda_i$, $e_2 = \sum_{\substack{i,j=1\\ i\neq j}}^{4}\lambda_i\lambda_j$, $e_3 = \sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4}\lambda_i\lambda_j\lambda_k$ and $e_4 = \prod_{i=1}^{4}\lambda_i$.

Suppose, for the near critically damped nonlinear system the two real eigen-values say $-\lambda_1$ and $-\lambda_2$ are almost equal and the other two eigen-values say $-\lambda_3$ and $-\lambda_4$ are real and different.

When $\varepsilon = 0$, the equation (3.2.1) becomes linear and the solution of the linear equation is

$$x(t,0) = a_{3,0}e^{-\lambda_3 t} + a_{4,0}e^{-\lambda_4 t} + \frac{1}{2}a_{1,0}(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_{2,0}\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right), \quad (3.2.2)$$

where $a_{i,0}$ (i=1, 2, 3, 4) are arbitrary constants.

When $\varepsilon \neq 0$, following the method presented by Alam [24], we choose the solution of (3.2.1) in the form

$$x(t,\varepsilon) = a_3(t)e^{-\lambda_3 t} + a_4(t)e^{-\lambda_4 t} + \frac{1}{2}a_1(t)(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2(t)\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right) \quad (3.2.3)$$
$$+ \varepsilon u_1(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \ldots,$$

where each a_i (i=1, 2, 3, 4) satisfies the following first order differential equations

$$\frac{da_1(t)}{dt} = \varepsilon A_1(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \cdots$$

$$\frac{da_2(t)}{dt} = \varepsilon A_2(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \cdots$$

$$\frac{da_3(t)}{dt} = \varepsilon A_3(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \cdots \quad (3.2.4)$$

and $$\frac{da_4(t)}{dt} = \varepsilon A_4(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \cdots$$

Confining only to a first few terms 1, 2, 3, ..., n in the series expansion of (3.2.3) and (3.2.4), we calculate the functions u_i and A_i (i=1, 2, 3,...., n), such that $a_i(t)$ (i=1, 2, 3, ... n), appearing in (3.2.3) and (3.2.4), satisfy the given differential equation (3.2.1) with an accuracy of order ε^{n+1}. To determine the unknown functions u_1, A_1, A_2, A_3, A_4 it is assumed (as customary in the KBM method) that the correction term u_1 does not contain secular-type terms $t\,e^{-\lambda_i t}$ and/or $t\,e^{-\lambda_j t}$ for different eigen-values, which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is, in general, confined to a lower order usually the first order. Differentiating the equation (3.2.3) four times with respect t, substituting the derivatives $\frac{d^4x}{dt^4}$, $\frac{d^3x}{dt^3}$, $\frac{d^2x}{dt^2}$, $\frac{dx}{dt}$ and x in the original equation (3.2.1), using the relations in (3.2.4) and then equating the coefficients of ε, we obtain

$$
\begin{aligned}
&(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1 \\
&+e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3 \\
&+e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4 \\
&+\frac{1}{2}\begin{pmatrix} e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4)A_1+ \\ e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_4)A_1 \end{pmatrix} \\
&(D+\lambda_4)\left(e^{-\lambda_1 t}(\lambda_1-\lambda_3-\frac{3}{2}D)+e^{-\lambda_2 t}(\lambda_2-\lambda_3-\frac{3}{2}D)\right)A_2 \\
&+\frac{1}{\lambda_1-\lambda_2}\left(e^{-\lambda_1 t}(\lambda_4-\lambda_1)+e^{-\lambda_2 t}(\lambda_2-\lambda_4)+(e^{-\lambda_1 t}-e^{-\lambda_2 t})D\right)\times \\
&D\left(D+\lambda_3-\frac{\lambda_1+\lambda}{2}\right)A_2=-f^{(0)},
\end{aligned}
\tag{3.2.5}
$$

where $f^{(0)} = f(x_0)$ and

$$
x_0 = a_3(t)e^{-\lambda_3 t}+a_4(t)e^{-\lambda_4 t}+\frac{1}{2}a_1(t)(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2(t)\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)
$$

It is assumed that the functional value $f^{(0)}$ can be expanded in power series as in the form (detail can be found in [24])

$$
f^{(0)}=\sum_{r=0}^{n}F_r(a_3e^{-\lambda_3 t},a_4e^{-\lambda_4 t})\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^r, \tag{3.2.6}
$$

where n is the order of polynomial of the nonlinear function f. The assumption is certainly valid when f is a polynomial function of x. Such polynomial functions of f cover some special and important systems in mechanics. Following Alam [16, 20] we assume that u_1 does not contain the terms F_0 and F_1 of $f^{(0)}$, since the system is considered near to a critically damped non-oscillatory nonlinear system. Substituting the value of $f^{(0)}$ into (3.2.5) and equating the coefficients of powers of $\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)$, we will obtain equations for u_1 and A_i (i = 1, 2, 3, 4). Krylov-Bogoliubov-Mitropolskii (KBM) [33, 50], Sattar [73], Alam [7, 9, 16, 20] imposed the condition that u_1 will not contain the fundamental terms (the solution presented in equation (3.2.2) is called the generating solution and its terms are called fundamental terms) of $f^{(0)}$. It is not easy to solve the equations for the unknown functions A_1, A_2, A_3 and A_4, if the nonlinear function f and the eigen-values $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$ of the corresponding linear equation of (3.2.1) are not specified. When these are specified, the values can be found subject to the condition that the coefficients in the solutions of them do not become large (Akbar *et al.* [4], Alam [16, 24]). For this reason, we have assumed that the relations, limit $\lambda_1 \to \lambda_2$ and $\lambda_3 \approx 3\lambda_4$, exist among the eigen-values. Imposing the relations, limit $\lambda_1 \to \lambda_2$ and $\lambda_3 \approx 3\lambda_4$, we can find the values of A_1, A_2, A_3 and A_4. Substituting the values of A_1, A_2, A_3 and A_4 in the equation (3.2.4), we obtain the results of $\frac{da_i}{dt}$ (i=1, 2, 3, 4), which are proportional to the small parameter ε, so they are slowly varying functions of time t, i.e., they are almost constants and by integrating the equations of (3.2.4), we obtain the values of a_i (i=1, 2, 3, 4). It is laborious to solve equation for u_1. However, as $\lambda_1 \to \lambda_2$ it takes the simple form and can be solved. Finally, substituting the values of a_i (i=1, 2, 3, 4) and u_1 in the equation (3.2.3), we will get the complete solution of (3.2.1).

Thus, the determination of the first approximate solution is completed. The above process can be applied to the higher order approximations.

CHAPTER 4

RESULTS AND DISCUSSIONS

In this chapter, we have shown the application of the methodology discussed in the previous Chapter 3. Here, examples have been chosen to utilize the tools devised earlier. In 4.1 we have solved a particular differential equation, which represents damped oscillatory system. We have chosen a differential equation of fourth order with small nonlinearities. The methodology adopted to solve this is discussed in 3.1. Here the results and findings are discussed in succession. In a similar fashion, with suitable example in 4.2, we have discussed the near critically damped non-oscillatory nonlinear system, whose methodology is discussed in 3.2.

4.1 Approximate Solutions of Fourth Order Damped Oscillatory Nonlinear Systems

4.1.1 Example

As an example to solve damped oscillatory nonlinear system, we consider the fourth order nonlinear differential equation of the form

$$\frac{d^4x}{dt^4}+c_1\frac{d^3x}{dt^3}+c_2\frac{d^2x}{dt^2}+c_3\frac{dx}{dt}+c_4x=-\varepsilon x^3 \tag{4.1.1.1}$$

The unperturbed solution of (4.1.1.1) as prescribed in (3.1.2) can be rewritten as

$$x(t,0)=ae^{-k_1t}\cosh(\omega_1t+\varphi_1)+be^{-k_2t}\cos(\omega_2t+\varphi_2).$$

Also, from (3.1.3), the first approximate solution is

$$x(t,\varepsilon)=ae^{-k_1t}\cosh(\omega_1t+\varphi_1)+be^{-k_2t}\cos(\omega_2t+\varphi_2)+\varepsilon u_1$$

For equation (4.1.1.1), we have, $f=-x^3$ and

$$f^{(0)}=-\left(a_1e^{-\lambda_1t}+a_2e^{-\lambda_2t}+a_3e^{-\lambda_3t}+a_4e^{-\lambda_4t}\right)^3$$

or,

$$f^{(0)} = -\{a_1^3 e^{-3\lambda_1 t} + a_2^3 e^{-3\lambda_2 t} + a_3^3 e^{-3\lambda_3 t} + a_4^3 e^{-3\lambda_4 t}$$
$$+ 3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1^2 a_3 e^{-(2\lambda_1+\lambda_3)t} + 3a_1^2 a_4 e^{-(2\lambda_1+\lambda_4)t}$$
$$+ 3a_2^2 a_1 e^{-(2\lambda_2+\lambda_1)t} + 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2 a_4 e^{-(2\lambda_2+\lambda_4)t}$$
$$+ 3a_3^2 a_1 e^{-(2\lambda_3+\lambda_1)t} + 3a_3^2 a_2 e^{-(2\lambda_3+\lambda_2)t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} \quad (4.1.1.2)$$
$$+ 3a_4^2 a_1 e^{-(2\lambda_4+\lambda_1)t} + 3a_4^2 a_2 e^{-(2\lambda_4+\lambda_2)t} + 3a_4^2 a_3 e^{-(2\lambda_4+\lambda_3)t}$$
$$+ 6a_1a_2a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 6a_1a_2a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t}$$
$$+ 6a_1a_3a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 6a_2a_3a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t}\}.$$

Now equating the like terms, i,e,. in terms of $e^{-\lambda_i t}$ (i =1,2,3,4), we have

$$e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4)A_1$$
$$= -\{3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 6a_1a_3a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t}\} \quad (4.1.1.3)$$

$$e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_4)A_2$$
$$= -\{3a_2^2 a_1 e^{-(2\lambda_2+\lambda_1)t} + 6a_2a_3a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t}\} \quad (4.1.1.4)$$

$$e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3$$
$$= -\{3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 6a_1a_2a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t}\} \quad (4.1.1.5)$$

$$e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4$$
$$= -\{3a_4^2 a_3 e^{-(2\lambda_4+\lambda_3)t} + 6a_1a_2a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t}\} \quad (4.1.1.6)$$

and

$$(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1$$
$$= -\{a_1^3 e^{-3\lambda_1 t} + a_2^3 e^{-3\lambda_2 t} + a_3^3 e^{-3\lambda_3 t} + a_4^3 e^{-3\lambda_4 t}$$
$$+ 3a_1^2 a_3 e^{-(2\lambda_1+\lambda_3)t} + 3a_1^2 a_4 e^{-(2\lambda_1+\lambda_4)t}$$
$$+ 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2 a_4 e^{-(2\lambda_2+\lambda_4)t} \quad (4.1.1.7)$$
$$+ 3a_3^2 a_1 e^{-(2\lambda_3+\lambda_1)t} + 3a_3^2 a_2 e^{-(2\lambda_3+\lambda_2)t}$$
$$+ 3a_4^2 a_1 e^{-(2\lambda_4+\lambda_1)t} + 3a_4^2 a_2 e^{-(2\lambda_4+\lambda_2)t}\}.$$

Solving the equations (4.1.1.3)-(4.1.1.7) and considering $\lambda_1 = k_1 - \omega_1$, $\lambda_2 = k_1 + \omega_1$, $\lambda_3 = k_2 - i\omega_2$ and $\lambda_4 = k_2 + i\omega_2$, and then substituting these values, we obtain

$$A_1 = \frac{3a_1 a_3 a_4 e^{-2k_2 t}}{(k_2 - \omega_1)(k_1 + k_2 - \omega_1 + i\omega_2)(k_1 + k_2 - \omega_1 - i\omega_2)} + \frac{3a_1^{\,2} a_2 e^{-2k_1 t}}{2(k_1 - \omega_1)(3k_1 + k_2 - \omega_1 + i\omega_2)(3k_1 - k_2 - \omega_1 - i\omega_2)},$$

$$A_2 = \frac{3a_2 a_3 a_4 e^{-2k_2 t}}{(k_2 + \omega_1)(k_1 + k_2 + \omega_1 + i\omega_2)(k_1 + k_2 + \omega_1 - i\omega_2)} + \frac{3a_1 a_2^{\,2} e^{-2k_1 t}}{2(k_1 + \omega_1)(3k_1 - k_2 + \omega_1 + i\omega_2)(3k_1 - k_2 + \omega_1 - i\omega_2)},$$

$$A_3 = \frac{3a_1 a_2 a_3 e^{-2k_1 t}}{(k_1 - i\omega_2)(k_1 + k_2 + \omega_1 - i\omega_2)(k_1 + k_2 - \omega_1 - i\omega_2)} + \frac{3a_3^{\,2} a_4 e^{-2k_{21} t}}{2(k_2 - i\omega_2)(3k_2 - k_1 + \omega_1 - i\omega_2)(3k_2 - k_1 - \omega_1 - i\omega_2)}, \qquad (4.1.1.8)$$

and

$$A_4 = \frac{3a_1 a_2 a_4 e^{-2k_1 t}}{(k_1 + i\omega_2)(k_1 + k_2 + \omega_1 + i\omega_2)(k_1 + k_2 - \omega_1 + i\omega_2)} + \frac{3a_3 a_4^{\,2} e^{-2k_{21} t}}{2(k_2 + i\omega_2)(3k_2 - k_1 + \omega_1 + i\omega_2)(3k_2 - k_1 - \omega_1 + i\omega_2)}.$$

Putting the values of (4.1.1.8) into equation (3.1.4) and neglecting the second and higher powers of ε (since ε is very small), we get

$$\frac{da_1}{dt} = \varepsilon \left\{ \frac{3a_1 a_3 a_4 e^{-2k_2 t}}{(k_2 - \omega_1)(k_1 + k_2 - \omega_1 + i\omega_2)(k_1 + k_2 - \omega_1 - i\omega_2)} + \frac{3a_1^{\,2} a_2 e^{-2k_1 t}}{2(k_1 - \omega_1)(3k_1 + k_2 - \omega_1 + i\omega_2)(3k_1 - k_2 - \omega_1 - i\omega_2)} \right\},$$

$$\frac{da_2}{dt} = \varepsilon \left\{ \begin{array}{l} \dfrac{3a_2 a_3 a_4 e^{-2k_2 t}}{(k_2+\omega_1)(k_1+k_2+\omega_1+i\omega_2)(k_1+k_2+\omega_1-i\omega_2)} \\ + \dfrac{3a_1 a_2{}^2 e^{-2k_1 t}}{2(k_1+\omega_1)(3k_1-k_2+\omega_1+i\omega_2)(3k_1-k_2+\omega_1-i\omega_2)} \end{array} \right\},$$

$$\frac{da_3}{dt} = \varepsilon \left\{ \begin{array}{l} \dfrac{3a_1 a_2 a_3 e^{-2k_1 t}}{(k_1-i\omega_2)(k_1+k_2+\omega_1-i\omega_2)(k_1+k_2-\omega_1-i\omega_2)} \\ + \dfrac{3a_3{}^2 a_4 e^{-2k_{21} t}}{2(k_2-i\omega_2)(3k_2-k_1+\omega_1-i\omega_2)(3k_2-k_1-\omega_1-i\omega_2)} \end{array} \right\} \qquad (4.1.1.9)$$

and

$$\frac{da_4}{dt} = \varepsilon \left\{ \begin{array}{l} \dfrac{3a_1 a_2 a_4 e^{-2k_1 t}}{(k_1+i\omega_2)(k_1+k_2+\omega_1+i\omega_2)(k_1+k_2-\omega_1+i\omega_2)} \\ + \dfrac{3a_3 a_4{}^2 e^{-2k_{21} t}}{2(k_2+i\omega_2)(3k_2-k_1+\omega_1+i\omega_2)(3k_2-k_1-\omega_1+i\omega_2)} \end{array} \right\}.$$

Now, substituting $a_1 = \dfrac{ae^{\varphi_1}}{2}$, $a_2 = \dfrac{ae^{-\varphi_1}}{2}$, $a_3 = \dfrac{be^{i\varphi_2}}{2}$, $a_4 = \dfrac{be^{-i\varphi_2}}{2}$ into the equation (4.1.1.9) and simplifying, we obtain,

$$\frac{da}{dt} = \varepsilon\left(l_1 a^3 e^{-2k_1 t} + l_2 ab^2 e^{-2k_2 t} \right),$$

$$\frac{db}{dt} = \varepsilon\left(r_1 a^2 b e^{-2k_1 t} + r_2 b^3 e^{-2k_2 t} \right),$$

$$\frac{d\varphi_1}{dt} = \varepsilon\left(m_1 a^2 e^{-2k_1 t} + m_2 b^2 e^{-2k_2 t} \right), \qquad (4.1.1.10)$$

and $$\frac{d\varphi_2}{dt} = \varepsilon\left(s_1 a^2 e^{-2k_1 t} + s_2 b^2 e^{-2k_2 t} \right),$$

where

$$l_1 = \frac{3}{8}\left[\frac{k_1(9k_1{}^2 + k_2{}^2 + \omega_1{}^2 + \omega_2{}^2 - 6k_1k_2) + 2\omega_1{}^2(3k_1 - k_2)}{\left(k_1{}^2 - \omega_1{}^2\right)\{(3k_1 - k_2 - \omega_1)^2 + \omega_2{}^2\}\{(3k_1 - k_2 + \omega_1)^2 + \omega_2{}^2\}} \right],$$

$$l_2 = \frac{3}{4}\left[\frac{k_2(k_1{}^2 + k_2{}^2 + \omega_1{}^2 + \omega_2{}^2 + 2k_1k_2) + 2\omega_1{}^2(k_1 + k_2)}{\left(k_2{}^2 - \omega_1{}^2\right)\{(k_1 + k_2 - \omega_1)^2 + \omega_2{}^2\}\{(k_1 + k_2 + \omega_1)^2 + \omega_2{}^2\}} \right],$$

$$r_1 = \frac{3}{4}\left[\frac{k_1(k_1^2 + k_2^2 - \omega_1^2 - \omega_2^2 - 2k_1k_2) - 2\omega_2^2(k_1 + k_2)}{(k_1^2 + \omega_2^2)\{(k_1 + k_2 + \omega_1)^2 + \omega_2^2\}\{(k_1 + k_2 - \omega_1)^2 + \omega_2^2\}}\right],$$

$$r_2 = \frac{3}{8}\left[\frac{k_2(k_1^2 + 9k_2^2 - \omega_1^2 - \omega_2^2 - 6k_1k_2) - 2\omega_2^2(3k_2 - k_1)}{(k_2^2 + \omega_2^2)\{(3k_2 - k_1 + \omega_1)^2 + \omega_2^2\}\{(3k_2 - k_1 - \omega_1)^2 + \omega_2^2\}}\right],$$

$$m_1 = \frac{3}{8}\left[\frac{2k_1\omega_1(3k_1 - k_2) + \omega_1(9k_1^2 + k_2^2 + \omega_1^2 + \omega_2^2 - 6k_1k_2)}{(k_1^2 - \omega_1^2)\{(3k_1 - k_2 - \omega_1)^2 + \omega_2^2\}\{(3k_1 - k_2 + \omega_1)^2 + \omega_2^2\}}\right],$$

$$m_2 = \frac{3}{4}\left[\frac{2k_2\omega_1(k_1 + k_2) + \omega_1(k_1^2 + k_2^2 + \omega_1^2 + \omega_2^2 + 2k_1k_2)}{(k_2^2 - \omega_1^2)\{(k_1 + k_2 - \omega_1)^2 + \omega_2^2\}\{(k_1 + k_2 + \omega_1)^2 + \omega_2^2\}}\right],$$

$$s_1 = \frac{3}{4}\left[\frac{2k_1\omega_2(k_1 + k_2) + \omega_2(k_1^2 + k_2^2 - \omega_1^2 - \omega_2^2 + 2k_1k_2)}{(k_1^2 + \omega_2^2)\{(k_1 + k_2 + \omega_1)^2 + \omega_2^2\}\{(k_1 + k_2 - \omega_1)^2 + \omega_2^2\}}\right]$$

and

$$s_2 = \frac{3}{8}\left[\frac{2k_2\omega_2(3k_2 - k_1) + \omega_2(k_1^2 + 9k_2^2 - \omega_1^2 - \omega_2^2 - 6k_1k_2)}{(k_2^2 + \omega_2^2)\{(3k_2 - k_2 + \omega_1)^2 + \omega_2^2\}\{(3k_2 - k_1 - \omega_1)^2 + \omega_2^2\}}\right].$$

In a similar manner the solution of (4.1.1.7) can be written as

$$\begin{aligned} u_1 = &-\frac{1}{16}\sum_{m,n,=1,m\neq n}^{2} a^{-m+2n} b^{2m-n} \cosh\left\{i^{m-1} 3(\omega_m t + \varphi_m)\right\} g_{m,n}\, e^{-3k_m t} \\ &-\frac{1}{16}\sum_{m,n,=1,m\neq n}^{2} a^{-m+2n} b^{2m-n} \sinh\left\{i^{m-1} 3(\omega_m t + \varphi_m)\right\} h_{m,n}\, e^{-3k_m t} \\ &-\frac{3ab}{16}\sum_{m,n,=1,m\neq n}^{2} a^{m+n-2} b^{m+n-3} \cos(\omega_2 t + \varphi_2) e^{-(2k_1 + k_2)t}\, c_{m,n}\, e^{(-1)^n(2\omega_1 t + 2\varphi_1)} \\ &+\frac{3ab}{16}\sum_{m,n,=1,m\neq n}^{2} a^{m+n-2} b^{m+n-3} \sin(\omega_2 t + \varphi_2) e^{-(2k_1 + k_2)t}\, d_{m,n}\, e^{(-1)^n(2\omega_1 t + 2\varphi_1)} \\ &-\frac{3ab}{16}\sum_{m,n,=1,m\neq n}^{2} a^{m+n-3} b^{m+n-2} \cos 2(\omega_2 t + \varphi_2) e^{-(k_1 + 2k_2)t}\, p_{m,n}\, e^{(-1)^n(\omega_1 t + \varphi_1)} \\ &+\frac{3ab}{16}\sum_{m,n,=1,m\neq n}^{2} a^{m+n-3} b^{m+n-2} \sin 2(\omega_2 t + \varphi_2) e^{-(k_1 + 2k_2)t}\, q_{m,n}\, e^{(-1)^n(\omega_1 t + \varphi_1)} \end{aligned} \quad (4.1.1.11)$$

where

$$g_{m,n}=\frac{\begin{gathered}(3k_m-k_n)^2\{k_m{}^2+(-1)^n 2\omega_m{}^2\}+\{2\omega_m{}^2+(-1)^n k_m{}^2\}(9\omega_m{}^2+\omega_n{}^2)\\ +(-1)^n 18k_m\omega_m{}^2(3k_m-k_n)\end{gathered}}{\begin{gathered}\{k_m{}^2+(-1)^m\omega_m{}^2\}\{k_m{}^2+(-1)^m 4\omega_m{}^2\}\{(3k_m-k_n+(-1)^m 3^{2-m}\omega_1)^2\\ +9^{2-n}\omega_2{}^2\}\{(3k_m-k_n+(-1)^n 3^{2-m}\omega_1)^2+9^{2-n}\omega_2{}^2\}\end{gathered}},$$

$$h_{m,n}=\frac{\begin{gathered}6\omega_m\{k_m{}^2+(-1)^n 2\omega_m{}^2\}(3k_m-k_n)+3k_m\omega_m(3k_m-k_n)^2\\ +(-1)^n 3k_m\omega_m(9\omega_m{}^2+\omega_n{}^2)\end{gathered}}{\begin{gathered}\{k_m{}^2+(-1)^m\omega_m{}^2\}\{k_m{}^2+(-1)^m 4\omega_m{}^2\}\{(3k_m-k_n+(-1)^m 3^{2-m}\omega_1)^2\\ +9^{2-n}\omega_2{}^2\}\{(3k_m-k_n+(-1)^n 3^{2-m}\omega_1)^2+9^{2-n}\omega_2{}^2\}\end{gathered}},$$

$$c_{m,n}=\frac{\begin{gathered}\{k_1+(-1)^m\omega_1\}\{(k_1+k_2)^2+(-1)^m 4\omega_1(k_1+k_2)+(3\omega_1{}^2-\omega_2{}^2)\}\\ -2\omega_2{}^2(k_1+k_2)+(-1)^n 4\omega_1\omega_2{}^2\end{gathered}}{\begin{gathered}\{k_1+(-1)^m\omega_1\}\{(k_1+k_2(-1)^m\omega_1)^2+\omega_2{}^2\}\{(k_1+k_2+(-1)^m 3\omega_1)^2+\omega_2{}^2\}\\ \times\{(k_1+(-1)^m\omega_1)^2+\omega_2{}^2\}\end{gathered}},$$

$$d_{m,n}=\frac{\begin{gathered}\{k_1+(-1)^m\omega_1\}\{2\omega_2(k_1+k_2)+(-1)^m 4\omega_1\omega_2\}+\omega_2(k_1+k_2)^2\\ +(-1)^m 4\omega_1\omega_2(k_1+k_2)+\omega_2(3\omega_1{}^2-\omega_2{}^2)\end{gathered}}{\begin{gathered}\{k_1+(-1)^m\omega_1\}\{(k_1+k_2(-1)^m\omega_1)^2+\omega_2{}^2\}\{(k_1+k_2+(-1)^m 3\omega_1)^2+\omega_2{}^2\}\\ \times\{(k_1+(-1)^m\omega_1)^2+\omega_2{}^2\}\end{gathered}},$$

$$p_{m,n}=\frac{\begin{gathered}\{k_2(k_2+(-1)^m\omega_1)-\omega_2{}^2\}\{k_1+k_2+(-1)^m\omega_1\}^2-3k_2\omega_2{}^2\{k_2+(-1)^m\omega_1\}\\ -4\omega_2{}^2\{2k_2+(-1)^m\omega_1\}\{k_1+k_2(-1)^m\omega_1\}+3\omega_2{}^4\end{gathered}}{\begin{gathered}\{k_2{}^2+\omega_2{}^2\}\{(k_1+k_2(-1)^m\omega_1)^2+\omega_2{}^2\}\{(k_1+k_2+(-1)^m\omega_1)^2+9\omega_2{}^2\}\\ \times\{(k_2+(-1)^m\omega_1)^2+\omega_2{}^2\}\end{gathered}},$$

and

$$q_{m,n}=\frac{\begin{gathered}\{4k_2\omega_2(k_2+(-1)^m\omega_1)-4\omega_2{}^3\}\{k_1+k_2+(-1)^m\omega_1\}+\{2k_2+(-1)^m\omega_1\}\\ \{\omega_2(k_1+k_2(-1)^m\omega_1)^2-3\omega_2{}^2\}\end{gathered}}{\begin{gathered}\{k_2{}^2+\omega_2{}^2\}\{(k_1+k_2(-1)^m\omega_1)^2+\omega_2{}^2\}\{(k_1+k_2+(-1)^m\omega_1)^2+9\omega_2{}^2\}\\ \times\{(k_2+(-1)^m\omega_1)^2+\omega_2{}^2\}\end{gathered}}.$$

Equation (4.1.1.10) has no exact solution. Since, $\frac{da}{dt}$, $\frac{db}{dt}$, $\frac{d\varphi_1}{dt}$ and $\frac{d\varphi_2}{dt}$ are proportional to the small parameter ε, therefore they are slowly varying functions of time t with the period $T=\frac{2\pi}{\omega}$. Moreover, by assuming a and b are constants in the right hand side of equation (4.1.1.10) and by integrating equation (4.1.1.10), we have

$$a = a_0 + \frac{\varepsilon}{2}\left\{\frac{l_1 a_0{}^3(1-e^{-2k_1 t})}{k_1} + \frac{l_2 a_0 b_0{}^2(1-e^{-2k_2 t})}{k_2}\right\},$$

$$b = b_0 + \frac{\varepsilon}{2}\left\{\frac{r_2 b_0{}^3(1-e^{-2k_2 t})}{k_2} + \frac{l_1 a_0{}^2 b_0(1-e^{-2k_1 t})}{k_1}\right\},$$

$$\varphi_1 = \varphi_1(0) + \frac{\varepsilon}{2}\left\{\frac{m_1 a_0{}^2(1-e^{-2k_1 t})}{k_1} + \frac{m_2 b_0{}^2(1-e^{-2k_2 t})}{k_2}\right\} \qquad (4.1.1.12)$$

and $$\varphi_2 = \varphi_2(0) + \frac{\varepsilon}{2}\left\{\frac{s_2 b_0{}^2(1-e^{-2k_2 t})}{k_2} + \frac{s_1 a_0{}^2(1-e^{-2k_1 t})}{k_1}\right\}.$$

Therefore, the first approximate solution of the equation (4.1.1.1) is

$$x(t,\varepsilon) = a e^{-k_1 t}\cosh(\omega_1 t + \varphi_1) + b e^{-k_2 t}\cos(\omega_2 t + \varphi_2) + \varepsilon u_1, \qquad (4.1.1.13)$$

where a, b, φ_1 and φ_2 are given by (4.1.1.12) and u_1 is given by (4.1.1.11).

4.1.2 Discussion

By balancing harmonic terms and separating equation (3.1.5), we have five equations in which the variational equations contain the first harmonics and the correction terms contain harmonics with multiple arguments. These assumptions for the second order and third order differential equations certainly hold. But these assumptions for the fourth order differential equations do not hold sufficiently. When one of the eigen-values of the corresponding unperturbed equation is a linear combination of the other eigen-values, both the variational equations and the correction terms contain secular type terms. Then the solutions fail to give the desired results. In these cases, to obtain the desired results, the technique in [22, 25, 26] is necessary. Following the KBM method, an asymptotic method is developed to obtain the solutions of a fourth order damped oscillatory nonlinear differential equation with small nonlinearities, when out of the four eigen-values of the corresponding linear equations two are real and the other two are complex. For some values of k_1, k_2, ω_1, ω_2, φ_1, φ_2 and ε we have evaluated x from (4.1.1.13), in which a, b, φ_1 and φ_2 are evaluated from (4.1.1.12). The corresponding second solutions of (4.1.1.1) are calculated by a fourth order Runge-Kutta formula with a small time increment Δt=0.05. Both the results are plotted in Fig.(4.1.2.1) to Fig.(4.1.2.5) to show the comparisons between the analytical and numerical results. From the figures it is observed that the analytical solution represented by equation (4.1.1.13) along with the equations (4.1.1.11) and (4.1.1.12) agrees well with the numerical solution.

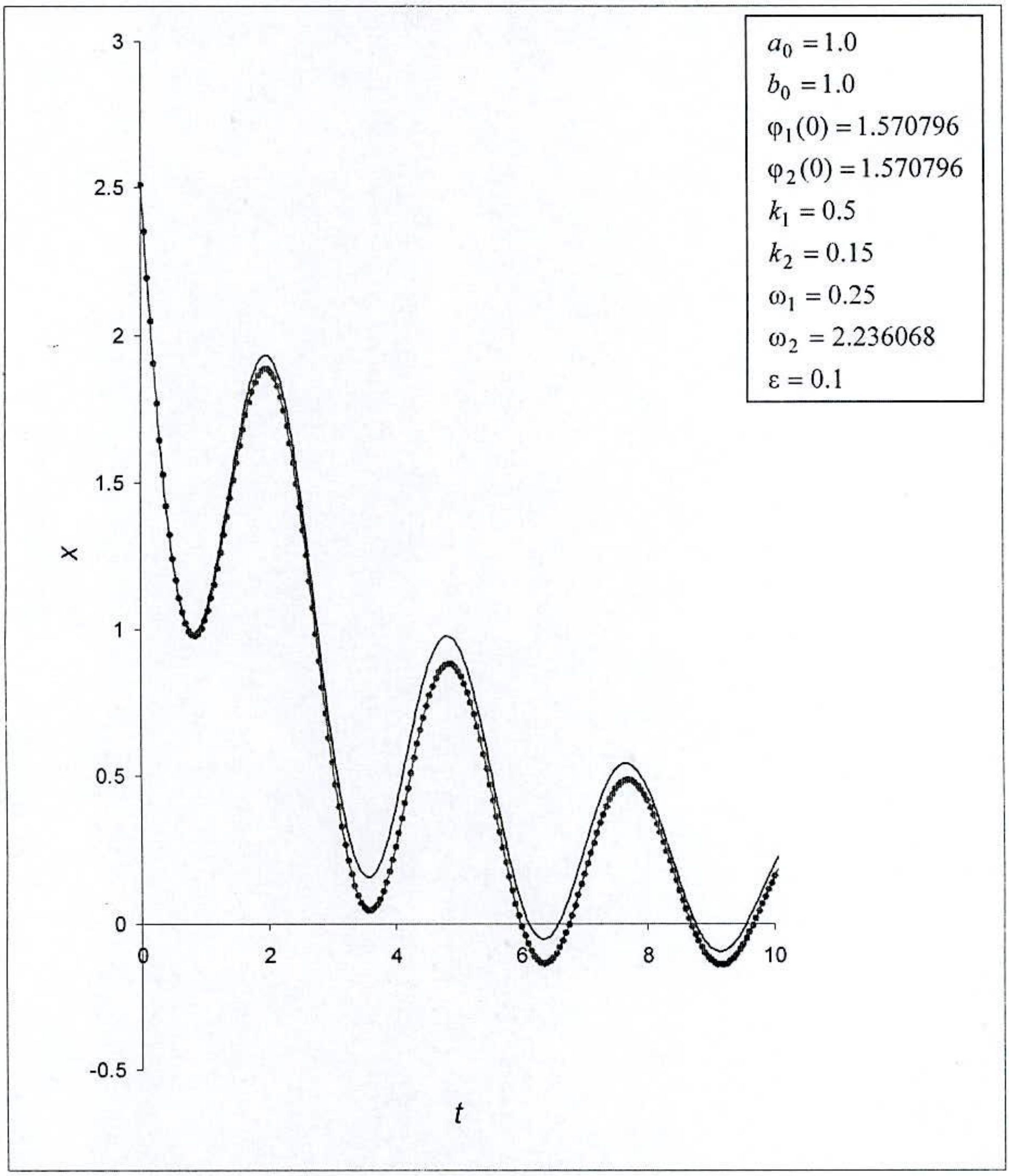

Fig. (4.1.2.1)

Fig.(4.1.2.1) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ○ ○ ○).

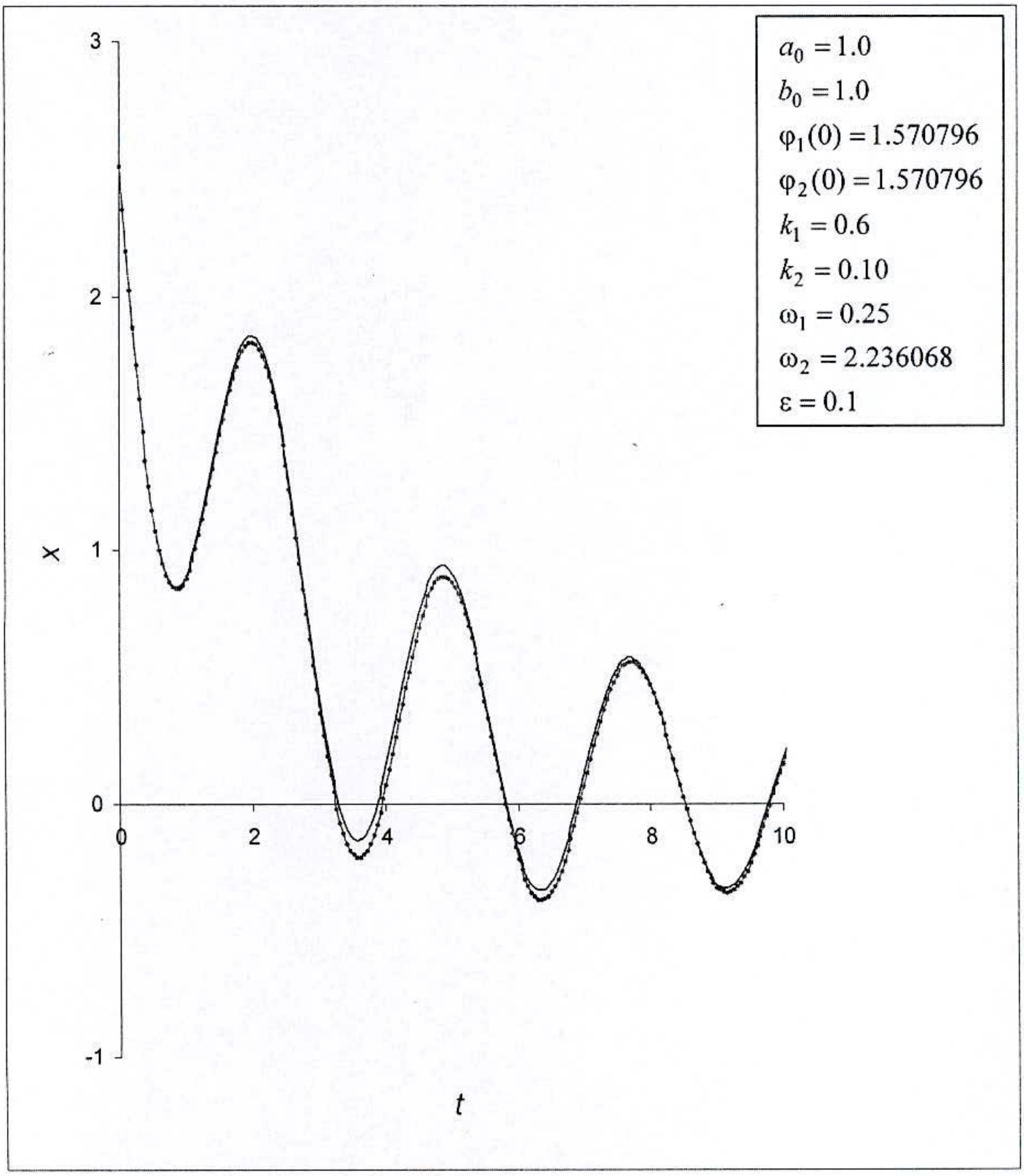

Fig.(4.1.2.2)

Fig.(4.1.2.2) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ∘ ∘ ∘).

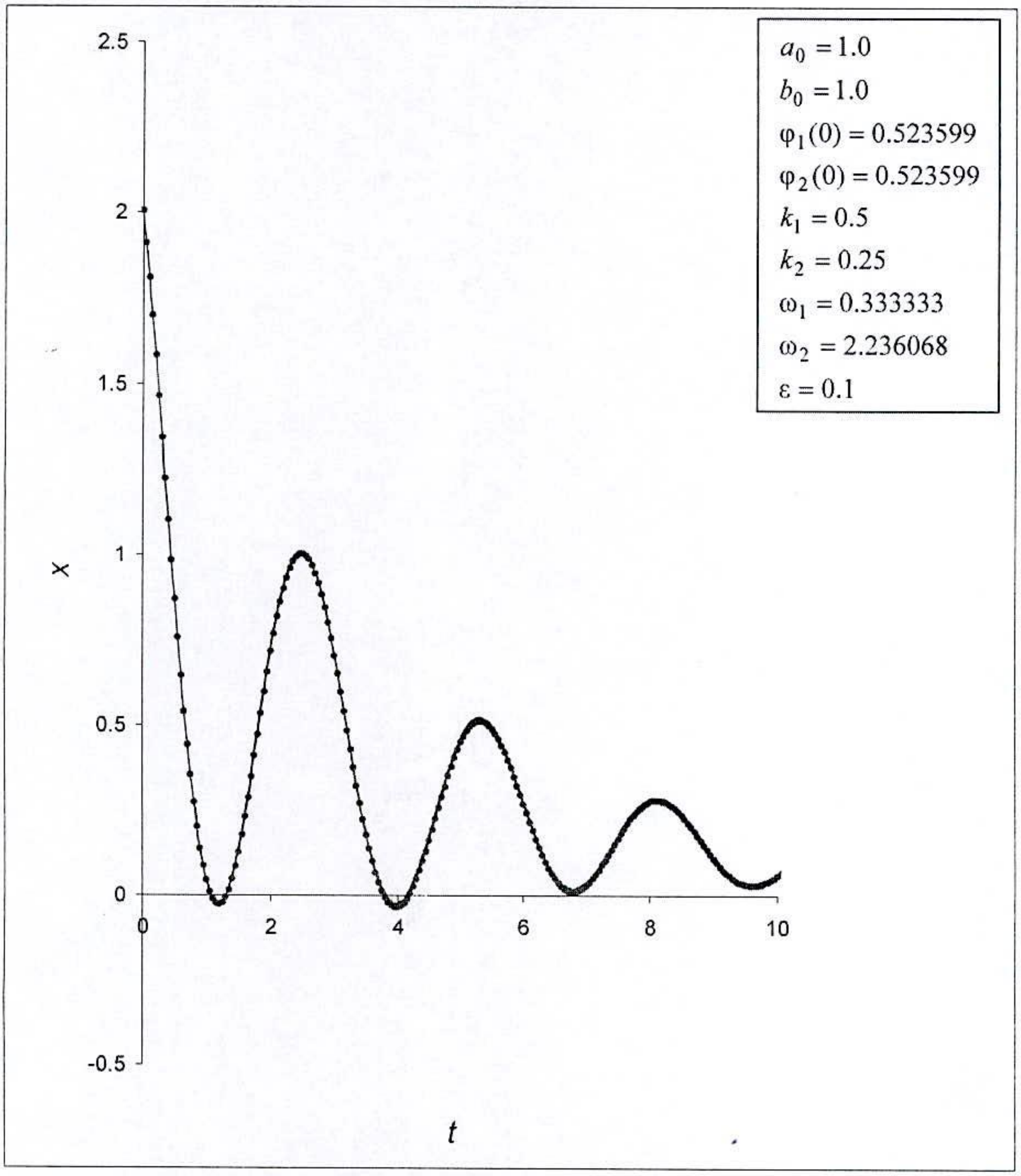

Fig.(4.1.2.3)

Fig.(4.1.2.3) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ○ ○ ○).

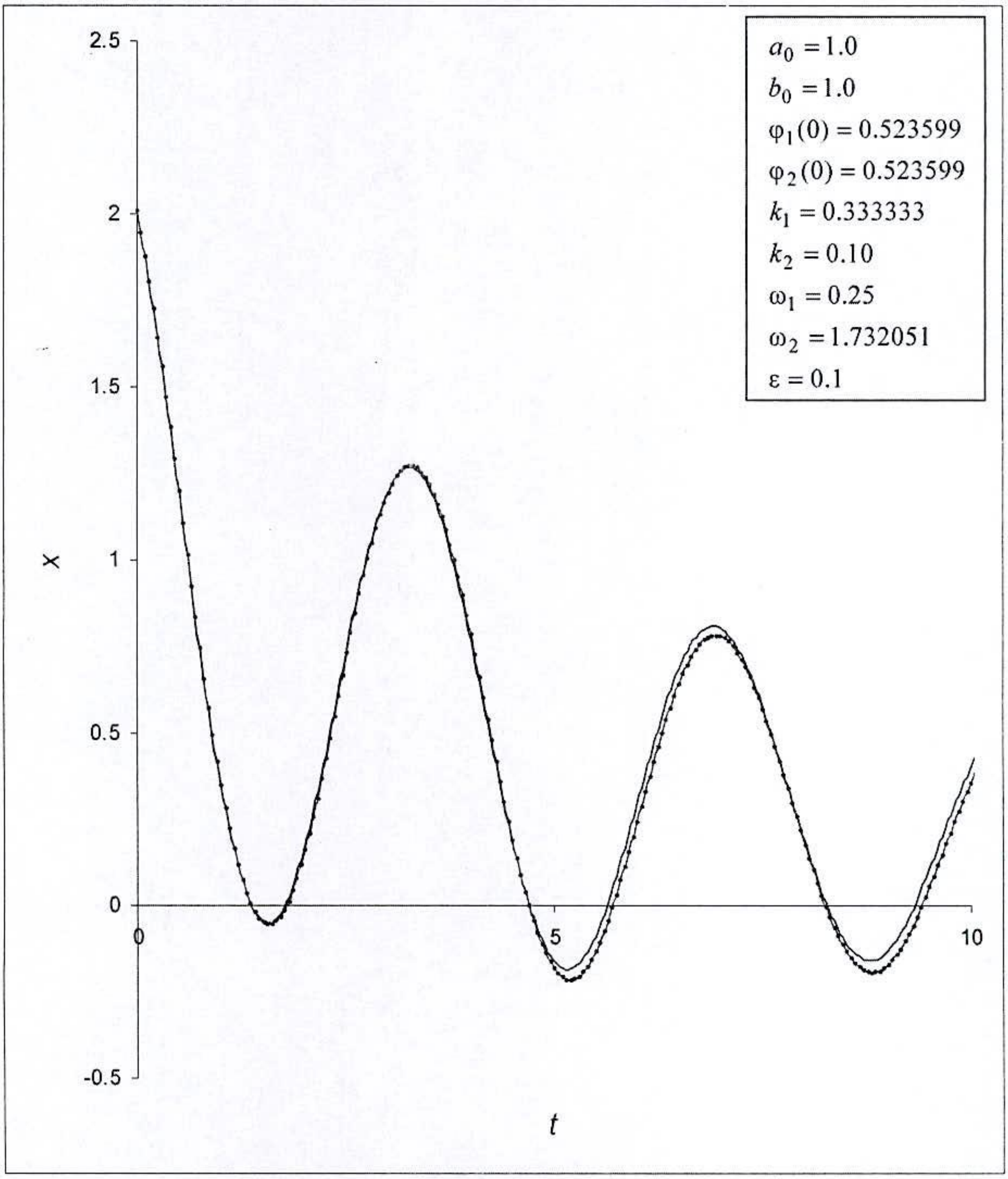

Fig.(4.1.2.4)

Fig.(4.1.2.4) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ∘ ∘ ∘).

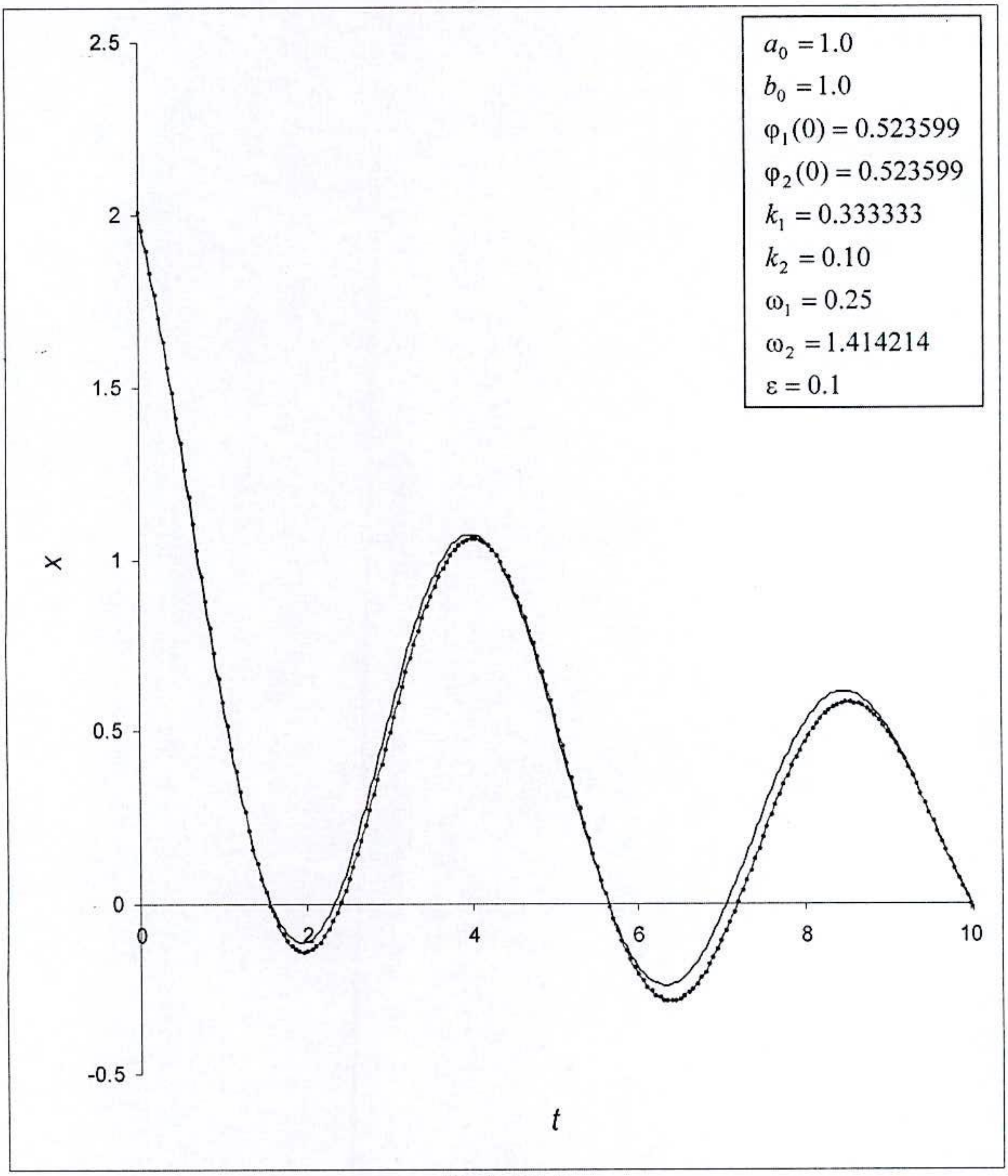

Fig.(4.1.2.5)

Fig.(4.1.2.5) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ○ ○ ○).

4.2 Asymptotic Solutions of Fourth Order Near Critically Damped Non-Oscillatory Nonlinear Systems

4.2.1 Example

As an example to solve fourth order near critically damped nonlinear system, we consider the fourth order nonlinear differential equation of the form

$$\frac{d^4x}{dt^4}+e_1\frac{d^3x}{dt^3}+e_2\frac{d^2x}{dt^2}+e_3\frac{dx}{dt}+e_4x=-\varepsilon x^3 \tag{4.2.1.1}$$

The unperturbed solution of (4.2.1.1) as prescribed in (3.2.2), we have

$$x_0=a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t}+\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)$$

Also, from (3.2.3) the first approximate solution is

$$x(t,\varepsilon)=a_3(t)e^{-\lambda_3 t}+a_4(t)e^{-\lambda_4 t}+\frac{1}{2}a_1(t)(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2(t)\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)$$
$$+\varepsilon u_1(a_1,a_2,a_3,a_4,t)$$

For equation (4.2.1.1), we have, $f=x^3$ and

$$f^{(0)}=\left\{a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t}+\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^3$$

or,

$$f^{(0)}=\left(a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t}\right)^3+3\left(a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t}\right)^2$$
$$\times\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}$$
$$+3\left(a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t}\right)\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^2 \tag{4.2.1.2}$$
$$+\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^3$$

Now equating the coefficients of powers of $\left(\dfrac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)$, we have

$$(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1 = -\sum_{r=2}^{n} F_r(a_3 e^{-\lambda_3 t}, a_4 e^{-\lambda_4 t}) \times \left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^r$$

i.e.,

$$(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1 = -[3(a_3 e^{-\lambda_3 t}+a_4 e^{-\lambda_4 t}) \times \left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^2 + \left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^3] \tag{4.2.1.3}$$

$$\begin{aligned}
&e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3 \\
&+e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4 \\
&+\frac{1}{2}\left\{\begin{matrix} e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4) \\ +e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_3)\end{matrix}\right\}A_1 \\
&+(D+\lambda_4)\left\{e^{-\lambda_1 t}(\lambda_1-\lambda_2-\frac{3}{2}D)+e^{-\lambda_2 t}(\lambda_2-\lambda_3-\frac{3}{2}D)\right\}A_2 \\
&+\left(\frac{(\lambda_1 e^{-\lambda_1 t}-\lambda_2 e^{-\lambda_2 t})}{(\lambda_1-\lambda_2)}\right)\times D(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2})A_2 \\
&=-[(a_3 e^{-\lambda_3 t}+a_4 e^{-\lambda_4 t})^3 \\
&\quad +3(a_3 e^{-\lambda_3 t}+a_4 e^{-\lambda_4 t})^2 \frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})]
\end{aligned} \tag{4.2.1.4}$$

and $$(\lambda_4+D)\times D(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2})A_2 = -3a_2(a_3 e^{-\lambda_3 t}+a_4 e^{-\lambda_4 t})^2 \tag{4.2.1.5}$$

Solving equation (4.2.1.5), we obtain

$$A_2 = a_2[n_1 a_3{}^2 e^{-2\lambda_3 t} + n_2 a_3 a_4 e^{-(\lambda_3+\lambda_4)t} + n_3 a_4{}^2 e^{-2\lambda_4 t}], \tag{4.2.1.6}$$

where

$$n_1 = \frac{3}{\lambda_3(\lambda_1+\lambda_2+2\lambda_3)(2\lambda_3-\lambda_4)},$$

$$n_2 = \frac{12}{\lambda_3(\lambda_3+\lambda_4)(\lambda_1+\lambda_2+2\lambda_4)},$$

$$n_3 = \frac{3}{\lambda_4^2(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)}.$$

Now substituting the value of A_2 from equation (4.2.1.6) into equation (4.2.1.4), we obtain

$$\begin{aligned}
&e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3 \\
&+e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4 \\
&+\frac{1}{2}\left\{\begin{matrix} e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4) \\ +e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_3) \end{matrix}\right\}A_1 \\
&=[a_2 n_1\{(\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)+(1-\lambda_2 t) \\
&\qquad \times(\lambda_1+\lambda_2+2\lambda_3)\lambda_3-\frac{3}{2}a_1]a_3^2 e^{(\lambda_1+2\lambda_3)t} \\
&+[\frac{1}{2}a_2 n_2\{(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3+3\lambda_4)+(1-\lambda_2 t) \\
&\qquad \times(2\lambda_4^2+2\lambda_3\lambda_4+\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_4)\lambda_3 \\
&\qquad -3a_1]a_3 a_4 e^{(\lambda_1+\lambda_3+\lambda_4)t} \\
&+[a_2 n_3\{(\lambda_1+\lambda_4)(\lambda_1-\lambda_3+3\lambda_4) \\
&\qquad +(1-\lambda_2 t)(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)\lambda_4-\frac{3}{2}a_1]a_4^2 e^{(\lambda_1+2\lambda_4)t} \\
&+[a_2 n_1\{(\lambda_2+2\lambda_3)(\lambda_2+2\lambda_3-\lambda_4)-\frac{3}{2}a_1]a_3^2 e^{(\lambda_2+2\lambda_3)t} \\
&+[\frac{1}{2}a_2 n_2\{(\lambda_2+\lambda_3)(2\lambda_2+\lambda_3+3\lambda_4)-3a_1]a_3 a_4 e^{(\lambda_2+\lambda_3+\lambda_4)t} \\
&+[a_2 n_3\{(\lambda_2+\lambda_4)(\lambda_2-\lambda_3+3\lambda_4)-\frac{3}{2}a_1]a_4^2 e^{(\lambda_2+2\lambda_4)t} \\
&-[a_3^3 e^{-3\lambda_3 t}+3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t}+3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t}+a_4^3 e^{-3\lambda_4 t}]
\end{aligned} \tag{4.2.1.7}$$

In order to separate the equation (4.2.1.7) for determining the unknown functions A_1, A_3 and A_4, we consider the most important relations among the eigen-values as limit $\lambda_1 \to \lambda_2$ and $\lambda_3 \approx 3\lambda_4$ (Akbar *et al.* [5], Alam [16, 24]). It is interesting to note that our solution approaches toward a critically damped solution (found by Alam [24]) if $\lambda_1 \to \lambda_2$. However, the equation (4.2.1.7) has no exact solution unless $\lambda_1 \to \lambda_2$. Under these imposed conditions and by equating like terms on the both sides of (4.2.1.7), we obtain

$$\begin{aligned} & e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4)A_1 \\ & = -a_2 a_3^2 n_1 \lambda_2 \lambda_3 (\lambda_1+\lambda_2+2\lambda_3) t e^{-(\lambda_1+2\lambda_3)t} \\ & \quad -\frac{1}{2} a_2 a_3 a_4 n_2 \lambda_2 (2\lambda_4^2 + 2\lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_4) \\ & \quad \times t e^{-(\lambda_1+\lambda_3+\lambda_4)t} - a_2 a_4^2 n_3 \lambda_2 \lambda_4 (\lambda_1+\lambda_2-2\lambda_3+4\lambda_4) t e^{-(\lambda_1+2\lambda_4)t} \end{aligned} \tag{4.2.1.8}$$

$$\begin{aligned} & e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3 \\ & = [a_2 n_1 \{(\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)+\lambda_3(\lambda_1+\lambda_2+2\lambda_3)\} - \frac{3}{2}a_1] a_3^2 e^{-(\lambda_1+2\lambda_3)t} \\ & \quad + [\frac{1}{2} a_2 n_2 \{(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3+3\lambda_4) \\ & \qquad + (2\lambda_4^2 + 2\lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\} - 3a_1] a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} \\ & \quad + [a_2 n_3 \{(\lambda_1+\lambda_4)(\lambda_1-\lambda_3+3\lambda_4)+\lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)\} \\ & \qquad - \frac{3}{2}a_1] a_4^2 e^{-(\lambda_1+2\lambda_4)t} \\ & \quad + [a_2 n_1 (\lambda_2+2\lambda_3)(\lambda_2+2\lambda_3-\lambda_4) - \frac{3}{2}a_1] a_3^2 e^{-(\lambda_2+2\lambda_3)t} \\ & \quad + [\frac{1}{2} a_2 n_2 (\lambda_2+\lambda_3)(2\lambda_2+\lambda_3+3\lambda_4) - 3a_1] a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} \\ & \quad + [a_2 n_3 (\lambda_2+\lambda_4)(\lambda_2-\lambda_3+3\lambda_4) - \frac{3}{2}a_1] a_4^2 e^{-(\lambda_2+2\lambda_4)t} \\ & \quad - [a_3^3 e^{-3\lambda_3 t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t}] \end{aligned} \tag{4.2.1.9}$$

and

$$e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4 = 0 \tag{4.2.1.10}$$

The particular solutions of equations (4.2.1.8), (4.2.1.9), (4.2.1.10) yield respectively

$$A_1 = I_1 a_2 a_3^2 t e^{-(\lambda_1-\lambda_2+2\lambda_3)t} + I_2 a_2 a_3^2 t e^{-(\lambda_1-\lambda_2+2\lambda_3)t} + I_3 a_2 a_3 a_4 t e^{-(\lambda_1-\lambda_2+\lambda_3+\lambda_4)t} + I_4 a_2 a_3 a_4 t e^{-(\lambda_1-\lambda_2+\lambda_3+\lambda_4)t} + I_5 a_2 a_4^2 t e^{-(\lambda_1-\lambda_2+2\lambda_4)t} + I_6 a_2 a_4^2 t e^{-(\lambda_1-\lambda_2+2\lambda_4)t} \quad (4.2.1.11)$$

$$A_3 = (M_1 a_2 + M_2 a_1) a_3^2 e^{-(\lambda_1+\lambda_3)t} + (M_3 a_2 + M_4 a_1) a_3 a_4 e^{-(\lambda_1+\lambda_4)t} + (M_5 a_2 + M_6 a_1) a_4^2 e^{-(\lambda_1+2\lambda_4-\lambda_3)t} + (M_7 a_2 + M_8 a_1) \times a_3^2 e^{-(\lambda_2+\lambda_3)t} + (M_9 a_2 + M_{10} a_1) a_3 a_4 e^{-(\lambda_2+\lambda_4)t} + (M_{11} a_2 + M_{12} a_1) a_4^2 e^{-(\lambda_2+2\lambda_4-\lambda_3)t} + M_{13} a_3^3 e^{-2\lambda_3 t} + M_{14} a_3^2 a_4 e^{-(\lambda_3+\lambda_4)t} + M_{15} a_3 a_4^2 e^{-2\lambda_4 t} + M_{16} a_4^3 e^{-(3\lambda_4-\lambda_3)t} \quad (4.2.1.12)$$

and

$$A_4 = 0, \quad (4.2.1.13)$$

where

$$I_1 = -\frac{r_1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)},$$

$$I_2 = -\frac{r_1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)}\left(\frac{1}{2\lambda_3}+\frac{1}{(\lambda_1+\lambda_3)}+\frac{1}{(\lambda_1+2\lambda_3-\lambda_4)}\right),$$

$$I_3 = -\frac{r_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)},$$

$$I_4 = -\frac{r_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)}\left(\frac{1}{(\lambda_1+\lambda_3)}+\frac{1}{(\lambda_1+\lambda_4)}+\frac{1}{(\lambda_3+\lambda_4)}\right),$$

$$I_5 = -\frac{r_3}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_3)},$$

$$I_6 = -\frac{r_3}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_3)}\left(\frac{1}{2\lambda_4}+\frac{1}{(\lambda_1+\lambda_4)}+\frac{1}{(\lambda_1+2\lambda_4-\lambda_3)}\right),$$

$$r_1 = -n_1\lambda_2\lambda_3(\lambda_1+\lambda_2+2\lambda_3),$$

$$r_2 = -\frac{1}{2}n_2\lambda_2(2\lambda^2{}_4+2\lambda_3\lambda_4+\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_4),$$

$$r_3 = -n_3\lambda_2\lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4),$$

$$n_1 = \frac{3}{\lambda_3(\lambda_1+\lambda_2+2\lambda_3)(2\lambda_3-\lambda_4)},$$

$$n_2 = \frac{12}{\lambda_3(\lambda_3+\lambda_4)(\lambda_1+\lambda_2+2\lambda_4)},$$

$$n_3 = \frac{3}{\lambda_4{}^2(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)},$$

$$M_1 = -\frac{m_1}{2\lambda_3(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+2\lambda_3-\lambda_4)},$$

$$M_2 = -\frac{m_2}{2\lambda_3(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+2\lambda_3-\lambda_4)},$$

$$M_3 = -\frac{l_1}{(\lambda_3+\lambda_4)(\lambda_1+\lambda_3+\lambda_4-\lambda_2)(\lambda_1+\lambda_3)},$$

$$M_4 = -\frac{l_2}{(\lambda_3+\lambda_4)(\lambda_1+\lambda_3+\lambda_4-\lambda_2)(\lambda_1+\lambda_3)},$$

$$M_5 = -\frac{p_1}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_2)},$$

$$M_6 = -\frac{p_2}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_2)},$$

$$M_7 = -\frac{q_1}{2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)(\lambda_2+2\lambda_3-\lambda_4)},$$

$$M_8 = -\frac{q_2}{2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)(\lambda_2+2\lambda_3-\lambda_4)},$$

$$M_9 = -\frac{h_1}{(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)(\lambda_2+\lambda_3+\lambda_4-\lambda_1)},$$

$$M_{10} = -\frac{h_2}{(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)(\lambda_2+\lambda_3+\lambda_4-\lambda_1)},$$

$$M_{11} = -\frac{s_1}{2\lambda_4(\lambda_2+\lambda_4)(\lambda_2+2\lambda_4-\lambda_1)},$$

$$M_{12} = -\frac{s_2}{2\lambda_4(\lambda_2+\lambda_4)(\lambda_2+2\lambda_4-\lambda_1)},$$

$$M_{13} = \frac{1}{(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)(3\lambda_3 - \lambda_4)},$$

$$M_{14} = \frac{3}{2\lambda_3(3\lambda_3 + \lambda_4 - \lambda_1)(2\lambda_3 + \lambda_4 - \lambda_2)},$$

$$M_{15} = \frac{3}{(\lambda_3 + \lambda_4)(2\lambda_4 + \lambda_3 - \lambda_1)(2\lambda_4 + \lambda_3 - \lambda_2)},$$

$$M_{16} = \frac{1}{2\lambda_4(3\lambda_4 - \lambda_1)(3\lambda_4 - \lambda_2)},$$

$$m_1 = n_1\{(\lambda_1 + 2\lambda_3)(\lambda_1 + 2\lambda_3 - \lambda_4) + \lambda_3(\lambda_1 + \lambda_2 + 2\lambda_3)\}, \qquad m_2 = -\frac{3}{2},$$

$$l_1 = \frac{1}{2}n_2\left\{\begin{matrix}(\lambda_1 + \lambda_3)(2\lambda_1 + \lambda_3 + 3\lambda_4) \\ + (2\lambda^2{}_4 + 2\lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\end{matrix}\right\}, \qquad l_2 = -3,$$

$$p_1 = n_3\{(\lambda_1 + \lambda_4)(\lambda_1 - \lambda_3 + 3\lambda_4) + \lambda_4(\lambda_1 + \lambda_2 - 2\lambda_3 + 4\lambda_4)\}, \qquad p_2 = -\frac{3}{2},$$

$$q_1 = n_1(\lambda_2 + 2\lambda_3)(\lambda_2 + 2\lambda_3 - \lambda_4), \qquad q_2 = -\frac{3}{2},$$

$$h_1 = \frac{1}{2}n_2(\lambda_2 + \lambda_3)(2\lambda_2 + \lambda_3 + 3\lambda_4), \qquad h_2 = -3,$$

$$s_1 = n_3(\lambda_2 + \lambda_4)(\lambda_2 - \lambda_3 + 3\lambda_4), \qquad s_2 = -\frac{3}{2},$$

The solution of the equation (4.2.1.3) is

$$\begin{aligned} u_1 = &-3a_3e^{-(2\lambda_1+\lambda_3)t}\left[d_0a_1{}^2 + 2d_1a_1a_2 + d_3a_2{}^2 + (d_2a_2{}^2 - 2d_0a_1a_2)t + d_0a_2{}^2t^2\right] \\ &- 3a_4e^{-(2\lambda_1+\lambda_4)t}\left[d_4a_1{}^2 + 2d_5a_1a_2 + d_7a_2{}^2 + (d_6a_2{}^2 - 2d_4a_1a_2)t + d_4a_2{}^2t^2\right] \\ &- e^{-3\lambda_1 t}\Big[d_8a_1{}^3 + 3a_1a_2(d_9a_1 + d_{10}a_2) + d_{12}a_2{}^3 + a_2(d_{11}a_2{}^2 - 3d_8a_1{}^2 \\ &\quad - 6d_9a_1a_2)t + 3a_2{}^2(d_8a_1 + d_9a_2)t^2 - d_8a_2{}^3t^3\Big] \end{aligned} \tag{4.2.1.14}$$

where

$$d_0 = \frac{1}{2\lambda_1(\lambda_1 + \lambda_3)^2(2\lambda_1 + \lambda_3 - \lambda_4)},$$

$$d_1 = -\frac{1}{2\lambda_1(\lambda_1+\lambda_3)^2(2\lambda_1+\lambda_3-\lambda_4)}\left(\frac{1}{2\lambda_1}+\frac{2}{(\lambda_1+\lambda_3)}+\frac{1}{(2\lambda_1+\lambda_3-\lambda_4)}\right),$$

$$d_2 = \frac{1}{2\lambda_1(\lambda_1+\lambda_3)^2(2\lambda_1+\lambda_3-\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{4}{(\lambda_1+\lambda_3)}+\frac{2}{(2\lambda_1+\lambda_3-\lambda_4)}\right),$$

$$d_3 = \frac{1}{2\lambda_1(\lambda_1+\lambda_3)^2(2\lambda_1+\lambda_3-\lambda_4)}$$
$$\times\left[\frac{1}{2\lambda_1{}^2}+\frac{2}{\lambda_1(\lambda_1+\lambda_3)}+\frac{6}{(\lambda_1+\lambda_3)^2}+\frac{2}{(2\lambda_1+\lambda_3-\lambda_4)^2}+\frac{1}{(2\lambda_1+\lambda_3-\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{4}{(\lambda_1+\lambda_3)}\right)\right],$$

$$d_4 = \frac{1}{2\lambda_1(\lambda_1+\lambda_4)^2(2\lambda_1-\lambda_3+\lambda_4)},$$

$$d_5 = -\frac{1}{2\lambda_1(\lambda_1+\lambda_4)^2(2\lambda_1-\lambda_3+\lambda_4)}\left(\frac{1}{2\lambda_1}+\frac{2}{(\lambda_1+\lambda_4)}+\frac{1}{(2\lambda_1-\lambda_3+\lambda_4)}\right),$$

$$d_6 = \frac{1}{2\lambda_1(\lambda_1+\lambda_4)^2(2\lambda_1-\lambda_3+\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{4}{(\lambda_1+\lambda_4)}+\frac{2}{(2\lambda_1-\lambda_3+\lambda_4)}\right),$$

$$d_7 = \frac{1}{2\lambda_1(\lambda_1+\lambda_4)^2(2\lambda_1-\lambda_3+\lambda_4)}$$
$$\times\left[\frac{1}{2\lambda_1{}^2}+\frac{2}{\lambda_1(\lambda_1+\lambda_4)}+\frac{6}{(\lambda_1+\lambda_4)^2}+\frac{2}{(2\lambda_1-\lambda_3+\lambda_4)^2}+\frac{1}{(2\lambda_1-\lambda_3+\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{4}{(\lambda_1+\lambda_4)}\right)\right],$$

$$d_8 = \frac{1}{4\lambda_1{}^2(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)},$$

$$d_9 = -\frac{1}{4\lambda_1{}^2(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{1}{3\lambda_1-\lambda_3}+\frac{1}{3\lambda_1-\lambda_4}\right),$$

$$d_{10}=\frac{1}{4\lambda_1^2(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\left[\frac{2}{(3\lambda_1-\lambda_3)^2}+\frac{2}{(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\right.$$
$$\left.+\frac{2}{(3\lambda_1-\lambda_4)^2}+\frac{2}{(3\lambda_1-\lambda_3)}+\frac{2}{(3\lambda_1-\lambda_4)}+\frac{3}{2\lambda_1^2}\right],$$

$$d_{11}=-\frac{1}{4\lambda_1^2(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\left[\frac{6}{(3\lambda_1-\lambda_3)^2}+\frac{6}{(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}+\frac{9}{2\lambda_1^2}\right.$$
$$\left.+\frac{6}{(3\lambda_1-\lambda_4)^2}+\frac{6}{\lambda_1(3\lambda_1-\lambda_3)}+\frac{6}{\lambda_1(3\lambda_1-\lambda_4)}\right],$$

$$d_{12}=-\left[\frac{6}{(3\lambda_1-\lambda_3)^3}+\frac{6}{(3\lambda_1-\lambda_3)^2(3\lambda_1-\lambda_4)}+\frac{6}{(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)^2}+\frac{6}{(3\lambda_1-\lambda_4)^3}\right.$$
$$+\frac{1}{\lambda_1}\left\{\frac{6}{(3\lambda_1-\lambda_3)^2}+\frac{6}{(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}+\frac{6}{(3\lambda_1-\lambda_4)^2}\right\}$$
$$\left.+\frac{3}{2\lambda_1^2}\left(\frac{3}{3\lambda_1-\lambda_3}+\frac{3}{3\lambda_1-\lambda_4}\right)+\frac{3}{\lambda_1^3}\right].$$

Putting the values of A_2, A_1, A_3 and A_4 from equations (4.2.1.6), (4.2.1.11), (4.2.1.12) and (4.2.1.13) into the equation (3.2.4), we have

$$\frac{da_1(t)}{dt}=\varepsilon\begin{bmatrix} I_1a_2a_3^2te^{-(\lambda_1-\lambda_2+2\lambda_3)t}+I_2a_2a_3^2te^{-(\lambda_1-\lambda_2+2\lambda_3)t}\\ +I_3a_2a_3a_4te^{-(\lambda_1-\lambda_2+\lambda_3+\lambda_4)t}+I_4a_2a_3a_4te^{-(\lambda_1-\lambda_2+\lambda_3+\lambda_4)t}\\ +I_5a_2a_4^2te^{-(\lambda_1-\lambda_2+2\lambda_4)t}+I_6a_2a_4^2te^{-(\lambda_1-\lambda_2+2\lambda_4)t}\end{bmatrix}$$

$$\frac{da_2(t)}{dt}=\varepsilon a_2[n_1a_3^2e^{-2\lambda_3t}+n_2a_3a_4e^{-(\lambda_3+\lambda_4)t}+n_3a_4^2e^{-2\lambda_4t}],$$

$$\frac{da_3(t)}{dt} = \varepsilon\Big[(M_1 a_2 + M_2 a_1)a_3{}^2 e^{-(\lambda_1+\lambda_3)t} + (M_3 a_2 + M_4 a_1)a_3 a_4 e^{-(\lambda_1+\lambda_4)t}$$
$$+ (M_5 a_2 + M_6 a_1)a_4{}^2 e^{-(\lambda_1+2\lambda_4-\lambda_3)t} + (M_7 a_2 + M_8 a_1)$$
$$\times a_3{}^2 e^{-(\lambda_2+\lambda_3)t} + (M_9 a_2 + M_{10} a_1)a_3 a_4 e^{-(\lambda_2+\lambda_4)t} \qquad (4.2.1.15)$$
$$+ (M_{11} a_2 + M_{12} a_1)a_4{}^2 e^{-(\lambda_2+2\lambda_4-\lambda_3)t} + M_{13} a_3{}^3 e^{-2\lambda_3 t}$$
$$+ M_{14} a_3{}^2 a_4 e^{-(\lambda_3+\lambda_4)t} + M_{15} a_3 a_4{}^2 e^{-2\lambda_4 t} + M_{16} a_4{}^3 e^{-(3\lambda_4-\lambda_3)t}\Big]$$

and

$$\frac{da_4(t)}{dt} = 0$$

The functions $\frac{da_1(t)}{dt}$, $\frac{da_2(t)}{dt}$, $\frac{da_3(t)}{dt}$ and $\frac{da_4(t)}{dt}$ are slowly varying functions of time t, because each of these functions varies as the small parameter ε, and are thus almost constant. Following Murty and Deekshatulu [61], Murty *et al.* [62], we assume that a_i (i=1, 2, 3, 4) presented in the right hand side in the relations of equation (4.2.1.15), are constants. Thus, by solving the relations of the equation (4.2.1.15), we obtain

$$a_1(t) = a_1(0) + \varepsilon\Big[\ a_2(0)\,a_3{}^2(0) \times \left\{\left(\frac{I_2\left(1-e^{(-\lambda_1+\lambda_2-2\lambda_3)t}\right) - I_1\left(t\,e^{(-\lambda_1+\lambda_2-2\lambda_3)t} + \frac{e^{(-\lambda_1+\lambda_2-2\lambda_3)t}-1}{\lambda_1-\lambda_2+2\lambda_3}\right)}{\lambda_1-\lambda_2+2\lambda_3}\right)\right\}$$

$$+ a_2(0)a_3(0)a_4(0) \times \left\{\frac{\left(I_4\left(1-e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t}\right) - I_3\left(t\,e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t} + \frac{e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t}-1}{\lambda_1-\lambda_2+\lambda_3+\lambda_4}\right)\right)}{\left(\lambda_1-\lambda_2+\lambda_3+\lambda_4\right)}\right\}$$

$$+ a_2(0)a_4{}^2(0) \times \left\{\frac{\left(I_6\left(1-e^{(-\lambda_1+\lambda_2-2\lambda_4)t}\right) - I_5\left(t\,e^{(-\lambda_1+\lambda_2-2\lambda_4)t} + \frac{e^{(-\lambda_1+\lambda_2-2\lambda_4)t}-1}{\lambda_1-\lambda_2+2\lambda_4}\right)\right)}{\left(\lambda_1-\lambda_2+2\lambda_4\right)}\right\}\Big],$$

$$a_2(t) = a_2(0) + \varepsilon a_2(0)\left[n_1 a_3{}^2(0)\left\{\left(\frac{1-e^{-2\lambda_3 t}}{2\lambda_3}\right)\right\} + n_2 a_3(0)a_4(0)\left\{\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right)\right\} + n_3 a_4{}^2(0)\left\{\left(\frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\right)\right\}\right],$$

$$
\begin{aligned}
a_3(t) &= a_3(0) \\
&+ \varepsilon\Bigg[a_3^2(0)\{M_1 a_2(0) + M_2 a_1(0)\}\left(\frac{1-e^{-(\lambda_1+\lambda_3)t}}{\lambda_1+\lambda_3}\right) \\
&+ a_3(0)a_4(0)\{M_3 a_2(0) + M_4 a_1(0)\}\left(\frac{1-e^{-(\lambda_1+\lambda_4)t}}{\lambda_1+\lambda_4}\right) \\
&+ a_4^2(0)\{M_5 a_2(0) + M_6 a_1(0)\}\left(\frac{1-e^{-(\lambda_1-\lambda_3+2\lambda_4)t}}{\lambda_1-\lambda_3+2\lambda_4}\right) \\
&+ a_3^2(0)\{M_7 a_2(0) + M_8 a_1(0)\}\left(\frac{1-e^{-(\lambda_2+\lambda_3)t}}{\lambda_2+\lambda_3}\right) \\
&+ a_3(0)a_4(0)\{M_9 a_2(0) + M_{10} a_1(0)\}\left(\frac{1-e^{-(\lambda_2+\lambda_4)t}}{\lambda_2+\lambda_4}\right) \\
&+ a_4^2(0)\{M_{11} a_2(0) + M_{12} a_1(0)\}\left(\frac{1-e^{-(\lambda_2-\lambda_3+2\lambda_4)t}}{\lambda_2-\lambda_3+2\lambda_4}\right) \\
&+ a_3^3(0)M_{13}\left(\frac{1-e^{-2\lambda_3 t}}{2\lambda_3}\right) + a_3^2(0)a_4(0)M_{14}\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right) \qquad (4.2.1.16) \\
&+ a_3(0)a_4^2(0)M_{15}\left(\frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\right) + a_4^3(0)M_{16}\left(\frac{1-e^{-(3\lambda_4-\lambda_3)t}}{3\lambda_4-\lambda_3}\right)\Bigg]
\end{aligned}
$$

and

$$a_4(t) = a_4(0)$$

Therefore, we obtain the first approximate solution of the equation (4.2.1.1) as

$$x(t,\varepsilon) = a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} + \frac{1}{2}a_1(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + \varepsilon u_1, \qquad (4.2.1.17)$$

where a_1, a_2, a_3 and a_4 are given by the equations of (4.2.1.16) and u_1 is given by the equation (4.2.1.14).

4.2.2 Discussion

An asymptotic method, based on the theory of Krylov-Bogoliubov-Mitropolskii (KBM), is developed for solving fourth order near critically damped systems under some specific conditions with small nonlinearities, when all of the four eigen-values of the corresponding linear equation are real. The relations, limit $\lambda_1 \to \lambda_2$ and $\lambda_3 \approx 3\lambda_4$ among the eigen-values are imposed to solve the system. We have compared the approximate solution obtained by using our proposed perturbation method to the numerical solution to test the performance of our approximate solution. Firstly, $x(t,\varepsilon)$ is calculated by (4.2.1.17) by imposing the conditions that limit $\lambda_1 \to \lambda_2$ and $\lambda_3 \approx 3\lambda_4$ in which a_1, a_2, a_3 and a_4 are calculated by the equation (4.2.1.16) and u_1 is calculated by the equation (4.2.1.14) for different sets of initial conditions and for various values of t. Secondly, a corresponding numerical solution of (4.2.1.1) is computed by fourth order Runge-Kutta method. The approximate analytical solutions and numerical solutions are plotted in the figures (From Fig.(4.2.2.1) to Fig.(4.2.2.5). From these figures, we observed that the analytical solutions and the numerical solutions are in good agreement.

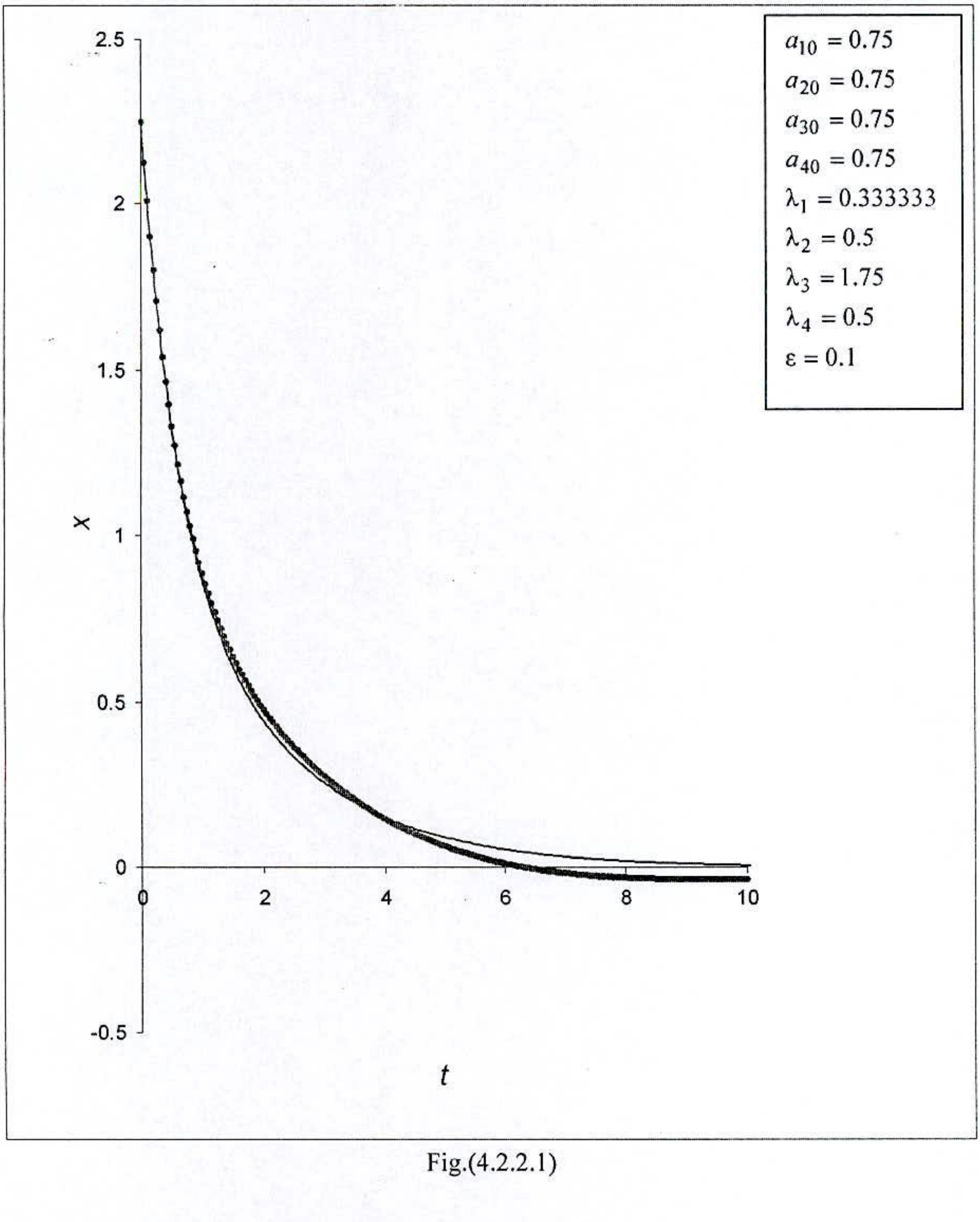

Fig.(4.2.2.1)

Fig.(4.2.2.1) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ○ ○ ○).

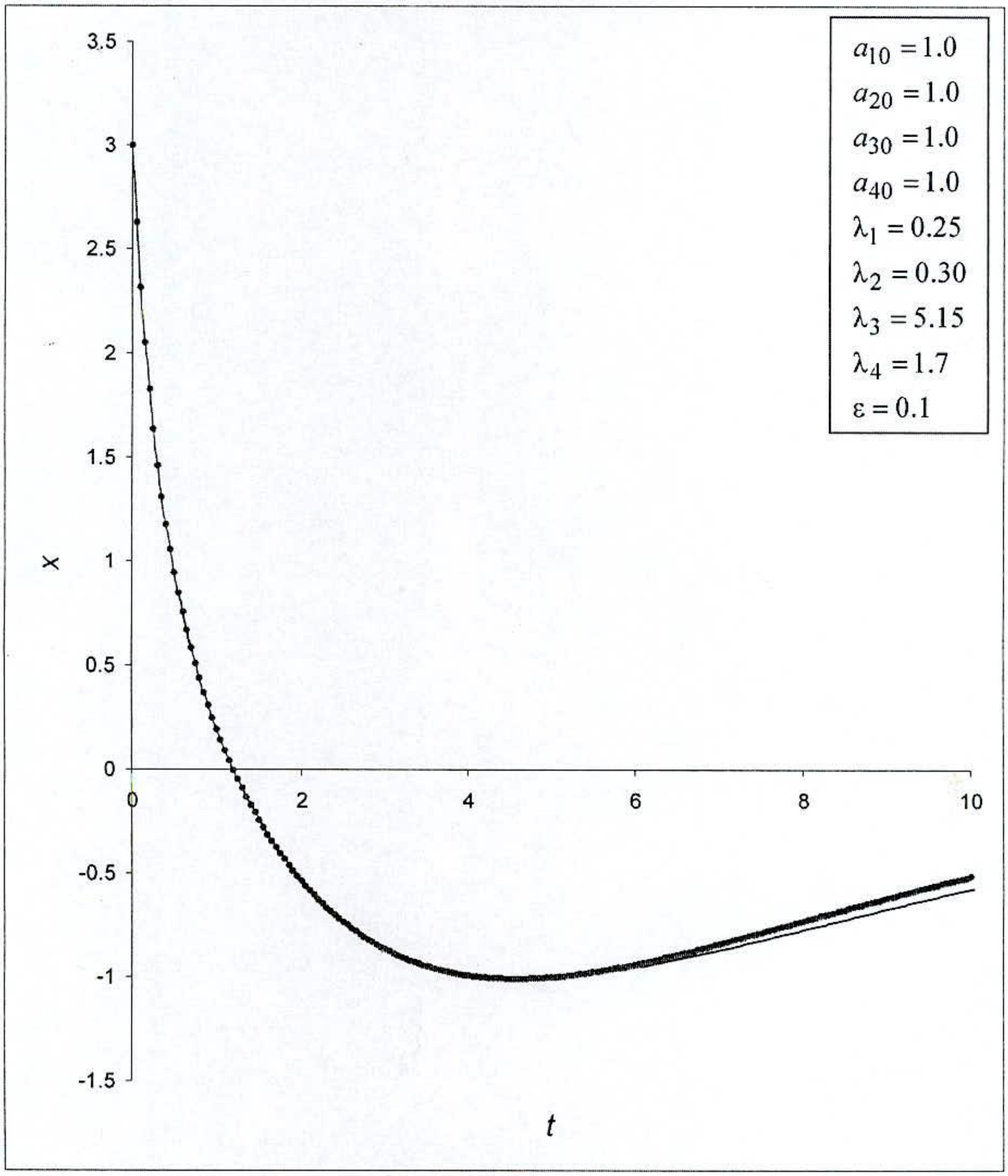

Fig.(4.2.2.2)

Fig.(4.2.2.2) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ○ ○ ○).

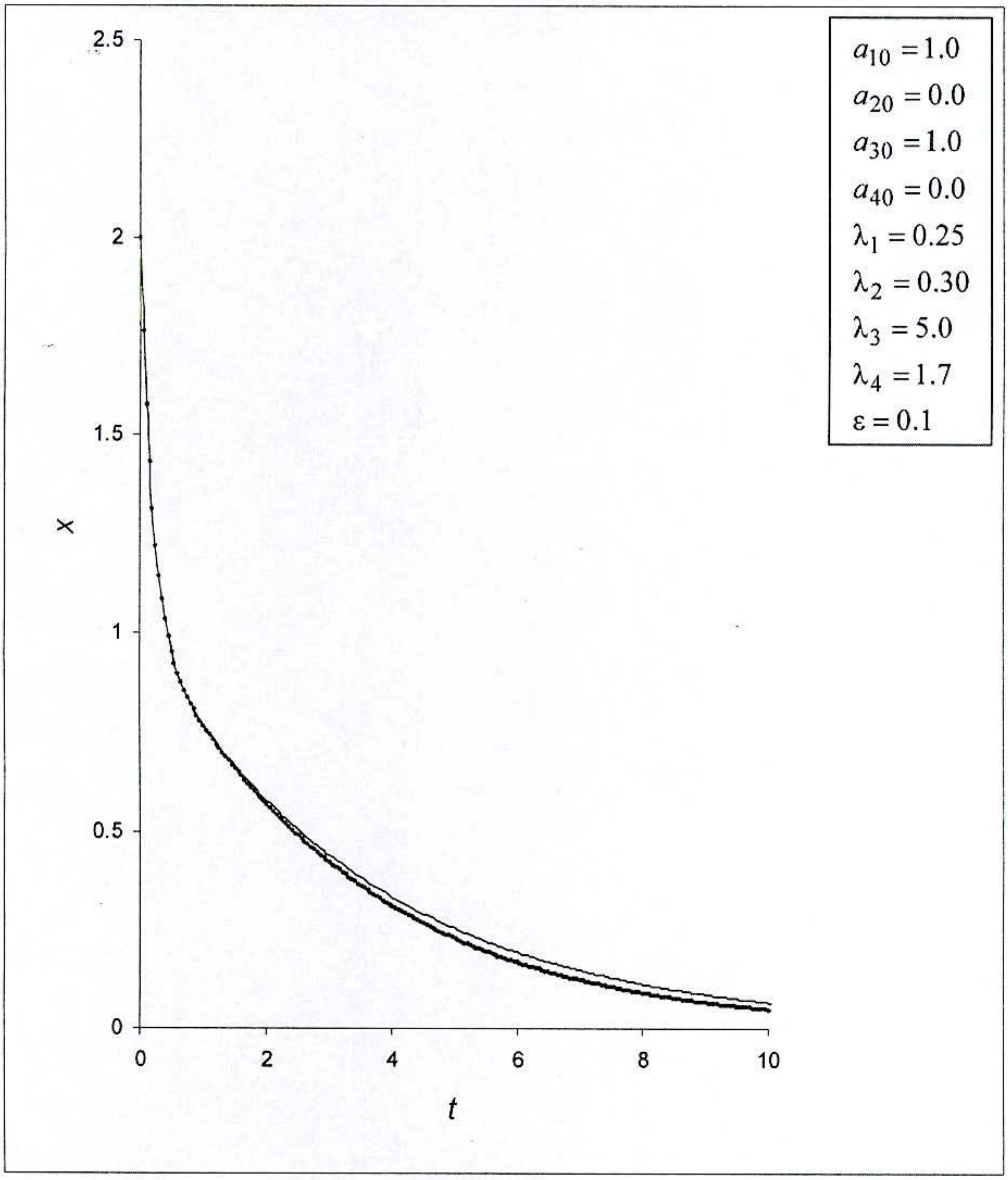

Fig.(4.2.2.3)

Fig.(4.2.2.3) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ∘∘∘).

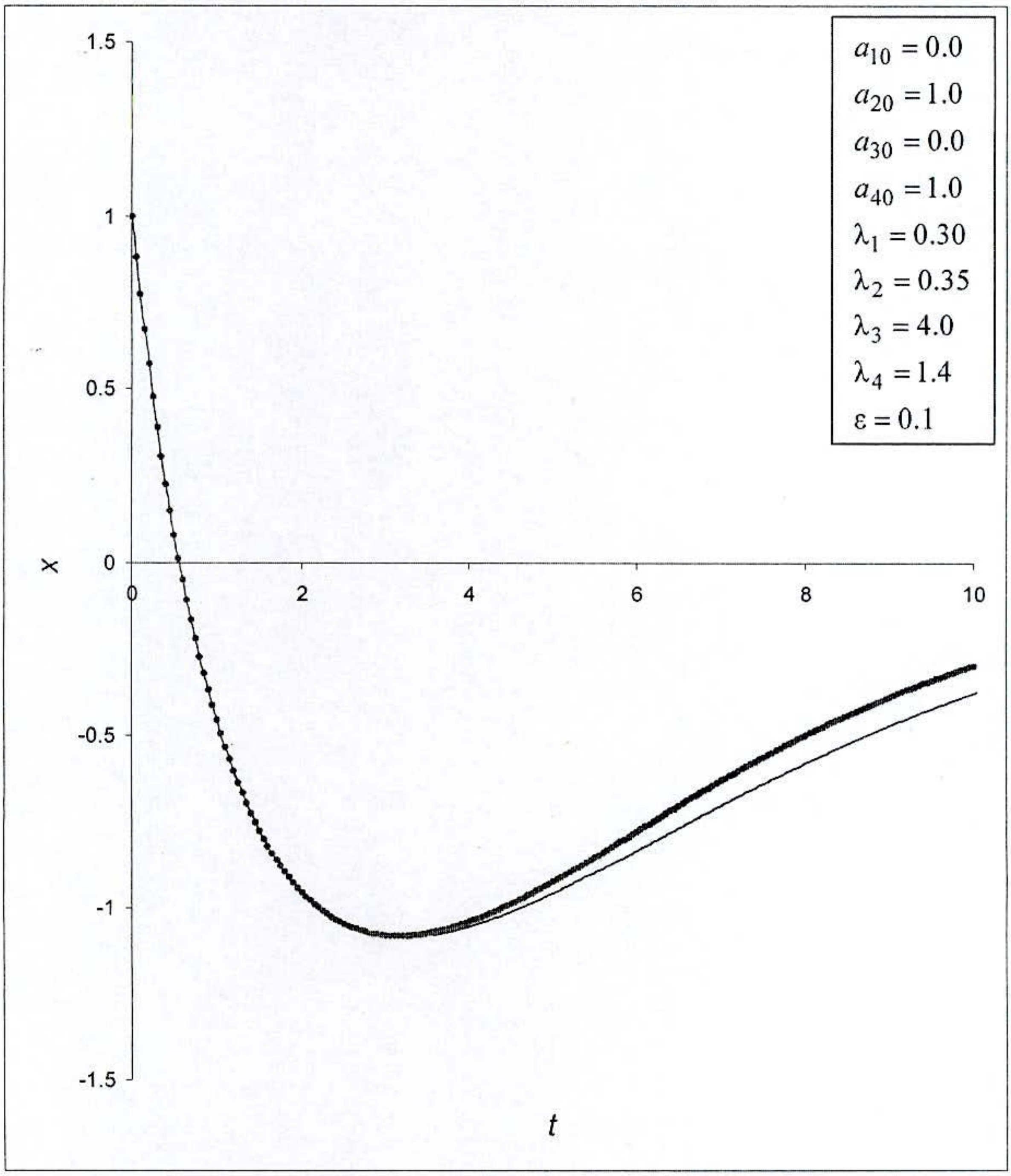

Fig.(4.2.2.4)

Fig.(4.2.2.4) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ∘ ∘ ∘).

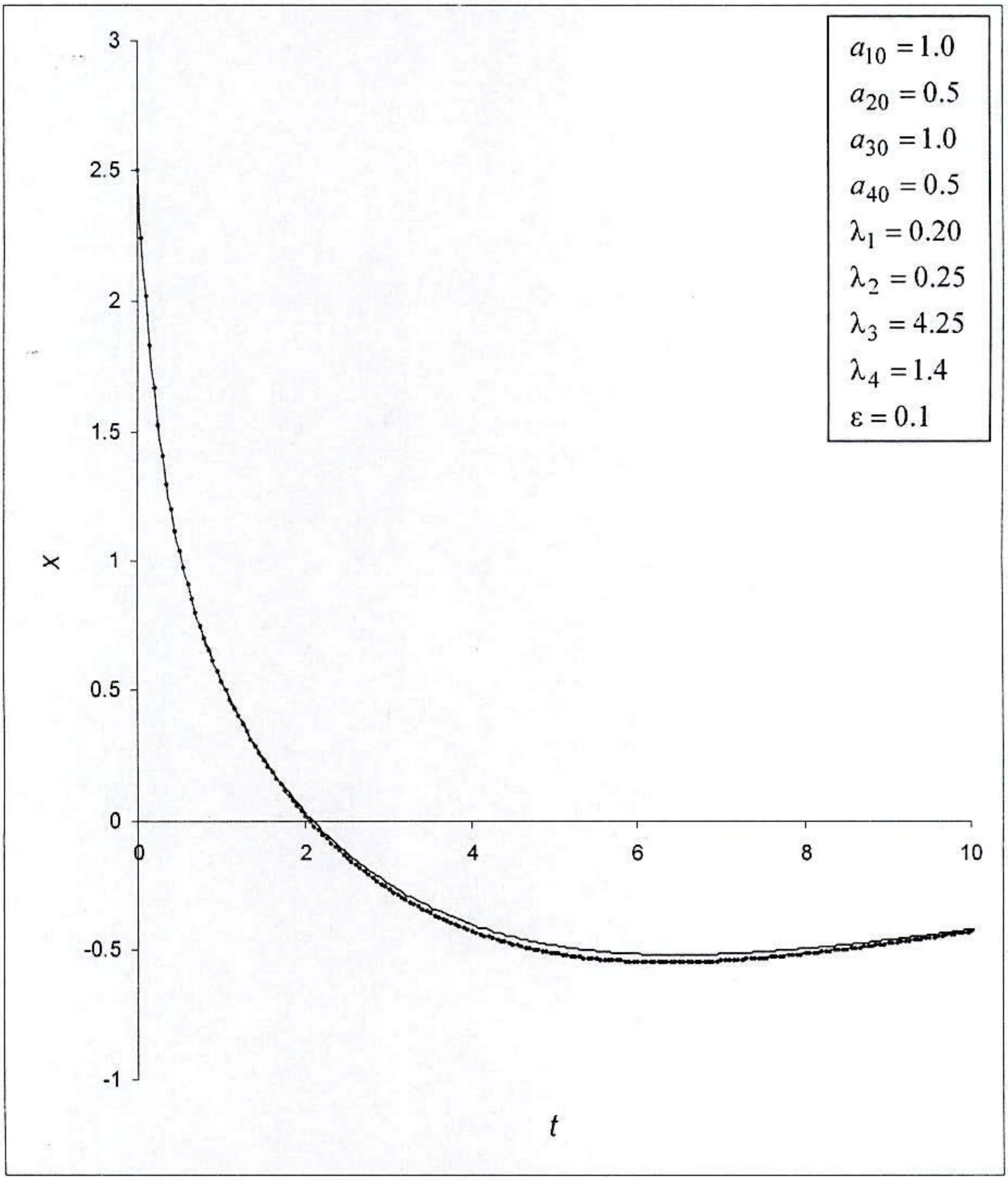

Fig.(4.2.2.5)

Fig.(4.2.2.5) Comparison between analytical solution and numerical solution for chosen values of arbitrary constants, eigen-values and small parameter (analytic solution in solid line — and numerical solution in dotted line ∘ ∘ ∘).

CHAPTER 5

CONCLUSION

Following the KBM method, the asymptotic methods are developed to obtain the solutions of fourth order nonlinear differential equations with small nonlinearities that represent damped oscillatory and near critically damped non-oscillatory systems under some specific conditions. For the damped oscillatory system out of the four eigen-values of the corresponding linear equation, two are assumed to be real and the other two are complex; where as, for the near critically damped non-oscillatory system all of the four eigen-values of the corresponding linear equation are real. The relations, limit $\lambda_1 \to \lambda_2$ and $\lambda_3 \approx 3\lambda_4$, among the eigen-values are imposed to solve the fourth order near critically damped non-oscillatory system. The results obtained by the perturbation method with the propositions about the eigen-values are in good agreement with those of the numerical method.

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