Entry NO. 34

A STUDY OF SINGLE AND MULTI STEP METHODS TO SOLVE DIFFERENTIAL EQUATIONS

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DECLARATION

This is declared that the thesis entitled "A study of single and multi step methods to solve differential equations" has been carried out by Mary Khanam in the Department of Mathematics of Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any other degree or diploma.

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Dedicated to

My Beloved Parents

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Abstract

Here the task was to study single and multi step methods to solve differential equations. As Runge-Kutta method, a single step method has some property to represent a family so attention has been given to that method. It was found that to establish the method of any order the number of unknowns are more than the number of available equations. Thus there are options to choose certain values as someone wants (keeping in mind about the conditions). The opportunity has been taken and FIVE formulas of different orders, two fifth, two sixth and one seventh, has been proposed. Problems are solved to verify their capability and strength and it is found that the percentage errors with respect to the exact values are comparable in magnitude with respect to some available same order methods. In case of multi step methods, extensions in both explicit and implicit methods in terms of order are done. It is done keeping in mind that the extension in order will reduce the error. Extensions in Adams-Bashforth and Adams-Moulton formulas up to TENth order are done and with eighth order of both of them a problem is solved to demonstrate the strength of the extension of these predictor-corrector methods.

Contents

Chapter 1

Introduction

Mathematics is the language of science. For the sake of generalization of scientific results, even for special cases mathematical models are essential. In modeling, the relation between the variables follows tri-chotomy i.e. either they are equal or unequal. When the equal situation arises then different equations arises viz. linear equations, system of linear equations, nonlinear equations, system of nonlinear equations, different type of differential equations etc. When change in one variable depends on the change in one or more other variables then differential equations (ordinary or partial) arises. In the first case the equations are ordinary and the other produces partial differential equations. The differential equations may be linear as well as nonlinear. The highest order derivative present in the equation is the order of the equation and after rationalization the degree of the highest order derivative is the degree of the equation (Jain, 2000). If in the equation the dependent variable and all its derivatives are of degree one then the equation is termed as linear, otherwise they will be nonlinear. The general solution of a nth order linear differential equation contains n linearly independent arbitrary constants. To get the solution of particular problem it is required to fixing up the values of the arbitrary constants. That can be done if n conditions are prescribed. If all conditions are prescribed at the initial point of the domain then the conditions are termed as initial conditions and the equation together with the initial conditions called initial value problem (I.V.P.). If the conditions are prescribed at more than one point then they are termed as boundary conditions and the equation together with the boundary conditions is called boundary value problem (B.V.P.). Clearly the classification as I.V.P. or B.V.P. depends only where the conditions are prescribed, not on the equation itself. Thus both I.V.P. and B.V.P. may come forward with linear and nonlinear differential equations. A nth order linear ordinary differential equation generally written as

$$\sum_{i=0}^{n} a_i(x) y^{(i)} = b(x)$$

and the general nonlinear differential equation of order n can be written as

$$y^{(n)}(x) = F(x, y, y', y'', \ldots\ldots\ldots\ldots, y^{(n-1)}).$$

Though the latter form is generally used for nonlinear case, but if F is linear in terms of the dependent variable and its derivatives then it will represent linear equation. Thus we will consider this representation for our use. If n initial conditions are prescribed at $x = x_0$ then we can pose an I.V.P. as

$$y^{(n)} = f(x, y, y', y'', \ldots\ldots\ldots\ldots\ldots\ldots, y^{(n-1)})$$
$$y^{(p)}(x_0) = y_0^{(p)}, \quad p = 0,1,2, \ldots\ldots\ldots\ldots\ldots\ldots, (n-1)$$

Let us consider the equation

$$\frac{d^3u}{dt^3} = F(t,u,\frac{du}{dt},\frac{d^2u}{dt^2})$$

with conditions $\quad u(t_0) = u_0\ ;\quad \left.\frac{du}{dt}\right|_{t=t_0} = u_0^1;\quad \left.\frac{d^2u}{dt^2}\right|_{t=t_0} = u_0^2$

Clearly the question of linearity will depend on the linearity of the dependent variable u and its derivative in F.

Now let $\quad \frac{du}{dt} = u' = v$

$$\frac{d^2u}{dt^2} = v' = w.$$

Thus we get a system of equations, along with the conditions as follows

$$u' = v,\quad u(t_0) = u_0$$
$$v' = w\quad v(t_0) = u_0^1$$
$$w' = F\quad w(t_0) = u_0^2$$

In vector form $\frac{dy}{dt} = f(t,y),\quad y(t_0) = \alpha$

where $\quad y = [u,v,w]',\ f(t,y) = [v,w.F]'$ and $\alpha = [u_0,u_0^1,u_0^2]'$.

The above example can be generalized for any finite nth order equation. Thus any nth order equation can be reduced to a system of 1st order equations. So theory are developed emphasizing to solve 1st order equations.

Linear differential equations are easy to handle, where as no general rule is found for solving nonlinear differential equations. Unfortunately, the differential equations that arise in most of the physical situations are nonlinear and their analytical solutions are difficult even sometimes not possible. Thus numerical procedure came forward.

In the case of numerical computation it is required to represent function in terms of polynomials and unfortunately too many terms can not be taken. So usually exact value can not be calculated rather it is forced to estimate of an approximate value of the dependent variable, though desired degree of accuracy can be achieved. To illustrate the fact, let us consider a simple differential equation

$$\frac{dy}{dx} = \lambda y,\ y(x_0) = y_0,\ x \in [x_0,b]$$

The exact solution of the above equation is given by

$$y(x) = ce^{\lambda x}$$

where c is an arbitrary constant. Using the initial condition $y(x_0) = y_0$, we can write the relation in the form $\quad y(x) = y(x_0)e^{\lambda(x-x_0)}$.

It may be noted that the numerical methods for the solution of differential equation are the algorithms which will produce a table of approximate values of the dependent variable at nodal points (certain equally spaced values of independent variables)(Jain, 2000).

In order to compute the values of $y(x_k)$ at the nodal points $x_k = x_0 + kh, \quad k = 1,2,\ldots\ldots, N$, we write a recurrence relation between the values of $y(x)$ at x_{n+1} and x_n as

$$y(x_{n+1}) = e^{\lambda h} y(x_n), \quad n = 0,1,2,\ldots\ldots\ldots\ldots N-1.$$

This gives an algorithm for determining the values of $y(x_1), y(x_2), \ldots\ldots\ldots\ldots, y(x_N)$ from the given value $y(x_0)$ *at* $x = x_0$. However, from the computational view-point each value has to be multiplied by $e^{\lambda h}$, which is an exponential function and difficult to calculate exactly. We, therefore, take suitable approximation of $e^{\lambda h}$. For example, for sufficiently small $|\lambda h|$, the polynomials approximating to $e^{\lambda h}$ can be written

$$e^{\lambda h} = 1 + \lambda h + 0(|\lambda h|^2)$$

$$e^{\lambda h} = 1 + \lambda h + \frac{1}{2}(\lambda h)^2 + 0(|\lambda h|^3)$$

$$e^{\lambda h} = 1 + \lambda h + \frac{1}{2!}(\lambda h)^2 + \ldots\ldots\ldots + \frac{1}{p!}(\lambda h)^p + 0(|\lambda h|^{p+1})$$

We can also take other types of approximation to $e^{\lambda h}$ as follows

$$e^{\lambda h} = \frac{1}{1 - \lambda h} + 0(|\lambda h|^2)$$

$$e^{\lambda h} = \frac{1 + \frac{1}{2}\lambda h}{1 - \frac{1}{2}\lambda h} + 0(|\lambda h|^3)$$

$$e^{\lambda h} = \frac{1 + \frac{1}{2}\lambda h + \frac{1}{12}(\lambda h)^2}{1 - \frac{1}{2}\lambda h + \frac{1}{12}(\lambda h)^2} + 0(|\lambda h|^5)$$

$$e^{\lambda h} = \frac{1 + \frac{1}{2}\lambda h + \frac{1}{10}(\lambda h)^2 + \frac{1}{120}(\lambda h)^3}{1 - \frac{1}{2}\lambda h + \frac{1}{10}(\lambda h)^2 - \frac{1}{120}(\lambda h)^3} + 0(|\lambda h|^7)$$

Let us denote the approximation to $e^{\lambda h}$ by $E(\lambda h)$. The numerical method for obtaining the approximate values y_n *of* $y(x_n)$ can be written as

$$y_{n+1} = E(\lambda h) y_n, \quad n = 0,1,2,\ldots\ldots\ldots\ldots\ldots\ldots\ldots, N-1$$

If $\quad E(\lambda h) = 1 + \lambda h + \frac{1}{2!}(\lambda h)^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{(\lambda h)^p}{p!},$

then the above equation becomes

$$y_{n+1} = y(x_{n+1}) + 0(h^{p+1})$$

The integer p is called the order of the method. The remainder term

$$\frac{(\lambda h)^{p+1}}{(p+1)!} e^{\theta \lambda h}, \quad 0 < \theta < 1$$

which is neglected, is the relative discritization or local truncation error.

Also numbers can be handled with finite number of digits. Rules are there to represent any number with fixed digits, where again we are forced to commit errors, the round off error.

The numerical methods for finding solutions of initial value problems are broadly classified into following two categories

(1) Single step methods and
(2) Multi step methods.

From the terminology itself it is clear that if the value of the dependent variable is known at some point then we will be able to calculate the value of that at very next point with the known step size in case of single step method. Where as, in multi step methods, to calculate the next nodal value, values at more than one previous nodal point are required. Many methods of both categories have been evolved. In this study both the single step methods and multi step methods will be addressed, but focusing will be limited within Runge-Kutta methods (in single step methods category) and Adams-Bashforth formula and Adams-Moulton formula (in multi step methods category).

Chapter 2 will deal with Runge-Kutta methods with introduction to the single step methods. There two different 5th and 6th order and one 7th order i.e. altogether five new formulas of Runge-Kutta methods are proposed along with discussion of the method itself. In Chapter 3 the underling idea of multi step methods will be discussed. In this chapter further extension in steps in both Adams-Bashforth formula and Adams-Moulton formula will be done. Both the formulas have been extended up to 10 steps. In Chapter 4 the new tools will be utilized to verify its ability. And finally drawn conclusions will be presented in Chapter 5.

Chapter 2

Single Step Methods

If to calculate the value of the dependent variable at some point with known step size it is required to know the value of dependent variable at the very previous point only, then the method is termed as **single step method**. In this chapter discussion about single step methods is presented. Lot of single step methods have been evolved but here only the Runge-Kutta family is dealt with.

In section 2.1 a brief description and related ideas of the single step method will be presented. Section 2.2 will contain the discussion on Runge-Kutta methods. Proposed five new formulas, two 5th and 6th order and one 7th order formulas, will be presented in section 2.3. Their utilization to solve a selected problem will be demonstrated in section 2.4.

2.1 SINGLE STEP METHOD

A single step method for the solution of the differential equation

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0, \quad x \in [x_0, b] \tag{2.1.1}$$

is one in which the solution of the differential equation is approximated by calculating the solution of a related first order difference equation. A good number of text books, including Ceschino and Kuntzmann(1966), Collatz(1966), Gear(1971), Henrici(1962), Lambert(1973), Lapidus and Seinfeld(1971)etc, has deal with the single step methods for solving initial value problems of ordinary differential equations.
A general single step method can be written in the form

$$y_{n+1} = y_n + h\phi(x_n, y_n, h), \quad n = 0,1,2,................, N-1 \tag{2.1.2}$$

where $\phi(x,y,h)$ is a function of the arguments x,y,h and, in addition, depends on the right hand side of (2.1.1). The function $\phi(x,y,h)$ is called the increment function. If y'_{n+1} can be obtained simply by evaluating the right hand-side of (2.1.2), then the single step method is called explicit otherwise it is called implicit. The true value $y(x_n)$ will satisfy

$$y(x_{n+1}) = y(x_n) + h\phi(x_n, y(x_n), h) + T_n, \quad n = 0,1,2,.................., N-1 \tag{2.1.3}$$

where T_n is the truncation error.

The largest integer p, such that $\left|h^{-1}T_n\right| = 0(h^p)$, is called the order of the single step method.

The single step method (2.1.2) is said to be regular if the function $\phi(x,y,h)$ is defined and continuous in the domain $x_0 \le x \le b, -\infty < y < \infty, 0 \le h \le h_0$ and if there exist a constant L such that $|\phi(x,y,h)-\phi(x,z,h)| \le L|y-z|$ for every $x \in [x_0,b]$; $y,z \in (-\infty,\infty)$; $h \in (0,h_0)$ (h_0 is a positive constant).

Again a single step method of the form (2.1.2) is said to be consistent if

$$\phi(x,y,0) = f(x,y).$$

It is to be also insured that the formula (2.1.2) be insensitive to small change in the local errors and this will be guaranteed by the stability condition.

A necessary and sufficient condition for convergence of a regular single step method of order $p \ge 1$ is consistency. This result ensures that the approximate solution converges to the exact solution like Ch^p.

For the application of the formula (2.1.2) to (2.1.1), we need a specific form of the increment function $\phi(x,y,h)$.

2.2 RUNGE-KUTTA METHODS.

Runge-Kutta (RK) method is the generalization of the concept used in modified Euler's method. This method was devised by the German mathematicians C. Runge about the year 1894 and extended by M.W. Kutta a few years later. Among many others Butcher(1964, 1965) and Luther(1966, 1968) have studied the RK methods of different orders. The stability of the method is discussed by Distefano(1968), whereas the extended region of stability is discussed by Lawson(1966, 1967). Ralston(1962), Chai(1968), Jain et.al.(1981) are among many others who have discussed about the truncation error and error bounds of the methods.

In numerical analysis, RK methods are an important family of implicit and explicit methods for the approximation of solutions of ordinary differential equations (Jain, 2000). These methods have the following useful properties:

1. To evaluate y_{n+1}, they need only information at the point (x_n, y_n).

2. They do not involve the derivatives of f(x,y) such as in Taylor's series method.

3. They agree with the Taylor's series solution up to the terms of h^r, where r differs from method to method and is known as the order of that Runge-kutta method.

Since Euler's method and its improved and modified forms satisfy all the three properties, they can be termed as RK methods of first and second order respectively. In these methods the accuracy increases at the cost of calculation. Of this family of methods, the most widely used method is RK of fourth order and so is why the name of RK is used generally for this method.

In modified Euler's method the slope of the solution curve has been approximated with the slopes of the curve at the end points of the each subinterval in computing the solution. The natural generalization of this concept is computing the slope by taking a weighted average of the slopes taken at more number of points in each sub interval. However, the implementation of the scheme differs from modified Euler's method so that the developed algorithm is explicit in nature.

By the Mean-value Theorem any solution of

$$y' = f(x,y), \quad y(x_0) = y_0, \qquad x \in [x_0, b] \tag{2.2.1}$$

satisfies

$$y(x_{n+1}) = y(x_n) + hy'(\xi_n)$$

$$= y(x_n) + hf(\xi_n, y(\xi_n))$$

where

$$\xi_n = x_n + \theta_n h, \quad 0 < \theta_n < 1$$

Let us consider $\theta_n = 1/2$, so that $\xi_n = x_n + h/2$.

By Euler's method with spacing h/2, we get

$$y\left(x_n + \frac{1}{2}h\right) \cong y_n + \frac{1}{2}hf(x_n, y_n)$$

Thus, we have the approximation

$$y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right) \tag{2.2.2}$$

Alternatively, again using Euler's method, we proceed as follows:

$$y'\left(x_n + \frac{1}{2}h\right) \cong \frac{1}{2}[y'(x_n) + y'(x_{n+1})]$$

$$\cong \frac{1}{2}[f(x_n, y_n) + f(x_{n+1}, y_n + hf_n)]$$

and thus we have the approximation

$$y_{n+1} = y_n + \frac{1}{2}h[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))] \tag{2.2.3}$$

Either (2.2.2) or (2.2.3) can be regarded as

$$y_{n+1} = y_n + h(average\ slope)$$

$$= y_n + kh \tag{2.2.4}$$

This is the underlying idea of the RK approach. In general, we find the slope at x_n and at several other points, average these slopes, multiply by *h*, and add the result to y_n. Thus the RK method with *r* slopes can be written as

$$k_i = hf\left(x_n + c_i h,\ y_n + \sum_{j=1}^{i-1} a_{ij} k_j\right), \quad c_1 = 0, i = 1,2,\ldots\ldots,r \tag{2.2.5}$$

or,
$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + c_2 h,\ y_n + a_{21}k_1)$$

$$k_3 = hf(x_n + c_3 h,\ y_n + a_{31}k_1 + a_{32}k_2)$$

$$k_4 = hf(x_n + c_4 h,\ y_n + a_{41}k_1 + a_{42}k_2 + a_{43}k_3)$$

:

:

:

and
$$y_{n+1} = y_n + \sum_{i=1}^{r} w_i k_i$$

where the parameters $c_2, c_3, \ldots\ldots\ldots, c_r, a_{2j}, \ldots\ldots\ldots, a_{r(r-1)}$ *and* w_i are arbitrary.

From (2.2.4), we may interpret the increment function as the linear combination of the slopes at x_n and at several other points between x_n and x_{n+1}. To determine the parameters *c*'s, *a*'s and *w*'s in the above equation, y_{n+1} defined in the scheme is expanded in terms of step length *h* and the resultant equation is then compared with Taylor series expansion of the solution of the differential equation up to a certain number of terms say *p*. Then the *i*-stage RK method will be of order p or is a *p*th order RK method. It is interesting to note that for any $i > 4$ the maximum possible order *p* of the RK method is always less than *i*. We need $i = 6$ for a 5th order method, $i = 7$ or 8 to give a 6th order method and $i = k$ to give a (k-2)th order method, $k \geq 9$. However, for any i less than or equal to 4, it is possible derive an RK method of order $p = i$ (Jain, 2000).

Before proceeding further we want to point out the advantages and disadvantages of the RK method and are listed below.

ADVANTAGES:

(1) RK formula over Taylor's method is that these do not require the prior calculation of higher order derivatives of y(x) while the Taylor's method needs.
(2) The computational formula demands only the functional values at some selected points.
(3) This method is simple and gives us a result of moderately better accuracy.

DISADVANTAGES:

The main disadvantage of this method is that there is no practically suitable way of checking the computation. So, if any computational error in any stage took place, it propagates in the subsequent stages and remains undetected.

Let us examine a second order method, for which let us define

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf(x_n + c_2 h, y_n + a_{21} k_1)$$

and
$$y_{n+1} = y_n + w_1 k_1 + w_2 k_2 \tag{2.2.6}$$

where the parameters c_2, a_{21}, w_1 *and* w_2 are chosen to make y_{n+1} closer to $y(x_{n+1})$.

Now Taylor series gives

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + \ldots\ldots\ldots \tag{2.2.7}$$

where $y' = f(x, y)$

$$y'' = f_x + ff_y$$

$$y''' = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y(f_x + ff_y)$$

The values of $y'(x_n), y''(x_n), \ldots\ldots$ are obtained by substituting $x = x_n$. We expand k_1 *and* k_2 about the point (x_n, y_n).

$$k_1 = hf$$

$$k_2 = hf(x_n + c_2h, y_n + a_{21}k_1)$$

$$= h\left(f_n + c_2hf_x + a_{21}k_1f_y + \frac{(c_2h)^2}{2!}f_{xx} + \frac{(a_{21}k_1)^2}{2!}f_{yy} + c_2ha_{21}k_1f_{xy} + \dots\dots\dots \right)$$

$$= h\left(f_n + c_2hf_x + a_{21}hf_nf_y + \frac{(c_2h)^2}{2!}f_{xx} + \frac{(a_{21}hf_n)^2}{2!}f_{yy} + c_2ha_{21}hf_nf_{xy} + \dots\dots\dots \right)$$

Substituting the values of k_1 and k_2 in (2.2.6) we get

$$y_{n+1} = y_n + (w_1 + w_2)hf_n + h^2w_2(c_2f_x + a_{21}ff_y) + \frac{h^3}{2}w_2(c_2^2f_{xx} + 2c_2a_{21}ff_{xy} + a_{21}^2f^2f_{yy}) + \dots \quad (2.2.8)$$

Comparing (2.2.7) with (2.2.8) and matching coefficients of powers of h, we obtained three equations for the parameters

$$w_1 + w_2 = 1$$

$$c_2w_2 = \frac{1}{2}$$

$$a_{21}w_2 = \frac{1}{2}$$

We note that we have four unknowns and only three equations. Therefore, there is no unique solution. However we can assume a value for one of the constants and determine the others. This implies that there is an infinite family of second order RK methods. For example, if we choose $w_1 = \frac{1}{2}$, then we get $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{2}$, $c_2 = 1$, $a_{21} = 1$.

With these values equation (2.2.6) becomes

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \quad (2.2.9)$$

where $k_1 = hf(x_n, y_n)$

$$k_2 = hf(x_n + h, y_n + h)$$

This equation is the Heun's formula.

Similarly, if we choose $w_1 = 0$, then we get,

$$w_1 = 0,\, w_2 = 1,\, c_2 = \frac{1}{2}, a_{21} = \frac{1}{2}$$

and equation (2.2.6) becomes

$$y_{n+1} = y_n + k_2 \tag{2.2.10}$$

where $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

This result is the midpoint or polygon method.

Again if c_2 is chosen arbitrarily (nonzero), then

$$a_{21} = c_2,\, w_2 = \frac{1}{(2c_2)} \; and \; w_1 = 1 - \frac{1}{(2c_2)} \tag{2.2.11}$$

Using (2.2.11) in (2.2.8), we get

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2}(f_x + f_n f_y) + \frac{c_2 h^3}{4}(f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy}) + \ldots.. \tag{2.2.12}$$

Subtracting (2.2.12) from (2.2.7), we obtained the local truncation error

$$T_n = y(x_{n+1}) - y_{n+1}$$

$$= h^3\left[(\frac{1}{6} - \frac{c_2}{4})(f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy}) + \frac{1}{6}f_y(f_x + f_n f_y)\right] + \ldots\ldots..$$

$$= \frac{h^3}{12}\left[(2 - 3c_2)y''' + 3c_2 f_y y_n''\right] + \ldots\ldots$$

We observe that no choice of the parameter c_2 will make the leading term of T_n vanish for all *f(x ,y)*. The local truncation error depends not only on derivatives of the solution *y(x)* but also on the function *f*. This is typical of all the RK methods; in most other methods the truncation error depends only on certain derivatives of *y(x)*. Generally, c_2 would be chosen between 0 and 1.

An alternative way of choosing the arbitrary parameters is to produce zero among w_i's, where possible, to simplify the final formula. The choice of $c_2 = \frac{1}{2}$, for example, makes $w_1 = 0$. Sometimes the free parameters are chosen to have as large as possible the interval of absolute stability or to minimize the sum of the absolute values of the coefficients in the term T_n. Such a choice is called optimal. In the latter case we may define

$$\left|\frac{\partial^{i+j} f}{\partial_x^i \partial_y^j}\right| < \frac{L^{i+j}}{M^{j-1}}, \quad i, j = 0,1,2, \ldots\ldots\ldots\ldots\ldots\ldots$$

We find

$$|f| < M, \quad |f_{xy}| < L^2$$
$$|f_x| < LM, \quad |f_{xx}| < L^2 M$$
$$|f_y| < L, \quad |f_{yy}| < \frac{L^2}{M}$$

and thus $|T_n|$ becomes

$$|T_n| < ML^2h^3\left[4\left|\frac{1}{6} - \frac{c_2}{4}\right| + \frac{1}{3}\right]$$

Obviously the minimum value of $|T_n|$ occurs for $c_2 = 2/3$ in which case $|T_n| < ML^2h^3/3$.

We will state the RK method by listing the coefficients as follows:

$$\begin{array}{c|cccc}
c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
c_4 & a_{41} & a_{42} & a_{43} & \\
. & . & . & . & \\
. & . & . & . & \\
. & . & . & . & \\
\hline
 & w_1 & w_2 & w_3 & . \quad . \quad .
\end{array}$$

As a result, we may list some second order methods as

$$\begin{array}{c|cc}
1/2 & 1/2 & \\
\hline
 & 0 & 1
\end{array}
\qquad
\begin{array}{c|cc}
2/3 & 2/3 & \\
\hline
 & 1/4 & 3/4
\end{array}
\qquad
\begin{array}{c|cc}
1 & 1 & \\
\hline
 & 1/2 & 1/2
\end{array}$$

Improved tangent *Optimal* *Euler-Cauchy*

The 4th order classical RK method can be tabulated as:

$$
\begin{array}{c|cccc}
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6}
\end{array}
$$

2.3 Proposed Formulas

In the previous section derivation of a second order RK method is presented and it is seen that the number of equations obtained are less than the number of introduced constants. We are to propose 5th, 6th and 7th order formulas. Derivation of each of them is not presented here but the derivation of 6th order formula is presented in the Appendix-A.

In the following we will list the coefficients and the weights of the proposed forms. They will be of orders: two 5th, two 6th and one 7th.

$$
\begin{array}{c|cccccc}
\frac{1}{2} & \frac{1}{2} & & & & & \\
\frac{1}{3} & 0 & \frac{1}{3} & & & & \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & & & \\
\frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} & & \\
1 & 0 & 0 & 0 & 0 & 1 & \\
\hline
 & \frac{11}{120} & \frac{-4}{15} & \frac{27}{40} & \frac{-4}{15} & \frac{27}{40} & \frac{11}{120}
\end{array}
$$

First 5th order form(proposed)

$\frac{1}{2}$	$\frac{1}{2}$					
$\frac{1}{2}$	$\frac{13}{48}$	$\frac{11}{48}$				
1	$\frac{-5}{33}$	0	$\frac{38}{33}$			
$\frac{1}{3}$	$\frac{20}{81}$	0	$\frac{10}{81}$	$\frac{-3}{81}$		
$\frac{2}{3}$	$\frac{25}{81}$	0	$\frac{-4}{81}$	$\frac{6}{81}$	$\frac{27}{81}$	
	$\frac{11}{120}$	0	$\frac{-64}{120}$	$\frac{11}{120}$	$\frac{81}{120}$	$\frac{81}{120}$

Second 5th order form(proposed)

$\frac{1}{2}$	$\frac{1}{2}$						
$\frac{1}{3}$	0	$\frac{1}{3}$					
$\frac{2}{3}$	0	0	$\frac{2}{3}$				
$\frac{1}{3}$	0	0	0	$\frac{1}{3}$			
$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$		
1	0	0	0	0	0	1	
	$\frac{11}{120}$	$\frac{-4}{15}$	$\frac{27}{80}$	$\frac{27}{40}$	$\frac{27}{80}$	$\frac{-4}{15}$	$\frac{11}{120}$

First 6th order form(proposed)

$$
\begin{array}{c|ccccccc}
\frac{1}{2} & \frac{1}{2} & & & & & & \\
\frac{2}{3} & \frac{155}{216} & \frac{-11}{216} & & & & & \\
\frac{1}{2} & \frac{191}{324} & \frac{-14}{81} & \frac{1}{12} & & & & \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & & & \\
\frac{1}{3} & \frac{67}{192} & 0 & 0 & \frac{1}{8} & \frac{-9}{64} & & \\
1 & \frac{17}{6} & 0 & \frac{-27}{8} & \frac{17}{3} & 0 & \frac{-33}{8} & \\
\hline
 & \frac{-1}{10} & 0 & \frac{24}{10} & \frac{-36}{10} & 0 & \frac{24}{10} & \frac{-1}{10}
\end{array}
$$

Second 6th order form(proposed)

$$
\begin{array}{c|cccccccc}
\frac{1}{2} & \frac{1}{2} & & & & & & & \\
\frac{1}{3} & 0 & \frac{1}{3} & & & & & & \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & & & & & \\
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & & & & \\
\frac{2}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & & & \\
\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\hline
 & \frac{11}{120} & \frac{-4}{15} & \frac{27}{80} & \frac{-4}{15} & \frac{27}{80} & \frac{27}{80} & \frac{27}{80} & \frac{11}{120}
\end{array}
$$

Proposed 7th order form

2.4: Demonstration:

To demonstrate the use of the proposed formulae we have chosen a nonlinear first order differential equation $\frac{dy}{dx} = y + xy^2$. The general solution of this equation will contain one arbitrary constant and to get any particular solution we require one condition and we have chosen that as $y(0)=1$. We will try to find the value of the dependent variable y at $x=0.1$ and at $x=0.2$ i.e. $y(0.1)$ and $y(0.2)$ will be estimated. The chosen equation is in the Bernoullie's form. This form is chosen as it can be reduced to a linear one and as such its exact solution can be obtained easily. The analytical or exact solution is important to us to estimate the percentage error.

2.4.1

Estimate the value of $y(0.1)$ and $y(0.2)$ when $\frac{dy}{dx} = y + xy^2$, $y(0)=1$ by the first 5th order proposed RK method.

The proposed listing is listed below.

$$
\begin{array}{c|cccccc}
\frac{1}{2} & \frac{1}{2} & & & & & \\
\frac{1}{3} & 0 & \frac{1}{3} & & & & \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & & & \\
\frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} & & \\
1 & 0 & 0 & 0 & 0 & 1 & \\
\hline
\frac{11}{120} & \frac{-4}{15} & \frac{27}{40} & \frac{-4}{15} & \frac{27}{40} & \frac{11}{120} &
\end{array}
$$

Proposed first 5th order RK method

Let us take h=0.1, Here $f(x,y) = \frac{dy}{dx} = y + xy^2$.

For $n = 0, \quad x_0 = 0, \quad y_0 = 1$

$$k_1 = hf(x_0, y_0) = 0.1$$

$$k_2 = hf(x_0 + \frac{h}{2}, \ y_0 + \frac{k_1}{2}) = 0.1f(0.05, 1.05) = 0.1105$$

$$k_3 = hf(x_0 + \frac{h}{3}, \ y_0 + \frac{k_2}{3}) = 0.1f(\frac{0.1}{3}, 1 + \frac{0.1105}{3}) = 0.1073$$

$$k_4 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}) = 0.0f(0.05, 1.0537) = 0.11092$$

$$k_5 = hf(x_0 + \frac{2h}{3}, y_0 + \frac{2k_4}{3}) = 0.1f(0.06667, 1.07395) = 0.1151$$

$$k_6 = hf(x_0 + h, y_0 + k_5) = 0.1f(0.1, 1.1151) = 0.1239$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + 81k_3 - 32k_4 + 81k_5 + 11k_6) = 0.10335$$

$$\therefore y_1 = y_0 + \Delta y = 1.10335$$

Percentage error: $(1.1111 - 1.10335)/1.1111 = 6.9751 \times 10^{-03}$

For the second interval we have,

$x_1 = 0.1, y_1 = 1.1032$ and $f(x,y) = y + xy^2$

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1032) = 0.1225$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1368$$

$$k_3 = hf(x_1 + \frac{h}{3}, y_1 + \frac{k_2}{3}) = 0.1325$$

$$k_4 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_3}{2}) = 0.1f(0.15, 1.1695) = 0.1239$$

$$k_5 = hf(x_1 + \frac{2h}{3}, y_1 + \frac{2k_4}{3}) = 0.1f(0.1667, 1.1858) = 0.1244$$

$$k_6 = hf(x_1 + h, y_1 + k_5) = 0.1f(0.2, 1.2276) = 0.1254$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + 81k_3 - 32k_4 + 81k_5 + 11k_6) = 0.1266$$

$x_2 = x_1 + h = 0.2$ and $y_2 = y_1 + \Delta y = 1.2298$

Percentage error: (1.25-1.2298)/1.25 = 0.01616 %

2.4.2

Estimate the value of $y(0.1)$ and $y(0.2)$ when $\frac{dy}{dx} = y + xy^2$, y(0)=1 by the second 5th order proposed RK method.

The proposed listing is listed below.

$\frac{1}{2}$	$\frac{1}{2}$					
$\frac{1}{2}$	$\frac{13}{48}$	$\frac{11}{48}$				
1	$\frac{-5}{33}$	0	$\frac{38}{33}$			
$\frac{1}{3}$	$\frac{20}{81}$	0	$\frac{10}{81}$	$\frac{-3}{81}$		
$\frac{2}{3}$	$\frac{25}{81}$	0	$\frac{-4}{81}$	$\frac{6}{81}$	$\frac{27}{81}$	
	$\frac{11}{120}$	0	$\frac{-64}{120}$	$\frac{11}{120}$	$\frac{81}{120}$	$\frac{81}{120}$

Let us take h=0.1, Here, $f(x, y) = y + xy^2$

For n=0, $x_0 = 0, y_0 = 1$.

Here we have,

$$k_1 = hf(x_0, y_0) = 0.1(1+0) = 0.1$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1f(0.05, 1.05) = 0.1105$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{13k_1}{48} + \frac{11k_2}{48}) = 0.1f(0.05, 1.028) = 0.1081$$

$$k_4 = hf(x_0 + h, y_0 - \frac{5k_1}{33} + \frac{38k_3}{33}) = 0.1f(0.1, 1.1093) = 0.1232$$

$$k_5 = hf(x_0 + \frac{h}{3}, y_0 + \frac{20k_1}{81} + \frac{10k_3}{81} - \frac{3k_4}{81}) = 0.1f(0.0333, 1.1011) = 0.1135$$

$$k_6 = hf(x_0 + \frac{2h}{3}, y_0 + \frac{25k_1}{81} - \frac{4k_3}{81} + \frac{6k_4}{81} + \frac{27k_5}{81}) = 0.1f(0.0667, 1.045) = 0.1118$$

$$\Delta y = \frac{1}{120}(11k_1 - 64k_3 + 11k_4 + 81k_5 + 81k_6) = 0.1066$$

Thus $x_1 = x_0 + h = 0.1$ and $y_1 = y_0 + \Delta y = 1.1066$

Percentage error $(1.1111 - 1.1066)/1.1111 = 4.054 \times 10^{-03}$ %

For the second interval, we have,

$x_1 = 0.1, \quad y_1 = 1.1066$ and $f(x, y) = y + xy^2$

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1066) = 0.1229$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1f(0.15, 1.1681) = 0.1373$$

$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{13k_1}{48} + \frac{11k_2}{48}) = 0.1f(0.15, 1.1714) = 0.1377$$

$$k_4 = hf(x_1 + h, y_1 - \frac{5k_1}{33} + \frac{38k_3}{33}) = 0.1f(0.2, 1.2465) = 0.1557$$

$$k_5 = hf(x_1 + \frac{h}{3}, y_1 + \frac{20k_1}{81} + \frac{10k_3}{81} - \frac{3k_4}{81}) = 0.1f(0.133, 1.1482) = 0.1323$$

$$k_6 = hf(x_1 + \frac{2h}{3}, y_1 + \frac{25k_1}{81} - \frac{4k_3}{81} + \frac{6k_4}{81} + \frac{27k_5}{81}) = 0.1f(0.1667, 1.1934) = 0.1431$$

$$\Delta y = \frac{1}{120}(11k_1 - 64k_3 + 11k_4 + 81k_5 + 81k_6) = 0.13799$$

Thus $x_2 = x_1 + h = 0.2$ and $y_2 = y_1 + \Delta y = 1.2446$

Percentage error: (1.25-1.2446)/1.25 = 0.00432 %

2.4.3

Estimate the value of $y(0.1)$ and $y(0.2)$ when $\frac{dy}{dx} = y + xy^2$, y(0)=1 by the first 6th order proposed RK method.

The proposed listing is listed below

$\frac{1}{2}$	$\frac{1}{2}$						
$\frac{1}{3}$	0	$\frac{1}{3}$					
$\frac{2}{3}$	0	0	$\frac{2}{3}$				
$\frac{1}{3}$	0	0	0	$\frac{1}{3}$			
$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$		
1	0	0	0	0	0	1	
	$\frac{11}{120}$	$\frac{-4}{15}$	$\frac{27}{80}$	$\frac{27}{40}$	$\frac{27}{80}$	$\frac{-4}{15}$	$\frac{11}{120}$

Let us take h=0.1, Here $f(x, y) = \frac{dy}{dx} = y + xy^2$.

For $n = 0, x_0 = 0, y_0 = 1$

$$k_1 = hf(x_0, y_0) = 0.1$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1f(0.05,1.05) = 0.1105$$

$$k_3 = hf(x_0 + \frac{h}{3}, y_0 + \frac{k_2}{3}) = 0.1073$$

$$k_4 = hf(x_0 + \frac{2h}{3}, y_0 + \frac{2k_3}{3}) = 0.1148$$

$$k_5 = hf(x_0 + \frac{h}{3}, y_0 + \frac{k_4}{3}) = 0.1074$$

$$k_6 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_5}{2}) = 0.11092$$

$$k_7 = hf(x_0 + h, y_0 + k_6) = 0.1234$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + \frac{81}{2}k_3 + 81k_4 + \frac{81}{2}k_5 - 32k_6 + 11k_7) = 0.10313$$

$\therefore \Delta y = 0.10313$ and $y_1 = y_0 + \Delta y = 1.10313$

Percentage error: $(1.11111 - 1.10313)/1.111 = 7.1731 \times 10^{-03}$

For the second interval we have,

$x_1 = 0.1, y_1 = 1.1031$ *and* $f(x, y) = y + xy^2$

$$k_1 = hf(x_1, y_1) = 0.1225$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1368$$

$$k_3 = hf(x_1 + \frac{h}{3}, y_1 + \frac{k_2}{3}) = 0.1325$$

$$k_4 = hf(x_1 + \frac{2h}{3}, y_1 + \frac{2k_3}{3}) = 0.1428$$

$$k_5 = hf(x_1 + \frac{h}{3}, y_1 + \frac{k_4}{3}) = 0.1327$$

$$k_6 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_5}{2}) = 0.1375$$

$$k_7 = hf(x_1 + h, y_1 + k_6) = 0.1548$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + \frac{81}{2}k_3 + 81k_4 + \frac{81}{2}k_5 - 32k_6 + 11k_7) = 0.1382$$

$\therefore \Delta y = 0.1382$ and $y_2 = y_1 + \Delta y = 1.2413$

Percentage error: $(1.25 - 1.2413)/1.25 = 6.96 \times 10^{-03}$ %

2.4.4

Estimate the value of $y(0.1)$ and $y(0.2)$ when $\frac{dy}{dx} = y + xy^2$, y(0)=1 by the second 6th order proposed RK method.

The proposed listing is listed below

$\frac{1}{2}$	$\frac{1}{2}$						
$\frac{2}{3}$	$\frac{155}{216}$	$\frac{-11}{216}$					
$\frac{1}{2}$	$\frac{191}{324}$	$\frac{-14}{81}$	$\frac{1}{12}$				
$\frac{1}{3}$	$\frac{1}{3}$	0	0	0			
$\frac{1}{3}$	$\frac{67}{192}$	0	0	$\frac{1}{8}$	$\frac{-9}{64}$		
1	$\frac{17}{6}$	0	$\frac{-27}{8}$	$\frac{17}{3}$	0	$\frac{-33}{8}$	
	$\frac{-1}{10}$	0	$\frac{24}{10}$	$\frac{36}{10}$	0	$\frac{24}{10}$	$\frac{-1}{10}$

Let us take h= 0.1 and f(x,y)= y +xy^2. Given, for $x_0 = 0, y_0 = 1$

Here we have,

$k_1 = hf(x_0, y_0) = 0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1) = 0.1f(.05, 1.05) = 0.1105$

$k_3 = hf(x_0 + \frac{2}{3}h, y_0 + \frac{155}{216}k_1 - \frac{11}{216}k_2) = = 0.1 f(0.0667, 1.066) = 0.1142$

$k_4 = hf(x_0 + \frac{h}{2}, y_0 + \frac{191}{324}k_1 - \frac{14}{81}k_2 + \frac{1}{12}k_3) = 0.1 f(0.05, 1.049) = 0.1104$

$k_5 = hf(x_0 + \frac{h}{3}, y_0 + \frac{1}{3}\ k_1) = 0.1\ f(0.0333, 1.0333) = 0.1069$

$k_6 = hf(x_0\ + \frac{h}{3}, y_0 + \frac{67}{192}\ k_1 + \frac{1}{8}\ k_4 - \frac{9}{64}\ k_5) = 0.1\ f(0.0333, 1.0337) = 0.1069$

$k_7 = hf(x_0 + h, y_0 + \frac{17}{6}\ k_1 - \frac{27}{8}\ k_3 + \frac{17}{3}\ k_4 - \frac{33}{8}\ k_6) = 0.1\ f(0.1, 1.0825) = 0.1199$

$\Delta y = \frac{1}{10}(-k_1 + 24k_3 - 36k_4 + 24k_6 - k_7) = 0.1112$

$\Delta y = 0.1112$ and $y_1 = y_0 + \Delta y = 1.1112$

Percentage error: $(1.1111 - 1.1112)/1.1112 = -9\times10^{-05}\%$

For the second interval we have, $x_1 = 0.1, y_1 = 1.1112)$ and $f(x, y) = y + xy^2$

$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1112) = 0.1235$

$k_2 = hf(x_1 + \frac{h}{2}\ , y_1 + \frac{1}{2}\ k_1) = 0.1f(.15, 1.1729) = 0.1379$

$k_3 = hf(x_1 + \frac{2}{3}\ h, y_1 + \frac{155}{216}\ k_1 - \frac{11}{216}\ k_2) == 0.1\ f(0.1667, 1.1928) = 0.1429$

$k_4 = hf(x_1 + \frac{h}{2}, y_1 + \frac{191}{324}\ k_1 - \frac{14}{81}\ k_2 + \frac{1}{12}\ k_3) = 0.1\ f(0.15, 1.1721) = 0.1378$

$k_5 = hf(x_1 + \frac{h}{3}, y_1 + \frac{1}{3}\ k_1) = 0.1\ f(0.1333, 1.1524) = 0.1329$

$k_6 = hf(x_1\ + \frac{h}{3}, y_1 + \frac{67}{192}\ k_1 + \frac{1}{8}\ k_4 - \frac{9}{64}\ k_5)) = 0.1\ f(0.1333, 1.1528) = 0.1329$

$k_7 = hf(x_1 + h, y_1 + \frac{17}{6}\ k_1 - \frac{27}{8}\ k_3 + \frac{17}{3}\ k_4 - \frac{33}{8}\ k_6)) = 0.1\ f(0.2, 1.2115) = 0.1505$

$\Delta y = \frac{1}{10}(-k_1 + 24k_3 - 36k_4 + 24k_6 - k_7) = 0.15079$

$\Delta y = 0.1508$ and $y_2 = y_1 + \Delta y = 1.2619$

Percentage error: $(1.25 - 1.2619)/1.25 = -9.52\times10^{-03}\%$

2.4.5

Estimate the value of $y(0.1)$ and $y(0.2)$ when $\frac{dy}{dx} = y + xy^2$, $y(0)=1$ by the 7th order proposed RK method.

The proposed listing is listed below

$\frac{1}{2}$	$\frac{1}{2}$							
$\frac{1}{3}$	0	$\frac{1}{3}$						
$\frac{1}{2}$	0	0	$\frac{1}{2}$					
$\frac{1}{3}$	0	0	0	$\frac{1}{3}$				
$\frac{2}{3}$	0	0	0	0	$\frac{2}{3}$			
$\frac{2}{3}$	0	0	0	0	0	$\frac{2}{3}$		
1	0	0	0	0	0	0	1	
	$\frac{11}{120}$	$\frac{-4}{15}$	$\frac{27}{80}$	$\frac{-4}{15}$	$\frac{27}{80}$	$\frac{27}{80}$	$\frac{27}{80}$	$\frac{11}{120}$

Let us take h=0.1, Here $f(x,y) = y + xy^2$

For n=0, $x_0 = 0$ and $y_0 = 1$

$$k_1 = hf(x_0, y_0) = 0.1$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1105$$

$$k_3 = hf(x_0 + \frac{h}{3}, y_0 + \frac{k_2}{3}) = 0.1073$$

$$k_4 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}) = 0.1109$$

$$k_5 = hf(x_0 + \frac{h}{3}, y_0 + \frac{k_4}{3}) = 0.1073$$

$$k_6 = hf(x_0 + \frac{2h}{3}, y_0 + \frac{2k_5}{3}) = 0.1148$$

$$k_7 = hf(x_0 + \frac{2h}{3}, y_0 + \frac{2k_6}{3}) = 0.1154$$

$$k_8 = hf(x_0 + h, y_0 + k_7) = 0.12397$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + \frac{81}{2}k_3 - 32k_4 + \frac{81}{2}k_5 + \frac{81}{2}k_6 + \frac{81}{2}k_7 + 11k_8) = 0.10336$$

$\therefore \Delta y = 0.10336$ and $y_1 = y_0 + \Delta y = 1.10336$

Percentage error: $(1.1111 - 1.1034)/1.1111 = 6.9301 \times 10^{-03}\,\%$

For the second interval we have, $x_1 = 0.1$ and $y_1 = 1.1034$

$$k_1 = hf(x_1, y_1) = 0.1225$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.13698$$

$$k_3 = hf(x_1 + \frac{h}{3}, y_1 + \frac{k_2}{3}) = 0.1325$$

$$k_4 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_3}{2}) = 0.1375$$

$$k_5 = hf(x_1 + \frac{h}{3}, y_1 + \frac{k_4}{3}) = 0.1325$$

$$k_6 = hf(x_1 + \frac{2h}{3}, y_1 + \frac{2k_5}{3}) = 0.1428$$

$$k_7 = hf(x_1 + \frac{2h}{3}, y_1 + \frac{2k_6}{3}) = 0.1438$$

$$k_8 = hf(x_1 + h, y_1 + k_7) = 0.1559$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + \frac{81}{2}k_3 - 32k_4 + \frac{81}{2}k_5 + \frac{81}{2}k_6 + \frac{81}{2}k_7 + 11k_8) = 0.13849$$

$$\therefore \Delta y = 0.13849 \text{ and } y_2 = y_1 + \Delta y = 1.24189$$

Percentage error: $(1.25 - 1.24189)/1.25 = 6.488 \times 10^{-03}\,\%$

Chapter 3

Multi Step Methods

If to calculate the value at a point requires the values of more than one previous point then the method is termed as **multi step method**. The multi step method may be explicit (or open or predictor) or implicit (or closed or corrector). The explicit and implicit multi step methods have been discussed in detail by Ceschino and Kuntzmann(1966), Gear(1971), Henrici(1962), Lambert(1973) and Lapidus and Seinfeld(1971).

To get the primitive of the differential equation $y' = f(x, y)$ we integrate between the limits x_{n-j} and x_{n+1}, and we get

$$y(x_{n+1}) = y(x_{n-j}) + \int_{x_{n-j}}^{x_{n+1}} f(x, y)dx$$

For different integer values of j different formula evolved. In case of explicit methods for j=0 we get Adams-Bashforth formulas while for j=1 we get Nystrom formulas. Again in case of implicit methods for j=0 we get Adams-Moulton formulas and for j=1 we get Milne-Simpson formulas.

In this chapter, general discussion on multi step methods will be done in section 3.1. In section 3.2 Adams-Bashforth formulas with the extension up to 10th order will be presented. Also section 3.3 Adams-Moulton formulas up to 10th order extension will be presented. Section 3.4 contains demonstrations of derived Adams-Bashforth and Adams-Moulton formula of 8th order.

3.1 Multi Step Methods

The numerical methods for the solution of the differential equations of the type

$$y' = f(x, y),\ y(x_0) = y_{0,}\ x \in [x_0, b] \qquad (3.1.1)$$

are called multi step methods if the value of $y(x)$ at $x = x_{n+1}$ uses the values of the dependent variable and its derivative at more than one grid or mesh points. Let us suppose that we have already obtained approximate values of y and $y' = f(x, y)$ at the points $x_m = x_0 + mh$, $m = 1,2,3,..........., n$. We denote the approximate values at these points by

$$y(x_m) = y_{m,}\ f(x_m, y(x_m)) = f_{m,}\ m = 0,1.........n$$

Then the general multi step or k-step method for the solution of (3.1.1) may be written as

$$y_{n+1}=a_1y_n+a_2y_{n-1}+....+a_ky_{n-k+1}+h\Phi(x_{n+1},x_n,....,x_{n-k+1},y'_{n+1},y'_n,....y'_{n-k+1},h) \quad (3.1.2)$$

where h is the constant step size and $a_1,a_2,......,a_k$ are real given constants. If Φ is independent of y'_{n+1}, then the general multi step is called an explicit, open or predictor method; otherwise an implicit, closed or corrector method.

The truncation or discretization error of the method (3.1.2) at $x=x_n$ is given by

$$T(y(x_n),h)=y(x_{n+1})-a_1y(x_n)-.........-a_ky(x_{n-k-1})$$

$$-h\Phi(x_{n+1},x_n,.....x_{n-k+1},y'(x_{n+1}),y'(x_n),....y'(x_{n-k+1})) \quad (3.1.3)$$

If p is the largest integer such that

$$\left|h^{-1}T(y(x_n),h)\right|=0(h^p), \quad (3.1.4)$$

then p is said to be the order of the largest multi step method.

A linear form

$$y_{n+1}=a_1y_n+a_2y_{n-1}+......+a_ky_{n-k+1}+h(b_0y'_{n+1}+b_1y'_n+......+b_ky'_{n-k+1}) \quad (3.1.5)$$

of (3.1.2) is called the general linear multi step method. The constants $a_i's\ and\ b_i's$ are real and known. The k-1 values $y_1,y_2,..........,y_{k-1}$ required to start the computation in (3.1.5) are obtained, using the single step methods. The special cases of the linear multi step method (3.1.5) are used for solving the initial value problem (3.1.1).

3.1.1 Explicit Multi Step Methods

By integrating the differential equation $y'=f(x,y)$ between the limits x_{n-j} and x_{n+1}, we get

$$\int_{x_{n-j}}^{x_{n+1}} y'dx=\int_{x_{n-j}}^{x_{n+1}} f(x,y)dx$$

$$or,\quad [y]_{x_{n-j}}^{x_{n+1}}=\int_{x_{n-j}}^{x_{n+1}} f(x,y)dx$$

$$or,\quad y(x_{n+1})=y(x_{n-j})+\int_{x_{n-j}}^{x_{n+1}} f(x,y)\,dx \quad (3.1.6)$$

To carryout integration in (3.1.6) we can approximate $f(x,y)$ by a polynomial which interpolates $f(x,y)$ at k points $x_{n,}\ x_{n-1,}................,x_{n-k+1}$. Let us use the Newton backward interpolation formula of degree (k-1) for this purpose. If $f(x,y)$ has k continuous derivatives, then we have

$$P_{k-1}(x)=f_n+(x-x_n)\frac{\nabla f_n}{h}+\frac{(x-x_n)(x-x_{n-1})}{2!}\frac{\nabla^2 f_n}{h^2}+...............+$$

$$\frac{(x-x_n)(x-x_{n-1})...........(x-x_{n-k+2})}{(k-1)!}\frac{\nabla^{k-1} f_n}{h^{k-1}}+$$

$$\frac{(x-x_n)(x-x_{n-1})...........(x-x_{n-k+1})}{k!}f^{(k)}(\xi) \qquad (3.1.7)$$

where $f^{(k)}(\xi)$ is the kth derivative of f evaluated at some ξ in an interval containing x, x_{n-k+1}, and x_n.

Substituting $u=\dfrac{x-x_n}{h}$ in (3.1.7) we get,

$$P_{k-1}(x_n+uh)=f_n+u\nabla f_n+\frac{u(u+1)}{2!}\nabla^2 f_n+........+\frac{u(u+1)........(u+k-2)}{(k-1)!}\nabla^{k-1}f_n$$
$$+\frac{u(u+1)........(u+k-1)}{k!}h^k f^{(k)}(\xi)$$

$$=\sum_{m=0}^{k-1}(-1)^m\binom{-u}{m}\nabla^m f_n+(-1)^k\binom{-u}{k}h^k f^{(k)}(\xi) \qquad (3.1.8)$$

where $\displaystyle\binom{-u}{m}=(-1)^m\frac{u(u+1)................(u+m-1)}{m!}$

Inserting (3.1.8) into (3.1.6) and putting $dx=h\ du$, we obtain

$$y(x_{n+1})=y(x_{n-j})+h\int_{-j}^{1}\left[\sum_{m=0}^{k-1}(-1)^m\binom{-u}{m}\nabla^m f_n+(-1)^k\binom{-u}{k}h^k f^{(k)}(\xi)\right]du$$

$$=y(x_{n-j})+h\sum_{m=0}^{k-1}\gamma_m^{(j)}\nabla^m f_n+T_k^{(j)} \qquad (3.1.9)$$

where $\displaystyle T_k^{(j)}=h^{k+1}\int_{-j}^{1}(-1)^k\binom{-u}{k}f^{(k)}(\xi)du$

and $$\gamma_m^{(j)} = \int_{-j}^{1} (-1)^m \binom{-u}{m} du \tag{3.1.10}$$

If we ignore the remainder term $T_k^{(j)}$ in (3.1.9), we get

$$y_{n+1} = y_{n-j} + h\sum_{m=o}^{k-1} \gamma_m^{(j)} \nabla^m f_n \tag{3.1.11}$$

An alternative form of formula (3.1.11) can be obtained if the differences $\nabla^m f_n$ are expressed in terms of the function values f_m.

From the definition of the backward difference operator ∇, we find

$$\nabla^m f_n = \sum_{\rho=o}^{m} (-1)^\rho \binom{m}{\rho} f_{n-\rho} \tag{3.1.12}$$

Substituting in (3.1.11) and regrouping, we obtain

$$y_{n+1} = y_{n-j} + h\sum_{m=o}^{k-1} \gamma_m^{*(j)} f_{n-m}. \tag{3.1.13}$$

It is obvious from (3.1.9) that with k computed values, we obtain explicit multi step methods of order k, since the truncation error is of the form ch^{k+1}, where c is independent of h. A number of interesting formulas can be obtained for various integer values of j.

3.1.2 Implicit Multi Step Methods

In the preceding subsection we have expressed y_{n+1} in terms of previously calculated ordinates and slopes. A formula similar to (3.1.11) or (3.1.13), which involves the unknown slope y'_{n+1} on the right hand side, can be obtained if we replace $f(x,y)$ in (3.1.6) by a polynomial which interpolates $f(x,y)$ at x_{n+1}, $x_n, \ldots\ldots, x_{n-k+1}$ for an integer $k > 0$. Let us assume that $f(x,y)$ has $k+1$ continuous derivatives. The Newton backward difference formula which interpolates at these $k+1$ points in terms of $u = (x - x_n)/h$ is given by

$$P_k(x_n + hu) = f_{n+1} + (u-1)\nabla f_{n+1} + \frac{u(u-1)}{2!}\nabla^2 f_{n+1} + \ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{(u-1)u(u+1)\ldots.(u+k-2)}{k!}\nabla^k f_{n+1} + \frac{(u-1)u(u+1)\ldots.(u+k-1)}{(k+1)!} h^{k+1} f^{(k+1)}(\xi)$$

i.e. $P_k(x_n + hu) = \sum_{m=0}^{k} (-1)^m \binom{1-u}{m} \nabla^m f_{n+1} + (-1)^{k+1} \binom{1-u}{k+1} h^{k+1} f^{(k+1)}(\xi)$ (3.1.14)

Substituting (3.1.14) into (3.1.6) and putting $dx = hdu$, we get

$$y(x_{n+1}) = y(x_{n-j}) + h\int_{-j}^{1} \left[\sum_{m=0}^{k} (-1)^m \binom{1-u}{m} \nabla^m f_{n+1} + (-1)^{k+1} \binom{1-u}{k+1} h^{k+1} f^{(k+1)}(\xi) \right] du$$

or, $y(x_{n+1}) = y(x_{n-j}) + h\sum_{m=0}^{k} \delta_m^{(j)} \nabla^m f_{n+1} + T_{k+1}^{*(j)}$ (3.1.15)

where $T_{k+1}^{*(j)} = h^{k+1} \int_{-j}^{1} (-1)^{k+1} \binom{1-u}{k+1} f^{(k+1)}(\xi) du$ and

$$\delta_m^{(j)} = \int_{-j}^{1} (-1)^m \binom{1-u}{m} du \tag{3.1.16}$$

Neglecting $T_{k+1}^{*(j)}$ in (3.1.15) get

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k} \delta_m^{(j)} \nabla^m f_{n+1} \tag{3.1.17}$$

where $\delta_0^{(j)} = \int_{-j}^{1} \binom{1-u}{0} du = \int_{-j}^{1} du = [u]_{-j}^{1} = 1 + j$

If we replace the difference operator $\nabla^m f_{n+1}$ in terms of the function values, we obtain,

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k} \delta_m^{*(j)} f_{n-m+1} \tag{3.1.18}$$

From (3.1.17) or (3.1.18) we can obtain a number of multi step formulas for various values of *j*. It is obvious from (3.1.15) that the implicit multi step methods are of one order higher than the corresponding explicit multi step methods with the same number of previously calculated ordinates and slopes.

3.2 Adams-Bashforth formula

Adams-Bashforth-Moulton method named after John couch Adams (1819-1892), Francis Bashforth (1819-1912), and Forest Ray Moulton (1812-1952). Adams was an astronomer who discovered the planet Neptune and refused a Knighthood in 1847. The predictor formula is known as Adams-Bashforth predictor. The corrector formula is known as Adams-Moulton corrector.

To derive the relations for the Adams-Bashforth method, we write the differential equation

$$\frac{dy}{dx} = f(x,y).$$

By integration the differential equation $y' = f(x,y)$ between the limits x_{n-j} and x_{n+1}, we get

$$\int_{x_{n-j}}^{x_{n+1}} y'dx = \int_{x_{n-j}}^{x_{n+1}} f(x,y)dx$$

$$or, \quad [y]_{x_{n-j}}^{x_{n+1}} = \int_{x_{n-j}}^{x_{n+1}} f(x,y)dx$$

$$or, \quad y(x_{n+1}) = y(x_{n-j}) + \int_{x_{n-j}}^{x_{n+1}} f(x,y)dx \tag{3.2.1}$$

To carryout integration in (3.2.1) we can approximate $f(x,y)$ by a polynomial which interpolates $f(x,y)$ at k points $x_{n,}\ x_{n-1,}$................., x_{n-k+1}. Let us use the Newton backward interpolation formula of degree (k-1) for this purpose. If $f(x,y)$ has k continuous derivatives, then we have

$$P_{k-1}(x) = f_n + (x-x_n)\frac{\nabla f_n}{h} + \frac{(x-x_n)(x-x_{n-1})}{2!}\frac{\nabla^2 f_n}{h^2} + ...$$

$$+\frac{(x-x_n)(x-x_{n-1})............(x-x_{n-k+2})}{(k-1)!}\frac{\nabla^{k-1} f_n}{h^{k-1}}$$

$$+\frac{(x-x_n)(x-x_{n-1})............(x-x_{n-k+1})}{k!} f^{(k)}(\xi) \tag{3.2.2}$$

where $f^{(k)}(\xi)$ is the kth derivative of f evaluated at some ξ in an interval containing x, x_{n-k+1}, and x_n.

Substituting $u = \frac{x-x_n}{h}$ in (3.2.2) we get,

$$P_{k-1}(x_n+uh) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + ...+\frac{u(u+1).........(u+k-2)}{(k-1)!}\nabla^{k-1} f_n +$$

$$\frac{u(u+1).........(u+k-1)}{k!} h^k f^{(k)}(\xi)$$

$$\text{i.e. } P_{k-1}(x_n+uh) = \sum_{m=0}^{k-1}(-1)^m\binom{-u}{m}\nabla^m f_n + (-1)^k\binom{-u}{k}h^k f^{(k)}(\xi) \tag{3.2.3}$$

where $\binom{-u}{m} = (-1)^m \frac{u(u+1)................(u+m-1)}{m!}$

Inserting (3.2.3) into (3.2.1) and putting $dx = h\ du$, we obtain

$$y(x_{n+1}) = y(x_{n-j}) + h\int_{-j}^{1}\left[\sum_{m=0}^{k-1}(-1)^m\binom{-u}{m}\nabla^m f_n + (-1)^k\binom{-u}{k}h^k f^{(k)}(\xi)\right]du$$

$$= y(x_{n-j}) + h\sum_{m=0}^{k-1}\gamma_m^{(j)}\nabla^m f_n + T_k^{(j)} \qquad (3.2.4)$$

where $T_k^{(j)} = h^{k+1}\int_{-j}^{1}(-1)^k\binom{-u}{k}f^{(k)}(\xi)du$

and $\gamma_m^{(j)} = \int_{-j}^{1}(-1)^m\binom{-u}{m}du \qquad (3.2.5)$

If we ignore the remainder term $T_k^{(j)}$ in (3.2.4), we get

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k-1}\gamma_m^{(j)}\nabla^m f_n \qquad (3.2.6)$$

On calculating a few of $\gamma_m^{(j)}$ from (3.2.5) we obtain

$$\gamma_0^{(j)} = \int_{-j}^{1}du = 1 + j$$

$$\gamma_1^{(j)} = \int_{-j}^{1}u\,du = \frac{1}{2}(1-j)(1+j)$$

$$\gamma_2^{(j)} = \frac{1}{2}\int_{-j}^{1}u(u+1)\,du = \frac{1}{12}(5 - 3j^2 + 2j^3)$$

$$\gamma_3^{(j)} = \frac{1}{6}\int_{-j}^{1}u(u+1)(u+2)\,du = \frac{1}{24}(3-j)(3 - j - j^2 + ...)$$

$$\gamma_4^{(j)} = \frac{1}{24}\int_{-j}^{1}u(u+1)(u+2)(u+3)du = \frac{1}{720}(251 - 90j^2 + 110j^3 - 45j^4 + 6j^5)$$

$$\gamma_5^{(j)} = \frac{1}{120}\int_{-j}^{1}u(u+1)(u+2)(u+3)(u+4)du = \frac{1}{1440}(5-j)\begin{pmatrix}95 + 19j - 25j^2 + 35j^3 - 14j^4 \\ + 2j^5\end{pmatrix}$$

$$\gamma_6^{(j)} = \frac{1}{720}\int_{-j}^{1}u(u+1)(u+2)(u+3)(u+4)(u+5)du$$

$$= \frac{1}{720}\int_{-j}^{1}(u^6 + 15u^5 + 85u^4 + 225u^3 + 274u^2 + 120u)du$$

$$= \frac{1}{720}\left[\frac{u^7}{7} + \frac{15}{6}u^6 + \frac{85}{5}u^5 + \frac{225}{4}u^4 + \frac{274}{3}u^3 + \frac{120}{2}u^2\right]_{-j}^{1}$$

$$= \frac{1}{720}\left[\frac{19087}{84} - \left(\frac{-j^7}{7} + \frac{15}{6}j^6 - \frac{85}{5}j^5 + \frac{225}{4}j^4 - \frac{274}{3}j^3 + 60j^2\right)\right]$$

$$\gamma_7^{(j)} = \frac{1}{5040}\int_{-j}^{1} u(u+1)(u+2)(u+3)(u+4)(u+5)(u+6)du$$

$$= \frac{1}{5040}\int_{-j}^{1} \left(u^7 + 21u^6 + 175u^5 + 735u^4 + 1624u^3 + 1764u^2 + 720u\right)du$$

$$= \frac{1}{5040}\left[\frac{u^8}{8} + \frac{21}{7}u^7 + \frac{175}{6}u^6 + \frac{735}{5}u^5 + \frac{1624}{4}u^4 + \frac{1764}{3}u^3 + \frac{720}{2}u^2\right]_{-j}^{1}$$

$$= \frac{1}{5040}\left[\frac{36799}{24} - \left(\frac{j^8}{8} - 3j^7 + \frac{175}{6}j^6 - 147j^5 + 406j^4 - 588j^3 + 360j^2\right)\right]$$

$$\gamma_8^{(j)} = \frac{1}{40320}\int_{-j}^{1} u(u+1)(u+2)(u+3)(u+4)(u+5)(u+6)(u+7)du$$

$$= \frac{1}{40320}\int_{-j}^{1} \left(u^8 + 28u^7 + 322u^6 + 1960u^5 + 6769u^4 + 13132u^3 + 13068u^2 + 5040u\right)du$$

$$= \frac{1}{40320}\left[\frac{u^9}{9} + \frac{28}{8}u^8 + \frac{322}{7}u^7 + \frac{1960}{6}u^6 + \frac{6769}{5}u^5 + \frac{13132}{4}u^4 + \frac{13068}{3}u^3 + \frac{5040}{2}u^2\right]_{-j}^{1}$$

$$= \frac{1}{4.320}\left[\frac{1070017}{3628800} + \left(\frac{j^9}{9} - \frac{7}{2}j^8 + 46j^7 - \frac{980}{3}j^6 + \frac{6769}{5}j^5 - 3283j^4 + 4356j^3 - 2520j^2\right)\right]$$

$$\gamma_9^{(j)} = \frac{1}{362880}\int_{-j}^{1}\begin{pmatrix} u^9 + 36u^8 + 546u^7 + 4536u^6 + 22449u^5 + 67284u^4 + 118124u^3 + 109584u^2 + \\ 40320u \end{pmatrix}du$$

$$= \frac{1}{362880}\left[\frac{2082753}{20} - \begin{pmatrix} \frac{j^{10}}{10} - 4j^9 + \frac{273}{4}j^8 - 648j^7 + \frac{22449}{6}j^6 - \frac{67284}{5}j^5 + 29531j^4 - \\ 36528j^3 + 20160j^2 \end{pmatrix}\right]$$

Thus when $j = 0$ we will have the following values

$$\gamma_0^{(0)} = 1,\ \gamma_1^{(0)} = \frac{1}{2},\ \gamma_2^{(0)} = \frac{5}{12},\ \gamma_3^{(0)} = \frac{3}{8},\ \gamma_4^{(0)} = \frac{251}{720},\ \gamma_5^{(0)} = \frac{475}{1440},\ \gamma_6^{(0)} = \frac{19087}{60480},$$

$$\gamma_7^{(0)} = \frac{36799}{120960},\ \gamma_8^{(0)} = \frac{1070017}{3628800},\ \gamma_9^{(0)} = \frac{2082735}{7257600}$$

On replacing the coefficients $\gamma_m^{(0)}$ by their values in (3.2.6) we get,

$$y_{n+1} = y_n + h\left[\begin{array}{l} f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n + \frac{251}{720}\nabla^4 f_n + \frac{475}{1440}\nabla^5 f_n + \frac{19087}{60480}\nabla^6 f_n \\ + \frac{36799}{120960}\nabla^7 f_n + \frac{1070017}{3628800}\nabla^8 f_n + \frac{2082735}{7257600}\nabla^9 f_n + \ldots\ldots\ldots\ldots\ldots \end{array}\right] \quad (3.2.7)$$

Using the following definition of the backward difference operator ∇

$$\nabla^m f_n = \sum_{\rho=0}^{m} (-1)^\rho \binom{m}{\rho} f_{n-\rho} \quad (3.2.8)$$

we can rewrite the above equation in the compact form as, where k is order,

$$y_{n+1} = y_n + h\sum_{m=0}^{k-1} \gamma_m^{*(0)} f_{n-m}. \quad (3.2.9)$$

Now from (3.2.8) for different values of m, we will have

$$\nabla^0 f_n = f_n$$

$$\nabla f_n = f_n + (-1)\binom{1}{1} f_{n-1} = f_n - f_{n-1}$$

$$\nabla^2 f_n = \sum_{\rho=0}^{2} (-1)^\rho \binom{2}{\rho} f_{n-\rho} = \binom{2}{0} f_n + (-1)\binom{2}{1} f_{n-1} + \binom{2}{2} f_{n-2} = f_n - 2f_{n-1} + f_{n-2}$$

$$\nabla^3 f_n = \sum_{\rho=0}^{3} (-1)^\rho \binom{3}{\rho} f_{n-\rho} = \binom{3}{0} f_n - \binom{3}{1} f_{n-1} + \binom{3}{2} f_{n-2} - \binom{3}{3} f_{n-3} = f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}$$

$$\nabla^4 f_n = \sum_{\rho=0}^{4} (-1)^\rho \binom{4}{\rho} f_{n-\rho} = \binom{4}{0} f_n - \binom{4}{1} f_{n-1} + \binom{4}{2} f_{n-2} - \binom{4}{3} f_{n-3} + \binom{4}{4} f_{n-4}$$

$$= f_n - 4f_{n-1} + 6f_{n-2} - 4f_{n-3} + f_{n-4}$$

$$\nabla^5 f_n = \sum_{\rho=0}^{5} (-1)^\rho \binom{5}{\rho} f_{n-\rho} = \binom{5}{0} f_n - \binom{5}{1} f_{n-1} + \binom{5}{2} f_{n-2} - \binom{5}{3} f_{n-3} + \binom{5}{4} f_{n-4} - \binom{5}{5} f_{n-5}$$

$$= f_n - 5f_{n-1} + 10f_{n-2} - 10f_{n-3} + 5f_{n-4} - f_{n-5}$$

$$\nabla^6 f_n = \sum_{\rho=0}^{6} (-1)^\rho \binom{6}{\rho} f_{n-\rho} = \binom{6}{0} f_n - \binom{6}{1} f_{n-1} + \binom{6}{2} f_{n-2} - \binom{6}{3} f_{n-3} + \binom{6}{4} f_{n-4} - \binom{6}{5} f_{n-5} + \binom{6}{6} f_{n-6}$$

$$= f_n - 6f_{n-1} + 15f_{n-2} - 20f_{n-3} + 15f_{n-4} - 6f_{n-5} + f_{n-6}$$

$$\nabla^7 f_n = \sum_{\rho=0}^{7} (-1)^\rho \binom{7}{\rho} f_{n-\rho}$$

$$= \binom{7}{0} f_n - \binom{7}{1} f_{n-1} + \binom{7}{2} f_{n-2} - \binom{7}{3} f_{n-3} + \binom{7}{4} f_{n-4} - \binom{7}{5} f_{n-5} + \binom{7}{6} f_{n-6} - \binom{7}{7} f_{n-7}$$

$$= f_n - 7f_{n-1} + 21f_{n-2} - 35f_{n-3} + 35f_{n-4} - 21f_{n-5} + 7f_{n-6} - f_{n-7}$$

$$\nabla^8 f_n = \sum_{\rho=0}^{8}(-1)^\rho \binom{8}{\rho} f_{n-\rho}$$

$$= \binom{8}{0} f_n - \binom{8}{1} f_{n-1} + \binom{8}{2} f_{n-2} - \binom{8}{3} f_{n-3} + \binom{8}{4} f_{n-4} - \binom{8}{5} f_{n-5} + \binom{8}{6} f_{n-6} - \binom{8}{7} f_{n-7} + \binom{8}{8} f_{n-8}$$

$$= f_n - 8f_{n-1} + 28f_{n-2} - 56f_{n-3} + 70f_{n-4} - 56f_{n-5} + 28f_{n-6} - 8f_{n-7} + f_{n-8}$$

$$\nabla^9 f_n = \sum_{\rho=0}^{9}(-1)^\rho \binom{9}{\rho} f_{n-\rho}$$

$$= \binom{9}{0} f_n - \binom{9}{1} f_{n-1} + \binom{9}{2} f_{n-2} - \binom{9}{3} f_{n-3} + \binom{9}{4} f_{n-4} - \binom{9}{5} f_{n-5} + \binom{9}{6} f_{n-6} - \binom{9}{7} f_{n-7} + \binom{9}{8} f_{n-8} - \binom{9}{9} f_{n-9}$$

$$= f_n - 9f_{n-1} + 36f_{n-2} - 84f_{n-3} + 126f_{n-4} - 126f_{n-5} + 84f_{n-6} - 36f_{n-7} + 9f_{n-8} - f_{n-9}$$

A comparison between (3.2.7) and (3.2.9) suggests that the values of $\gamma_m^{*(0)}$, $m = 1,2,3,......$ will be different for different values of *k* for same *m*.

Now we will estimate the values of $\gamma_m^{*(0)}$ for different values of *k*.

For *k*=2,

From (3.2.7) we have

$$y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n\right] = y_n + h\left[f_n + \frac{1}{2}(f_n - f_{n-1})\right]$$

$$= y_n + h\left(\frac{3}{2}f_n - \frac{1}{2}f_{n-1}\right)$$

and from (3.2.9)

$$y_{n+1} = y_n + h\sum_{m=o}^{1}\gamma_m^{*(0)} f_{n-m}$$

$$= y_n + h\left[\gamma_0^{*(0)} f_n + \gamma_1^{*(0)} f_{n-1}\right]$$

Thus for $k = 2$ $\qquad \gamma_0^{*(0)} = \frac{3}{2}, \quad \gamma_1^{*(0)} = \frac{-1}{2}$

For *k* = *3*

$$y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n\right] = y_n + h\left[f_n + \frac{1}{2}(f_n - f_{n-1}) + \frac{5}{12}(f_n - 2f_{n-1} + f_{n-2})\right]$$

$$= y_n + h\left[\frac{23}{12}f_n - \frac{8}{6}f_{n-1} + \frac{5}{12}f_{n-2}\right]$$

and

$$y_{n+1} = y_n + h\sum_{m=0}^{2}\gamma_m^{*(0)} f_{n-m} = y_n + h\left[\gamma_0^{*(0)} f_n + \gamma_1^{*(0)} f_{n-1} + \gamma_2^{*(0)} f_{n-2}\right]$$

Thus when $k = 3$ then $\gamma_0^{*(0)} = \frac{23}{12}, \gamma_1^{*(0)} = \frac{-8}{6}, \gamma_2^{*(0)} = \frac{5}{12}$

For $k = 4$

$$y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n\right]$$

$$= y_n + h\left[f_n + \frac{1}{2}(f_n - f_{n-1}) + \frac{5}{12}(f_n - 2f_{n-1} + f_{n-2}) + \frac{3}{8}(f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3})\right]$$

$$= y_n + h\left[\frac{55}{24}f_n - \frac{59}{24}f_{n-1} + \frac{37}{24}f_{n-2} - \frac{9}{24}f_{n-3}\right]$$

Also

$$y_{n+1} = y_n + h\sum_{m=0}^{3}\gamma_m^{*(0)} f_{n-m} = y_n + h\left[\gamma_0^{*(0)} f_n + \gamma_1^{*(0)} f_{n-1} + \gamma_2^{*(0)} f_{n-2} + \gamma_3^{*(0)} f_{n-3}\right]$$

Hence when $k = 4$ then $\gamma_0^{*(0)} = \frac{55}{24}, \gamma_1^{*(0)} = \frac{-59}{24}, \gamma_2^{*(0)} = \frac{37}{24}, \gamma_3^{*(0)} = \frac{-9}{24}$

For $k = 5$

$$y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n + \frac{251}{720}\nabla^4 f_n\right]$$

$$= y_n + h\left[\frac{55}{24}f_n - \frac{59}{24}f_{n-1} + \frac{37}{24}f_{n-2} - \frac{9}{24}f_{n-3} + \frac{251}{720}(f_n - 4f_{n-1} + 6f_{n-2} - 4f_{n-3} + f_{n-4})\right]$$

$$= y_n + h\left[\frac{1901}{720}f_n - \frac{2774}{720}f_{n-1} + \frac{2616}{720}f_{n-2} - \frac{1274}{720}f_{n-3} + \frac{251}{720}f_{n-4}\right]$$

Again

$$y_{n+1} = y_n + h\sum_{m=0}^{4}\gamma_m^{*(0)} f_{n-m} = y_n + h\left[\gamma_0^{*(0)} + \gamma_1^{*(0)} f_{n-1} + \gamma_2^{*(0)} f_{n-2} + \gamma_3^{*(0)} f_{n-3} + \gamma_4^{*(0)} f_{n-4}\right]$$

Thus for k=5 $\gamma_0^{*(0)} = \frac{1901}{720}, \gamma_1^{*(0)} = \frac{-2774}{720}, \gamma_2^{*(0)} = \frac{2616}{720}, \gamma_3^{*(0)} = \frac{-1274}{720}, \gamma_4^{*(0)} = \frac{251}{720}$

For $k = 6$

From (3.2.7) we will have

$$y_{n+1} = y_n + h\begin{bmatrix} \frac{1901}{720}f_n - \frac{2774}{720}f_{n-1} + \frac{2616}{720}f_{n-2} - \frac{1274}{720}f_{n-3} + \frac{251}{720}f_{n-4} + \\ \frac{475}{1440}\left(f_n - 5f_{n-1} + 10f_{n-2} - 10f_{n-3} + 5f_{n-4} - f_{n-5}\right) \end{bmatrix}$$

$$= y_n + h\left[\frac{4277}{1440}f_n - \frac{7923}{1440}f_{n-1} + \frac{9982}{1440}f_{n-2} - \frac{7298}{1440}f_{n-3} + \frac{2877}{1440}f_{n-4} - \frac{475}{1440}f_{n-5}\right]$$

and from (3.2.9)

$$y_{n+1} = y_n + h\sum_{m=0}^{5}\gamma_m^{*(0)}f_{n-m} = y_n + h\left[\gamma_0^{*(0)}f_n + \gamma_1^{*(0)}f_{n-1} + \gamma_2^{*(0)}f_{n-2} + \gamma_3^{*(0)}f_{n-3} + \gamma_4^{*(0)}f_{n-4} + \gamma_5^{*(0)}f_{n-5}\right]$$

Hence for k = 6 $\quad \gamma_0^{*(0)} = \frac{4277}{1440}, \quad \gamma_1^{*(0)} = \frac{-7923}{1440}, \quad \gamma_2^{*(0)} = \frac{9982}{1440}, \quad \gamma_3^{*(0)} = \frac{-7298}{1440},$

$\gamma_4^{*(0)} = \frac{2877}{1440}, \ \gamma_5^{*(0)} = \frac{-475}{1440}$

For $k = 7$

$$y_{n+1} = y_n + h\begin{bmatrix} \frac{4277}{1440}f_n - \frac{7923}{1440}f_{n-1} + \frac{9982}{1440}f_{n-2} - \frac{7298}{1440}f_{n-3} + \frac{2877}{1440}f_{n-4} - \frac{475}{1440}f_{n-5} + \\ \frac{19087}{60480}(f_n - 6f_{n-1} + 15f_{n-2} - 20f_{n-3} + 15f_{n-4} - 6f_{n-5} + f_{n-6}) \end{bmatrix}$$

$$= y_n + h\begin{bmatrix} \frac{198721}{60480}f_n - \frac{447288}{60480}f_{n-1} + \frac{705549}{60480}f_{n-2} - \frac{688256}{60480}f_{n-3} + \frac{407139}{60480}f_{n-4} - \\ \frac{134472}{60480}f_{n-5} + \frac{19087}{60480}f_{n-6} \end{bmatrix}$$

and

$$y_{n+1} = y_n + h\sum_{m=0}^{6}\gamma_m^{*(0)}f_{n-m}$$

$$= y_n + h\left[\gamma_0^{*(0)}f_n + \gamma_1^{*(0)}f_{n-1} + \gamma_2^{*(0)}f_{n-2} + \gamma_3^{*(0)}f_{n-3} + \gamma_4^{*(0)}f_{n-4} + \gamma_5^{*(0)}f_{n-5} + \gamma_6^{*(0)}f_{n-6}\right]$$

Hence comparing for $k = 7$ we get $\quad \gamma_0^{*(0)} = \frac{198721}{60480}, \quad \gamma_1^{*(0)} = \frac{-447288}{60480}, \quad \gamma_2^{*(0)} = \frac{705549}{60480},$

$\gamma_3^{*(0)} = \frac{-688256}{60480}, \quad \gamma_4^{*(0)} = \frac{407139}{60480}, \quad \gamma_5^{*(0)} = \frac{-134472}{60480}, \quad \gamma_6^{*(0)} = \frac{19087}{60480}$

For $k = 8$

$$y_{n+1} = y_n + h\left[\begin{array}{l} \frac{198721}{60480} f_n - \frac{447288}{60480} f_{n-1} + \frac{705549}{60480} f_{n-2} - \frac{688256}{60480} f_{n-3} + \frac{407139}{60480} f_{n-4} - \\ \frac{134472}{60480} f_{n-5} + \frac{19087}{60480} f_{n-6} + \frac{36799}{120960}\left(\begin{array}{l} f_n - 7f_{n-1} + 21f_{n-2} - 35f_{n-3} + 35f_{n-4} \\ -21f_{n-5} + 7f_{n-6} - f_{n-7} \end{array}\right) \end{array}\right]$$

$$= y_n + h\left[\begin{array}{l} \frac{434241}{120960} f_n - \frac{1152169}{120960} f_{n-1} + \frac{2183877}{120960} f_{n-2} - \frac{2664477}{120960} f_{n-3} + \frac{2102243}{120960} f_{n-4} \\ -\frac{1041723}{120960} f_{n-5} + \frac{295767}{120960} f_{n-6} - \frac{36799}{120960} f_{n-7} \end{array}\right]$$

Also

$$y_{n+1} = y_n + h\sum_{m=0}^{7} \gamma_m^{*(0)} f_{n-m}$$

$$= y_n + h\left[\gamma_0^{*(0)} f_n + \gamma_1^{*(0)} f_{n-1} + \gamma_2^{*(0)} f_{n-2} + \gamma_3^{*(0)} f_{n-3} + \gamma_4^{*(0)} f_{n-4} + \gamma_5^{*(0)} f_{n-5} + \gamma_6^{*(0)} f_{n-6} + \gamma_7^{*(0)} f_{n-7}\right]$$

Hence for $k = 8$ $\gamma_0^{*(0)} = \frac{434241}{120960}$, $\gamma_1^{*(0)} = \frac{-1152169}{120960}$, $\gamma_2^{*(0)} = \frac{2183877}{120960}$, $\gamma_3^{*(0)} = \frac{-2664477}{120960}$,

$\gamma_4^{*(0)} = \frac{2102243}{120960}$, $\gamma_5^{*(0)} = \frac{-1041723}{120960}$, $\gamma_6^{*(0)} = \frac{295767}{120960}$, $\gamma_7^{*(0)} = \frac{-36799}{120960}$

For $k = 9$

$$y_{n+1} = y_n + h\left[\begin{array}{l} \frac{434241}{120960} f_n - \frac{1152169}{120960} f_{n-1} + \frac{2183877}{120960} f_{n-2} - \frac{2664477}{120960} f_{n-3} + \frac{2102243}{120960} f_{n-4} \\ -\frac{1041723}{120960} f_{n-5} + \frac{295767}{120960} f_{n-6} - \frac{36799}{120960} f_{n-7} + \\ \frac{1070017}{3628800}\left(\begin{array}{l} f_n - 8f_{n-1} + 28f_{n-2} - 56f_{n-3} + 70f_{n-4} - 56f_{n-5} + 28f_{n-6} - 8f_{n-7} \\ + f_{n-8} \end{array}\right) \end{array}\right]$$

$$= y_n + h\left[\begin{array}{l} \frac{14097247}{3628800} f_n - \frac{43125206}{3628800} f_{n-1} + \frac{95476786}{3628800} f_{n-2} - \frac{139855262}{3628800} f_{n-3} + \\ \frac{137968480}{3628800} f_{n-4} - \frac{91172642}{3628800} f_{n-5} + \frac{38833486}{3628800} f_{n-6} - \frac{9664106}{3628800} f_{n-7} + \\ \frac{1070017}{3628800} f_{n-8} \end{array}\right]$$

Again

$$y_{n+1} = y_n + h\sum_{m=0}^{8} \gamma_m^{*(0)} f_{n-m}$$

i.e. $y_{n+1} = y_n + h\begin{bmatrix} \gamma_0^{*(0)} f_n + \gamma_1^{*(0)} f_{n-1} + \gamma_2^{*(0)} f_{n-2} + \gamma_3^{*(0)} f_{n-3} + \gamma_4^{*(0)} f_{n-4} + \gamma_5^{*(0)} f_{n-5} + \gamma_6^{*(0)} f_{n-6} + \\ \gamma_7^{*(0)} f_{n-7} + \gamma_8^{*(0)} f_{n-8} \end{bmatrix}$

Thus for k = 9 we have $\gamma_0^{*(0)} = \dfrac{14097247}{3628800}$, , $\gamma_2^{*(0)} = \dfrac{95476786}{3628800}$,

$\gamma_3^{*(0)} = \dfrac{-139855262}{3628800}$, $\gamma_4^{*(0)} = \dfrac{137968480}{3628800}$, $\gamma_5^{*(0)} = \dfrac{-91172642}{3628800}$, $\gamma_6^{*(0)} = \dfrac{38833486}{3628800}$,

$\gamma_7^{*(0)} = \dfrac{-9664106}{3628800}$, $\gamma_8^{*(0)} = \dfrac{1070017}{3628800}$

For $k = 10$

$$y_{n+1} = y_n + h\begin{bmatrix} \frac{14097247}{3628800} f_n - \frac{43125206}{3628800} f_{n-1} + \frac{95476786}{3628800} f_{n-2} - \frac{139855262}{3628800} f_{n-3} + \\ \frac{137968480}{3628800} f_{n-4} - \frac{91172642}{3628800} f_{n-5} + \frac{38833486}{3628800} f_{n-6} - \frac{9664106}{3628800} f_{n-7} + \\ \frac{1070017}{3628800} f_{n-8} + \frac{2082753}{7257600}\begin{pmatrix} f_n - 9f_{n-1} + 36f_{n-2} - 84f_{n-3} + 126f_{n-4} - 126f_{n-5} \\ +84f_{n-6} - 36f_{n-7} + 9f_{n-8} - f_{n-9} \end{pmatrix} \end{bmatrix}$$

$$= y_n + h\begin{bmatrix} \frac{30277247}{7257600} f_n - \frac{104995189}{7257600} f_{n-1} + \frac{265932680}{7257600} f_{n-2} - \frac{454661776}{7257600} f_{n-3} + \\ \frac{538363838}{7257600} f_{n-4} - \frac{444772162}{7257600} f_{n-5} + \frac{252618224}{7257600} f_{n-6} - \frac{94307320}{7257600} f_{n-7} + \\ \frac{20884811}{7257600} f_{n-8} - \frac{2082753}{7257600} f_{n-9} \end{bmatrix}$$

Also

$$y_{n+1} = y_n + h\sum_{m=0}^{9} \gamma_m^{*(0)} f_{n-m}$$

$$= y_n + h\begin{bmatrix} \gamma_0^{*(0)} f_n + \gamma_1^{*(0)} f_{n-1} + \gamma_2^{*(0)} f_{n-2} + \gamma_3^{*(0)} f_{n-3} + \gamma_4^{*(0)} f_{n-4} + \gamma_5^{*(0)} f_{n-5} + \gamma_6^{*(0)} f_{n-6} + \\ \gamma_7^{*(0)} f_{n-7} + \gamma_8^{*(0)} f_{n-8} + \gamma_9^{*(0)} f_{n-9} \end{bmatrix}$$

Thus for $k = 10$ $\gamma_0^{*(0)} = \dfrac{30277247}{7257600}$, $\gamma_1^{*(0)} = \dfrac{-104995189}{7257600}$, $\gamma_2^{*(0)} = \dfrac{265932680}{7257600}$,

$\gamma_3^{*(0)} = \dfrac{-454661776}{7257600}$, $\gamma_4^{*(0)} = \dfrac{538363838}{7257600}$, $\gamma_5^{*(0)} = \dfrac{-444772162}{7257600}$, $\gamma_6^{*(0)} = \dfrac{252618224}{7257600}$,

$\gamma_7^{*(0)} = \dfrac{-94307320}{7257600}$, $\gamma_8^{*(0)} = \dfrac{20884811}{7257600}$, $\gamma_9^{*(0)} = \dfrac{-2082753}{7257600}$

The above calculated values of $\gamma_m^{*(0)}$ for different values of k are tabulated in Table-3.1.

Table-3.1-The coefficients $\gamma_m^{*(0)}$ in the formula $y_{n+1} = y_n + h\sum_{m=0}^{k-1} \gamma_m^{*(0)} f_{n-m}$

k	$\gamma_0^{*(0)}$	$\gamma_1^{*(0)}$	$\gamma_2^{*(0)}$	$\gamma_3^{*(0)}$	$\gamma_4^{*(0)}$	$\gamma_5^{*(0)}$	$\gamma_6^{*(0)}$	$\gamma_7^{*(0)}$	$\gamma_8^{*(0)}$	$\gamma_9^{*(0)}$
2.	$\frac{3}{2}$	$-\frac{1}{2}$								
3.	$\frac{23}{12}$	$\frac{-16}{12}$	$\frac{5}{12}$							
4.	$\frac{55}{24}$	$\frac{-59}{24}$	$\frac{37}{24}$	$\frac{-9}{24}$						
5.	$\frac{1901}{720}$	$\frac{-2774}{720}$	$\frac{2616}{720}$	$\frac{-1274}{720}$	$\frac{251}{720}$					
6.	$\frac{4277}{1440}$	$\frac{-7923}{1440}$	$\frac{9982}{1440}$	$\frac{-7298}{1440}$	$\frac{2877}{1440}$	$\frac{-475}{1440}$				
7.	$\frac{198721}{60480}$	$\frac{-447288}{60480}$	$\frac{705549}{60480}$	$\frac{-688256}{60480}$	$\frac{407139}{60480}$	$\frac{19087}{60480}$				
8.	$\frac{434241}{120960}$	$\frac{-11521069}{120960}$	$\frac{-2664477}{120960}$	$\frac{2102243}{120960}$	$\frac{-1041723}{120960}$	$\frac{295767}{120960}$	$\frac{-36799}{120960}$			
9.	$\frac{14097247}{3628800}$	$\frac{-43125206}{3628800}$	$\frac{95476786}{3628800}$	$\frac{-139855262}{3628800}$	$\frac{137968480}{3628800}$	$\frac{-91172642}{3628800}$	$\frac{38833486}{3628800}$	$\frac{-9664106}{3628800}$	$\frac{1070017}{3628800}$	
10.	$\frac{30277247}{7257600}$	$\frac{-104995189}{7257600}$	$\frac{265932680}{7257600}$	$\frac{-454661776}{7257600}$	$\frac{538363838}{7257600}$	$\frac{-444772162}{7257600}$	$\frac{252618224}{7257600}$	$\frac{-94307320}{7257600}$	$\frac{20884811}{7257600}$	$\frac{-2082753}{7257600}$

3.3 Adams- Moulton formula

To derive the relations for the Adams-Moulton formula, we write the differential equation

$$\frac{dy}{dx} = f(x,y).$$

By integration the differential equation $y' = f(x,y)$ between the limits x_{n-j} and x_{n+1}, we get

$$\int_{x_{n-j}}^{x_{n+1}} y'dx = \int_{x_{n-j}}^{x_{n+1}} f(x,y)dx$$

$$or, \quad [y]_{x_{n-j}}^{x_{n+1}} = \int_{x_{n-j}}^{x_{n+1}} f(x,y)dx$$

$$or, \quad y(x_{n+1}) = y(x_{n-j}) + \int_{x_{n-j}}^{x_{n+1}} f(x,y)dx \qquad (3.3.1)$$

To carryout integration in (3.3.1) we can approximate $f(x,y)$ by a polynomial which interpolates $f(x,y)$ at k points x_{n+1}, $x_n, \ldots\ldots, x_{n-k+1}$ for an integer $k > 0$. Let us assume that $f(x,y)$ has $k+1$ continuous derivatives. The Newton backward difference formula which interpolates at these k+1 points in terms of $u = (x - x_n)/h$ is given by

$$P_k(x_n + hu) = f_{n+1} + (u-1)\nabla f_{n+1} + \frac{(u-1)u}{2!}\nabla^2 f_{n+1} + \ldots\ldots\ldots\ldots$$

$$+ \frac{(u-1)u(u+1)\ldots\ldots(u+k-2)}{k!}\nabla^k f_{n+1} + \frac{(u-1)u(u+1)\ldots\ldots.(u+k-1)}{(k+1)!}h^{k+1} f^{(k+1)}(\xi)$$

$$= \sum_{m=0}^{k} (-1)^m \binom{1-u}{m} \nabla^m f_{n+1} + (-1)^{k+1}\binom{1-u}{k+1} h^{k+1} f^{(k+1)}(\xi) \qquad (3.3.2)$$

substituting (3.3.2) into (3.3.1), we get

$$y(x_{n+1}) = y(x_{n-j}) + h\int_{-j}^{1}\left[\sum_{m=0}^{k}(-1)^m\binom{1-u}{m}\nabla^m f_{n+1} + (-1)^{k+1}\binom{1-u}{k+1}h^{k+1} f^{(k+1)}\right]du$$

$$\text{or, } y(x_{n+1}) = y(x_{n-j}) + h\sum_{m=0}^{k} \delta_m^{(j)}\nabla^m f_{n+1} + T_{k+1}^{*(j)} \qquad (3.3.3)$$

Where $T_{k+1}^{*(j)} = h^{k+1}\int_{-j}^{1}(-1)^{k+1}\binom{1-u}{k+1} f^{(k+1)}(\xi)du$

and $\delta_m^{(j)} = \int_{-j}^{1}(-1)^m\binom{1-u}{m}du$

Neglecting $T_{k+1}^{*(j)}$ in (3.3.3) we get

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k}\delta_m^{(j)}\nabla^m f_{n+1} \qquad (3.3.4)$$

Now $\delta_0^{(j)} = \int_{-j}^{1} \binom{1-u}{0} du = \int_{-j}^{1} du = [u]_{-j}^{1} = 1 + j$

$$\delta_1^{(j)} = \int_{-j}^{1} (-1)\binom{1-u}{1} du = -\int_{-j}^{1} (1-u)du = -\left[u - \frac{u^2}{2}\right]_{-j}^{1} = \frac{-1}{2}(1+j)^2$$

$$\delta_2^{(j)} = \int_{-j}^{1} (-1)^2\binom{1-u}{2} du = \int_{-j}^{1} \frac{(1-u)(-u)}{2!} du = \frac{-1}{2\times 6}\left(1 - 3j^2 - 2j^3\right) = \frac{-1}{12}(1+j)^2(1-2j)$$

$$\delta_3^{(j)} = \int_{-j}^{1} -\binom{1-u}{3} du = -\int_{-j}^{1} \frac{(1-u)(-u)(1-u-2)}{3!} du = \int_{-j}^{1} u(1-u)(1+u)du$$

$$= \int_{-j}^{1} (1-u^2)udu = \int_{-j}^{1} \frac{1}{6}(u^3 - u)du = \frac{1}{6}\left[\frac{u^4}{4} - \frac{u^2}{2}\right]_{-j}^{1} = \frac{-1}{24}(1 - 2j^2 + j^4)$$

$$\delta_4^{(j)} = \int_{-j}^{1} \binom{1-u}{4} du = \int_{-j}^{1} \frac{(1-u)(-u)(1-u-2)(1-u-3)}{4!} du = \int_{-j}^{1} \frac{1}{24}(u^3 - u)(-2-u)du$$

$$= \frac{1}{24}\int_{-j}^{1} -(u^4 + 2u^3 - u^2 - 2u)du = -\frac{1}{24}\left[\frac{u^5}{5} + \frac{u^4}{2} - \frac{u^3}{3} - u^2\right]_{-j}^{1}$$

$$= \frac{-1}{720}(1+j)^2(19 - 38j + 27j^2 - 6j^3)$$

$$\delta_5^{(j)} = \int_{-j}^{1} -\binom{1-u}{5} du = -\int_{-j}^{1} \frac{(1-u)(-u)(1-u-2)(1-u-3)(1-u-4)}{5!} du$$

$$= \frac{1}{120}\int_{-j}^{1} \left(u^5 + 5u^4 + 5u^3 - 5u^2 - 6u\right)du = \frac{1}{120}\left[\frac{u^6}{6} + u^5 + \frac{5}{4}u^4 - \frac{5}{3}u^3 - \frac{6}{2}u^2\right]_{-j}^{1}$$

$$= \frac{-1}{1440}(1+j)^2\left(27 - 54j + 45j^2 - 16j^3 + 2j^4\right)$$

$$\delta_6^{(j)} = \int_{-j}^{1} \binom{1-u}{6} du = \int_{-j}^{1} \frac{(u^5 + 5u^4 + 5u^3 - 5u^2 - 6u)(u+4)du}{6!}$$

$$= \frac{1}{720}\int_{-j}^{1} \left(u^6 + 9u^5 + 25u^4 + 15u^3 - 26u^2 - 24u\right)du$$

$$= \frac{1}{720}\left[\frac{u^7}{7} + \frac{9}{6}u^6 + \frac{25}{5}u^5 + \frac{15}{4}u^4 - \frac{26}{3}u^3 - \frac{24}{2}u^2\right]_{-j}^{1}$$

$$= \frac{1}{720}\left[-\frac{863}{84} - \left(\frac{-j^7}{7} + \frac{9}{6}j^6 - 5j^5 + \frac{15}{4}j^4 + \frac{26}{3}j^3 - 12j^2\right)\right]$$

$$\delta_7^{(j)} = -\int_{-j}^{1}\binom{1-u}{7}du = \frac{1}{7!}\int_{-j}^{1}\left(u^6 + 9u^5 + 25u^4 + 15u^3 - 26u^2 - 24u\right)(u+5)du$$

$$= \frac{1}{5040}\int_{-j}^{1}\left(u^7 + 14u^6 + 70u^5 + 140u^4 + 49u^3 - 154u^2 - 120u\right)du$$

$$= \frac{1}{5040}\left[\frac{u^8}{8} + \frac{14}{7}u^7 + \frac{70}{6}u^6 + \frac{140}{5}u^5 + \frac{49}{4}u^4 - \frac{154}{3}u^3 - \frac{120}{2}u^2\right]_{-j}^{1}$$

$$= \frac{1}{5040}\left[\frac{-1235}{4} - \left(\frac{j^8}{8} - 2j^7 + \frac{35}{3}j^6 - 28j^5 + \frac{49}{4}j^4 + \frac{154}{3}j^3 - 60j^2\right)\right]$$

$$\delta_8^{(j)} = \int_{-j}^{1}\binom{1-u}{8}du = \frac{1}{8!}\int_{-j}^{1}(u^7 + 14u^6 + 70u^5 + 140u^4 + 49u^3 - 154u^2 - 120u)(u+6)du$$

$$= \frac{1}{40320}\int_{-j}^{1}\left(u^8 + 20u^7 + 154u^6 + 560u^5 + 889u^4 + 140u^3 - 1044u^2 - 720u\right)du$$

$$= \frac{1}{40320}\left[\frac{u^9}{9} + \frac{20}{8}u^8 + \frac{154}{7}u^7 + \frac{560}{6}u^6 + \frac{889}{5}u^5 + \frac{140}{4}u^4 - \frac{1044}{3}u^3 - \frac{720}{2}u^2\right]_{-j}^{1}$$

$$= \frac{1}{40320}\left[\frac{-33953}{90} - \left(\frac{-j^9}{9} + \frac{20}{8}j^8 - 22j^7 + \frac{280}{3}j^6 - \frac{889}{5}j^5 + 35j^4 + 348j^3 - 360j^2\right)\right]$$

$$\delta_9^{(j)} = \frac{1}{9!}\int_{-j}^{1}\left(u^8 + 20u^7 + 154u^6 + 560u^5 + 889u^4 + 140u^3 - 1044u^2 - 720u\right)(u+7)du$$

$$= \frac{1}{9!}\int_{-j}^{1}\left(u^9 + 27u^8 + 294u^7 + 1638u^6 + 4809u^5 + 6363u^4 - 64u^3 - 8028u^2 - 5040u\right)du$$

$$= \frac{1}{9!}\left[\frac{u^{10}}{10} + \frac{27}{9}u^9 + \frac{294}{8}u^8 + \frac{1638}{7}u^7 + \frac{4809}{6}u^6 + \frac{6363}{5}u^5 - \frac{64}{4}u^4 - \frac{8028}{3}u^3 - \frac{5040}{2}u^2\right]_{-j}^{1}$$

$$= \frac{1}{362880}\left[\frac{-57281}{20} - \left(\begin{matrix}\frac{j^{10}}{10} - 3j^9 + \frac{294}{8}j^8 - \frac{1638}{7}j^7 + \frac{4809}{6}j^6 - \frac{6363}{5}j^5 - 16j^4 + \\ \frac{8028}{3}j^3 - 2520j^2\end{matrix}\right)\right]$$

$$\delta_{10}^{(j)} = \frac{1}{10!}\int_{-j}^{1}\begin{matrix}\left(u^9 + 27u^8 + 294u^7 + 1638u^6 + 4809u^5 + 6363u^4 - 64u^3 - 8028u^2 - 5040u\right) \\ (u+8)du\end{matrix}$$

$$=\frac{1}{10!}\left[\begin{array}{l}\frac{u^{11}}{11}+\frac{35}{10}u^{10}+\frac{510}{9}u^{9}+\frac{3990}{8}u^{8}+\frac{17913}{7}u^{7}+\frac{44835}{6}u^{6}+\frac{50840}{5}u^{5}-\frac{8540}{4}u^{4}\\-\frac{69264}{3}u^{3}-\frac{40320}{2}u^{2}\end{array}\right]_{-j}^{1}$$

$$=\frac{1}{3628800}\left[\frac{-9751299}{396}-\left(\begin{array}{l}\frac{-j^{11}}{11}+\frac{35}{10}j^{10}-\frac{510}{9}j^{9}+\frac{3990}{8}j^{8}-\frac{17913}{7}j^{7}+\frac{44835}{6}j^{6}-\frac{50840}{5}j^{5}-\frac{8540}{4}j^{4}\\+\frac{69264}{3}j^{3}-\frac{40320}{2}j^{2}\end{array}\right)\right]$$

Thus when $j=0$ we will have the following values-

$$\delta_1^{(0)}=\frac{-1}{2},\ \delta_2^{(0)}=\frac{-1}{12},\ \delta_3^{(0)}=\frac{-1}{24},\ \delta_4^{(0)}=\frac{-19}{720},\ \delta_5^{(0)}=\frac{-27}{1440},\ \delta_6^{(0)}=\frac{-863}{60480},$$

$$\delta_7^{(0)}=\frac{-1235}{120960},\ \delta_8^{(0)}=\frac{-33953}{3628800},\ \delta_9^{(0)}=\frac{-57281}{7257600},\ \delta_{10}^{(0)}=\frac{-9751299}{1437004800}$$

Considering $k=10$ with $j=0$ from (3.3.4) we will have

$$y_{n+1}=y_n+h\left[\begin{array}{l}\delta_0^{(0)}\nabla^0 f_{n+1}+\delta_1^{(0)}\nabla f_{n+1}+\delta_2^{(0)}\nabla^2 f_{n+1}+\delta_3^{(0)}\nabla^3 f_{n+1}+\delta_4^{(0)}\nabla^4 f_{n+1}+\delta_5^{(0)}\nabla^5 f_{n+1}+\\\delta_6^{(0)}\nabla^6 f_{n+1}+\delta_7^{(0)}\nabla^7 f_{n+1}+\delta_8^{(0)}\nabla^8 f_{n+1}+\delta_9^{(0)}\nabla^9 f_{n+1}+\delta_{10}^{(0)}\nabla^{10} f_{n+1}\end{array}\right]$$

Using the values of $\delta_m^{(0)}$ we get

$$y_{n+1}=y_n+h\left[\begin{array}{l}f_{n+1}-\frac{1}{2}\nabla f_{n+1}-\frac{1}{12}\nabla^2 f_{n+1}-\frac{1}{24}\nabla^3 f_{n+1}-\frac{19}{720}\nabla^4 f_{n+1}-\frac{27}{1440}\nabla^5 f_{n+1}-\\\frac{863}{60480}\nabla^6 f_{n+1}-\frac{1235}{120960}\nabla^7 f_{n+1}-\frac{33953}{3628800}\nabla^8 f_{n+1}-\frac{57281}{7257600}\nabla^9 f_{n+1}-\\\frac{9751299}{1437004800}\nabla^{10} f_{n+1}\end{array}\right] \quad (3.3.5)$$

Using the following definition of the backward difference operator ∇

$$\nabla^m f_{n+1}=\sum_{\rho=0}^{m}(-1)^{\rho}\binom{m}{\rho}f_{n+1-\rho} \quad (3.3.6)$$

we can rewrite (3.3.5) as

$$y_{n+1}=y_n+h\sum_{m=0}^{k}\delta_m^{*(0)}f_{n-m+1} \quad (3.3.7)$$

Now from (3.3.6) for different values of m, we will have

$$\nabla^0 f_{n+1}=f_{n+1}$$

$$\nabla f_{n+1}=f_{n+1}+(-1)\binom{1}{1}f_n=f_{n+1}-f_n$$

$$\nabla^2 f_{n+1} = \sum_{\rho=0}^{2} (-1)^\rho \binom{2}{\rho} f_{n+1-\rho} = \binom{2}{0} f_{n+1} + (-1)\binom{2}{1} f_n + \binom{2}{2} f_{n-1} = f_{n+1} - 2f_n + f_{n-1}$$

$$\nabla^3 f_{n+1} = \sum_{\rho=0}^{3} (-1)^\rho \binom{3}{\rho} f_{n+1-\rho} = \binom{3}{0} f_{n+1} - \binom{3}{1} f_n + \binom{3}{2} f_{n-1} - \binom{3}{3} f_{n-2} = f_{n+1} - 3f_n + 3f_{n-1} - f_{n-2}$$

$$\nabla^4 f_{n+1} = \sum_{\rho=0}^{4} (-1)^\rho \binom{4}{\rho} f_{n+1-\rho} = \binom{4}{0} f_{n+1} - \binom{4}{1} f_n + \binom{4}{2} f_{n-1} - \binom{4}{3} f_{n-2} + \binom{4}{4} f_{n-3}$$

$$= f_{n+1} - 4f_n + 6f_{n-1} - 4f_{n-2} + f_{n-3}$$

$$\nabla^5 f_{n+1} = \sum_{\rho=0}^{5} (-1)^\rho \binom{5}{\rho} f_{n+1-\rho} = \binom{5}{0} f_{n+1} - \binom{5}{1} f_n + \binom{5}{2} f_{n-1} - \binom{5}{3} f_{n-2} + \binom{5}{4} f_{n-3} - \binom{5}{5} f_{n-4}$$

$$= f_{n+1} - 5f_n + 10f_{n-1} - 10f_{n-2} + 5f_{n-3} - f_{n-4}$$

$$\nabla^6 f_{n+1} = \sum_{\rho=0}^{6} (-1)^\rho \binom{6}{\rho} f_{n+1-\rho} = \binom{6}{0} f_{n+1} - \binom{6}{1} f_n + \binom{6}{2} f_{n-1} - \binom{6}{3} f_{n-2} + \binom{6}{4} f_{n-3} - \binom{6}{5} f_{n-4} + \binom{6}{6} f_{n-5}$$

$$= f_{n+1} - 6f_n + 15f_{n-1} - 20f_{n-2} + 15f_{n-3} - 6f_{n-4} + f_{n-5}$$

$$\nabla^7 f_{n+1} = \sum_{\rho=0}^{7} (-1)^\rho \binom{7}{\rho} f_{n+1-\rho}$$

$$= \binom{7}{0} f_{n+1} - \binom{7}{1} f_n + \binom{7}{2} f_{n-1} - \binom{7}{3} f_{n-2} + \binom{7}{4} f_{n-3} - \binom{7}{5} f_{n-4} + \binom{7}{6} f_{n-5} - \binom{7}{7} f_{n-6}$$

$$= f_{n+1} - 7f_n + 21f_{n-1} - 35f_{n-2} + 35f_{n-3} - 21f_{n-4} + 7f_{n-5} - f_{n-6}$$

$$\nabla^8 f_{n+1} = \sum_{\rho=0}^{8} (-1)^\rho \binom{8}{\rho} f_{n+1-\rho}$$

$$= \binom{8}{0} f_{n+1} - \binom{8}{1} f_n + \binom{8}{2} f_{n-1} - \binom{8}{3} f_{n-2} + \binom{8}{4} f_{n-3} - \binom{8}{5} f_{n-4} + \binom{8}{6} f_{n-5} - \binom{8}{7} f_{n-6} + \binom{8}{8} f_{n-7}$$

$$= f_{n+1} - 8f_n + 28f_{n-1} - 56f_{n-2} + 70f_{n-3} - 56f_{n-4} + 28f_{n-5} - 8f_{n-6} + f_{n-7}$$

$$\nabla^9 f_{n+1} = \sum_{\rho=0}^{9} (-1)^\rho \binom{9}{\rho} f_{n+1-\rho}$$

$$= \binom{9}{0} f_{n+1} - \binom{9}{1} f_n + \binom{9}{2} f_{n-1} - \binom{9}{3} f_{n-2} + \binom{9}{4} f_{n-3} - \binom{9}{5} f_{n-4} + \binom{9}{6} f_{n-5} - \binom{9}{7} f_{n-6} + \binom{9}{8} f_{n-7} - \binom{9}{9} f_{n-8}$$

$$= f_{n+1} - 9f_n + 36f_{n-1} - 84f_{n-2} + 126f_{n-3} - 126f_{n-4} + 84f_{n-5} - 36f_{n-6} + 9f_{n-7} - f_{n-8}$$

$$\nabla^{10} f_{n+1} = \sum_{\rho=0}^{10} (-1)^\rho \binom{10}{\rho} f_{n+1-\rho}$$

$$= \binom{10}{0} f_{n+1} - \binom{10}{1} f_n + \binom{10}{2} f_{n-1} - \binom{10}{3} f_{n-2} + \binom{10}{4} f_{n-3} - \binom{10}{5} f_{n-4} + \binom{10}{6} f_{n-5} - \binom{10}{7} f_{n-6}$$
$$+ \binom{10}{8} f_{n-7} - \binom{10}{9} f_{n-8} + \binom{10}{10} f_{n-9}$$

$$= f_{n+1} - 10f_n + 45f_{n-1} - 120f_{n-2} + 210f_{n-3} - 252f_{n-4} + 210f_{n-5} - 120f_{n-6} + 45f_{n-7} - 10f_{n-8} + f_{n-9}$$

A comparison between (3.3.5) and (3.3.7) suggests that the values of $\delta_m^{*(0)}$, $m = 1,2,3,......$ will be different for different values of k for same m.

Now we will estimate the values of $\delta_m^{*(0)}$ for different values of k.

For $k = 1$

From (3.3.5) we have

$$y_{n+1} = y_n + h\left[f_{n+1} - \frac{1}{2}\nabla f_{n+1}\right] = y_n + h\left[f_{n+1} - \frac{1}{2}(f_{n+1} - f_n)\right] = y_n + h\left[\frac{1}{2}f_{n+1} + \frac{1}{2}f_n\right]$$

and from (3.3.7)

$$y_{n+1} = y_n + h\sum_{m=0}^{1}\delta_m^{*(0)} f_{n-m+1} = y_n + h\left[\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n\right]$$

Thus for $k = 1$ $\qquad \delta_0^{*(0)} = \frac{1}{2}, \ \ \delta_1^{*(0)} = \frac{1}{2}$

For $k = 2$

$$y_{n+1} = y_n + h\left[f_{n+1} - \frac{1}{2}\nabla f_{n+1} - \frac{1}{12}\nabla^2 f_{n+1}\right] = y_n + h\left[\frac{1}{2}f_{n+1} + \frac{1}{2}f_n - \frac{1}{12}(f_{n+1} - 2f_n + f_{n-1})\right]$$

$$= y_n + h\left[\frac{5}{12}f_{n+1} + \frac{8}{12}f_n - \frac{1}{12}f_{n-1}\right]$$

Also

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{m=0}^{2}\delta_m^{*(0)} f_{n-m+1} \\ &= y_n + h\left[\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1}\right] \end{aligned}$$

Thus when $k = 2$ then $\qquad \delta_0^{*(0)} = \frac{5}{12} \ \ \delta_1^{*(0)} = \frac{8}{12} \ \ \delta_2^{*(0)} = \frac{-1}{12}$

For $k = 3$

$$y_{n+1} = y_n + h\left[\frac{5}{12}f_{n+1} + \frac{8}{12}f_n - \frac{1}{12}f_{n-1} - \frac{1}{24}(f_{n+1} - 3f_n + 3f_{n-1} - f_{n-2})\right]$$

$$= y_n + h\left[\frac{9}{24}f_{n+1} + \frac{19}{24}f_n - \frac{5}{24}f_{n-1} + \frac{1}{24}f_{n-2}\right]$$

Also from (3.3.7)

$$y_{n+1} = y_n + h\sum_{m=0}^{3}\delta_m^{*(0)} f_{n-m+1} = y_n + h\left[\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1} + \delta_3^{*(0)} f_{n-2}\right]$$

Hence when $k = 3$ then $\delta_0^{*(0)} = \frac{9}{24} \quad \delta_1^{*(0)} = \frac{19}{24} \quad \delta_2^{*(0)} = \frac{-5}{24} \quad \delta_3^{*(0)} = \frac{1}{24}$

For k =4

$$y_{n+1} = y_n + h\left[\frac{9}{24} f_{n+1} + \frac{19}{24} f_n - \frac{5}{24} f_{n-1} + \frac{1}{24} f_{n-2} - \frac{19}{720}(f_{n+1} - 4f_n + 6f_{n-1} - 4f_{n-2} + f_{n-3})\right]$$

$$= y_n + h\left[\frac{251}{720} f_{n+1} + \frac{646}{720} f_n - \frac{264}{720} f_{n-1} + \frac{106}{720} f_{n-2} - \frac{19}{720} f_{n-3}\right]$$

and $y_{n+1} = y_n + h\sum_{m=0}^{4}\delta_m^{*(0)} f_{n-m+1} = y_n + h\left[\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1} + \delta_3^{*(0)} f_{n-2} + \delta_4^{*(0)} f_{n-3}\right]$

Hence, when k =4 $\delta_0^{*(0)} = \frac{251}{720},\ \delta_1^{*(0)} = \frac{646}{720},\ \delta_2^{*(0)} = \frac{-264}{720},\ \delta_3^{*(0)} = \frac{106}{720},\ \delta_4^{*(0)} = \frac{-19}{720}$

For k = 5

$$y_{n+1} = y_n + h\left[\frac{251}{720} f_{n+1} + \frac{646}{720} f_n - \frac{264}{720} f_{n-1} + \frac{106}{720} f_{n-2} - \frac{19}{720} f_{n-3} - \frac{27}{1440}\begin{pmatrix} f_{n+1} - 5f_n + 10f_{n-1} - \\ 10f_{n-2} + 5f_{n-3} - f_{n-4}\end{pmatrix}\right]$$

$$= y_n + h\left[\frac{475}{1440} f_{n+1} + \frac{1427}{1440} f_n - \frac{798}{1440} f_{n-1} + \frac{482}{1440} f_{n-2} - \frac{173}{1440} f_{n-3} + \frac{27}{1440} f_{n-4}\right]$$

Also $y_{n+1} = y_n + h\sum_{m=0}^{5}\delta_m^{*(0)} f_{n-m+1}$

$$= y_n + h\left[\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1} + \delta_3^{*(0)} f_{n-2} + \delta_4^{*(0)} f_{n-3} + \delta_5^{*(0)} f_{n-4}\right]$$

Comparing we get, for $k = 5$

$$\delta_0^{*(0)} = \frac{475}{1440},\ \delta_1^{*(0)} = \frac{1427}{1440},\ \delta_2^{*(0)} = \frac{-798}{1440},\ \delta_3^{*(0)} = \frac{482}{1440},\ \delta_4^{*(0)} = \frac{-173}{1440},\ \delta_5^{*(0)} = \frac{27}{1440}$$

For k = 6

$$y_{n+1} = y_n + h\left[\begin{array}{l}\frac{475}{1440} f_{n+1} + \frac{1427}{1440} f_n - \frac{798}{1440} f_{n-1} + \frac{482}{1440} f_{n-2} - \frac{173}{1440} f_{n-3} + \frac{27}{1440} f_{n-4} \\ -\frac{863}{60480}(f_{n+1} - 6f_n + 15f_{n-1} - 20f_{n-2} + 15f_{n-3} - 6f_{n-4} + f_{n-5})\end{array}\right]$$

$$= y_n + h\left[\begin{array}{l}\frac{19087}{60480}f_{n+1} + \frac{65112}{60480}f_n - \frac{46461}{60480}f_{n-1} + \frac{37504}{60480}f_{n-2} - \frac{20211}{60480}f_{n-3} + \\ \frac{6312}{60480}f_{n-4} - \frac{863}{60480}f_{n-5}\end{array}\right]$$

Again $y_{n+1} = y_n + h \sum_{m=0}^{6} \delta_m^{*(0)} f_{n-m+1}$

$$= y_n + h\left[\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1} + \delta_3^{*(0)} f_{n-2} + \delta_4^{*(0)} f_{n-3} + \delta_5^{*(0)} f_{n-4} + \delta_6^{*(0)} f_{n-5}\right]$$

Thus when $k = 6$ then $\delta_0^{*(0)} = \frac{19087}{60480}$, $\delta_1^{*(0)} = \frac{65112}{60480}$, $\delta_2^{*(0)} = \frac{-46461}{60480}$, $\delta_3^{*(0)} = \frac{37504}{60480}$,

$\delta_4^{*(0)} = \frac{-20211}{60480}$, $\delta_5^{*(0)} = \frac{6312}{60480}$, $\delta_6^{*(0)} = \frac{-863}{60480}$

For $k = 7$

$$y_{n+1} = y_n + h\left[\begin{array}{l}\frac{19087}{60480}f_{n+1} + \frac{65112}{60480}f_n - \frac{46461}{60480}f_{n-1} + \frac{37504}{60480}f_{n-2} - \frac{20211}{60480}f_{n-3} + \\ \frac{6312}{60480}f_{n-4} - \frac{863}{60480}f_{n-5} - \frac{1235}{120960}\left(\begin{array}{l}f_{n+1} - 7f_n + 21f_{n-1} - 35f_{n-2} + 35f_{n-3} \\ -21f_{n-4} + 7f_{n-5} - f_{n-6}\end{array}\right)\end{array}\right]$$

$$= y_n + h\left[\begin{array}{l}\frac{36939}{120960}f_{n+1} + \frac{138869}{120960}f_n - \frac{118857}{120960}f_{n-1} + \frac{118233}{120960}f_{n-2} - \frac{83647}{120960}f_{n-3} \\ + \frac{38559}{120960}f_{n-4} - \frac{10371}{120960}f_{n-5} + \frac{1235}{120960}f_{n-6}\end{array}\right]$$

Also from (3.3.7) $y_{n+1} = y_n + h\sum_{m=0}^{7} \delta_m^{*(0)} f_{n-m+1}$

$$= y_n + h[\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1} + \delta_3^{*(0)} f_{n-2} + \delta_4^{*(0)} f_{n-3} + \delta_5^{*(0)} f_{n-4} + \delta_6^{*(0)} f_{n-5} + \delta_7^{*(0)} f_{n-6}]$$

Hence for $k = 7$ $\delta_0^{*(0)} = \frac{36939}{120960}$, $\delta_1^{*(0)} = \frac{138869}{120960}$, $\delta_2^{*(0)} = \frac{-118857}{120960}$, $\delta_3^{*(0)} = \frac{118233}{120960}$,

$\delta_4^{*(0)} = \frac{-83647}{120960}$, $\delta_5^{*(0)} = \frac{38559}{120960}$, $\delta_6^{*(0)} = \frac{-10371}{120960}$, $\delta_7^{*(0)} = \frac{1235}{120960}$

For $k = 8$

$$y_{n+1} = y_n + h\left[\begin{array}{l}\frac{36939}{120960}f_{n+1} + \frac{138869}{120960}f_n - \frac{118857}{120960}f_{n-1} + \frac{118233}{120960}f_{n-2} - \frac{83647}{120960}f_{n-3} + \\ \frac{38559}{120960}f_{n-4} - \frac{10371}{120960}f_{n-5} + \frac{1235}{120960}f_{n-6} - \frac{33953}{3628800} \\ \left(f_{n+1} - 8f_n + 28f_{n-1} - 56f_{n-2} + 70f_{n-3} - 56f_{n-4} + 28f_{n-5} - 8f_{n-6} + f_{n-7}\right)\end{array}\right]$$

$$= y_n + h\left[\begin{array}{l}\frac{1074217}{3628800}f_{n+1} + \frac{4437694}{3628800}f_n - \frac{4516394}{3628800}f_{n-1} + \frac{5448358}{3628800}f_{n-2} - \\ \frac{4886120}{3628800}f_{n-3} + \frac{3058138}{3628800}f_{n-4} - \frac{1261814}{3628800}f_{n-5} + \frac{308674}{3628800}f_{n-6} - \frac{33953}{3628800}f_{n-7}\end{array}\right]$$

And from (3.3.7) $\qquad y_{n+1} = y_n + h\sum_{m=0}^{8}\delta_m^{*(0)} f_{n-m+1}$

i.e. $y_{n+1} = y_n + h[\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1} + \delta_3^{*(0)} f_{n-2} + \delta_4^{*(0)} f_{n-3} + \delta_5^{*(0)} f_{n-4} + \delta_6^{*(0)} f_{n-5} +$
$\delta_7^{*(0)} f_{n-6} + \delta_8^{*(0)} f_{n-7}]$

Thus for $k = 8$ $\qquad \delta_0^{*(0)} = \frac{1074217}{3628800},\ \delta_1^{*(0)} = \frac{4437694}{3628800},\ \delta_2^{*(0)} = \frac{-4516394}{3628800},\ \delta_3^{*(0)} = \frac{5448358}{3628800},$

$\delta_4^{*(0)} = \frac{-4886120}{3628800},\ \delta_5^{*(0)} = \frac{3058138}{3628800},\ \delta_6^{*(0)} = \frac{-1261814}{3628800},\ \delta_7^{*(0)} = \frac{308674}{3628800},\ \delta_8^{*(0)} = \frac{-33953}{3628800}$

For $k = 9$

$$y_{n+1} = y_n + h\left[\begin{array}{l}\frac{1074217}{3628800}f_{n+1} + \frac{4437694}{3628800}f_n - \frac{4516394}{3628800}f_{n-1} + \frac{5448358}{3628800}f_{n-2} - \frac{4886120}{3628800}f_{n-3} \\ + \frac{3058138}{3628800}f_{n-4} - \frac{1261814}{3628800}f_{n-5} + \frac{308674}{3628800}f_{n-6} - \frac{33953}{3628800}f_{n-7} - \frac{57281}{7257600} \\ (f_{n+1} - 9f_n + 36f_{n-1} - 84f_{n-2} + 126f_{n-3} - 126f_{n-4} + 84f_{n-5} - 36f_{n-6} + 9f_{n-7} - f_{n-8})\end{array}\right]$$

$$= y_n + h\left[\begin{array}{l}\frac{2091153}{7257600}f_{n+1} + \frac{9390917}{7257600}f_n - \frac{11094904}{7257600}f_{n-1} + \frac{15708320}{7257600}f_{n-2} - \frac{16989646}{7257600}f_{n-3} + \\ \frac{13333682}{7257600}f_{n-4} - \frac{7335232}{7257600}f_{n-5} + \frac{2679464}{7257600}f_{n-6} - \frac{583435}{7257600}f_{n-7} + \frac{57281}{7257600}f_{n-8}\end{array}\right]$$

Again when $k = 9$ then $\qquad y_{n+1} = y_n + h\sum_{m=0}^{9}\delta_m^{*(0)} f_{n-m+1}$

i.e. $y_{n+1} = y_n + h\left[\begin{array}{l}\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1} + \delta_3^{*(0)} f_{n-2} + \delta_4^{*(0)} f_{n-3} + \delta_5^{*(0)} f_{n-4} + \delta_6^{*(0)} f_{n-5} + \\ \delta_7^{*(0)} f_{n-6} + \delta_8^{*(0)} f_{n-7} + \delta_9^{*(0)} f_{n-8}\end{array}\right]$

Thus for $k = 9$ $\qquad \delta_0^{*(0)} = \frac{2091153}{7257600},\ \delta_1^{*(0)} = \frac{9390917}{7257600},\ \delta_2^{*(0)} = \frac{-11094904}{7257600},$

$\delta_3^{*(0)} = \frac{15708320}{7257600},\ \delta_4^{*(0)} = \frac{-16989646}{7257600},\ \delta_5^{*(0)} = \frac{13333682}{7257600},\ \delta_6^{*(0)} = \frac{-7335232}{7257600},$

$\delta_7^{*(0)} = \frac{2679464}{7257600},\ \delta_8^{*(0)} = \frac{-583435}{7257600},\ \delta_9^{*(0)} = \frac{57281}{7257600}$

For $k = 10$

$$y_{n+1} = y_n + h\left[\begin{array}{l} \frac{2091153}{7257600} f_{n+1} + \frac{9390917}{7257600} f_n - \frac{11094904}{7257600} f_{n-1} + \frac{15708320}{7257600} f_{n-2} - \\ \frac{16989646}{7257600} f_{n-3} + \frac{13333682}{7257600} f_{n-4} - \frac{7335232}{7257600} f_{n-5} + \frac{2679464}{7257600} f_{n-6} - \\ \frac{583435}{7257600} f_{n-7} + \frac{57281}{7257600} f_{n-8} - \frac{9751299}{1437004800} (f_{n+1} - 10 f_n + 45 f_{n-1} - 120 f_{n-2} \\ + 210 f_{n-3} - 252 f_{n-4} + 210 f_{n-5} - 120 f_{n-6} + 45 f_{n-7} - 10 f_{n-8} + f_{n-9}) \end{array}\right]$$

$$= y_n + h\left[\begin{array}{l} \frac{404296995}{1437004800} f_{n+1} + \frac{1956914556}{1437004800} f_n - \frac{2635599447}{1437004800} f_{n-1} + \frac{4280403240}{1437004800} f_{n-2} \\ - \frac{5411722698}{1437004800} f_{n-3} + \frac{5097396384}{1437004800} f_{n-4} - \frac{3500148726}{1437004800} f_{n-5} + \frac{1700689752}{1437004800} \\ f_{n-6} - \frac{554328585}{1437004800} f_{n-7} + \frac{108854628}{1437004800} f_{n-8} - \frac{9751299}{1437004800} f_{n-9} \end{array}\right]$$

Also from (3.3.7) for $k = 10$ $\quad y_{n+1} = y_n + h \sum_{m=0}^{10} \delta_m^{*(0)} f_{n-m+1}$

$$= y_n + h\left(\begin{array}{l} \delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1} + \delta_3^{*(0)} f_{n-2} + \delta_4^{*(0)} f_{n-3} + \delta_5^{*(0)} f_{n-4} + \delta_6^{*(0)} f_{n-5} + \delta_7^{*(0)} f_{n-6} \\ + \delta_8^{*(0)} f_{n-7} + \delta_9^{*(0)} f_{n-8} + \delta_{10}^{*(0)} f_{n-9} \end{array}\right)$$

Hence, comparing the two values of y_{n+1} we get, for $k = 10$

$$\delta_0^{*(0)} = \frac{404296995}{1437004800}, \; \delta_1^{*(0)} = \frac{1956914556}{1437004800}, \; \delta_2^{*(0)} = \frac{-2635599447}{1437004800}, \; \delta_3^{*(0)} = \frac{4280403240}{1437004800},$$

$$\delta_4^{*(0)} = \frac{-5411722698}{1437004800}, \; \delta_5^{*(0)} = \frac{5097396384}{1437004800}, \; \delta_6^{*(0)} = \frac{-3500148726}{1437004800}, \; \delta_7^{*(0)} = \frac{1700689752}{1437004800},$$

$$\delta_8^{*(0)} = \frac{-554328585}{1437004800}, \; \delta_9^{*(0)} = \frac{108854628}{1437004800}, \; \delta_{10}^{*(0)} = \frac{-9751299}{1437004800}$$

The above calculated values of $\delta_m^{*(0)}$ for different values of k are tabulated in Table-3.2.

sTable-3.2-The coefficients $\delta_m^{*(0)}$ in the formula $y_{n+1} = y_n + h \sum_{m=0}^{k} \delta_m^{*(0)} f_{n-m+1}$

	$\delta_0^{*(0)}$	$\delta_1^{*(0)}$	$\delta_2^{*(0)}$	$\delta_3^{*(0)}$	$\delta_4^{*(0)}$	$\delta_5^{*(0)}$	$\delta_6^{*(0)}$	$\delta_7^{*(0)}$	$\delta_8^{*(0)}$	$\delta_9^{*(0)}$	
0.	1										
1.	$\frac{1}{2}$	$\frac{1}{2}$									
2.	$\frac{5}{12}$	$\frac{8}{12}$	$\frac{1}{12}$								
3.	$\frac{9}{24}$	$\frac{19}{24}$	$\frac{-5}{24}$	$\frac{1}{24}$							
4.	$\frac{251}{720}$	$\frac{646}{720}$	$\frac{-264}{720}$	$\frac{-19}{720}$							
5.	$\frac{475}{1440}$	$\frac{1427}{1440}$	$\frac{-798}{1440}$	$\frac{482}{1440}$	$\frac{-173}{1440}$	$\frac{27}{1440}$]					
6.	$\frac{19087}{60480}$	$\frac{65112}{60180}$	$\frac{-46461}{60480}$	$\frac{37501}{60480}$	$\frac{-20211}{60480}$	$\frac{-863}{60480}$					
7.	$\frac{36939}{120960}$	$\frac{138869}{120960}$	$\frac{-118857}{120960}$	$\frac{118233}{120960}$	$\frac{-83617}{120960}$	$\frac{38559}{120960}$	$\frac{-10371}{120960}$	$\frac{1235}{120960}$			
8.	$\frac{1074217}{3628800}$	$\frac{4437694}{3628800}$	$\frac{-4516394}{3628800}$	$\frac{5448358}{3628800}$	$\frac{-48861201}{3628800}$	$\frac{3058135}{3628800}$	$\frac{-1261814}{3628800}$	$\frac{308674}{3628800}$	$\frac{-33953}{3628800}$		
9.	$\frac{2091153}{7257600}$	$\frac{9390917}{7257600}$	$\frac{-11094904}{7257600}$	$\frac{15708320}{7257600}$	$\frac{-16989646}{7257600}$	$\frac{13333682}{7257600}$	$\frac{-7335232}{7257600}$	$\frac{2679464}{7257600}$	$\frac{-583435}{7257600}$	$\frac{57281}{7257600}$	
10.	$\frac{404296995}{1437004800}$	$\frac{1956914556}{1437004800}$	$\frac{-2635599447}{1437004800}$	$\frac{4280403240}{1437004800}$	$\frac{-5411722698}{1437004800}$	$\frac{5097396384}{1437004800}$	$\frac{-3500148726}{1437004800}$	$\frac{170068975}{143700480}$	$\frac{-554328585}{1437004800}$	$\frac{108854628}{1437004800}$	$\frac{-9751299}{1437004800}$

3.4 Demonstrations

In this section the extended orders of Adams-Bashforth and Adams-Moulton formula will be utilised to solve a selected problem. Though we have extended both the formula up to 10th order but for the demonstration purpose we will use the 8th order of them. We will try to solve $y' = -y^2$ with the given condition $y(0) = 1$ in the domain $x \in [0, 2]$ taking h = 0.2. For calculations values correct up to four decimal places will be taken.

As we will use 8th order formula where as to get the required result we require 10 steps so repeated use of the predictor-corrector formula will be required.

The eight order Adams-Bashforth formula is given by

$$y_{n+1} = y_n + h\sum_{m=0}^{7} \gamma_m^{*(0)} f_{n-m}$$

i.e. $y_{n+1} = y_n + h\left[\gamma_0^{*(0)} f_n + \gamma_1^{*(0)} f_{n-1} + \gamma_2^{*(0)} f_{n-2} + \gamma_3^{*(0)} f_{n-3} + \gamma_4^{*(0)} f_{n-4} + \gamma_5^{*(0)} f_{n-5} + \gamma_6^{*(0)} f_{n-6} + \gamma_7^{*(0)} f_{n-7}\right]$

Using the values of $\gamma_0^{*(0)}, \gamma_1^{*(0)}, \gamma_2^{*(0)}$, ... from the Table-3.1 we get

$$y_{n+1} = y_n + h\begin{bmatrix} \frac{434241}{120960} f_n - \frac{1152169}{120960} f_{n-1} + \frac{2183877}{120960} f_{n-2} - \frac{2664477}{120960} f_{n-3} + \frac{2102243}{120960} f_{n-4} \\ -\frac{1041723}{120960} f_{n-5} + \frac{295767}{120960} f_{n-6} - \frac{36799}{120960} f_{n-7} \end{bmatrix}$$

As $y_n' = f(x_n, y_n) = f_n$, so we can write

$$\begin{aligned} y_{n+1} = y_n + 0.2(3.5899 y_n' - 9.5252 y_{n-1}' + 18.0545 y_{n-2}' - 22.0278 y_{n-3}' + 17.3797 y_{n-4}' \\ - 8.6122 y_{n-5}' + 2.4452 y_{n-6}' - 0.3042 y_{n-7}') \end{aligned} \tag{3.4.1}$$

and is valid for $n \geq 7$. Now as $h = 0.2$, so we have

$$x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1, x_6 = 1.2, x_7 = 1.4, x_8 = 1.6, x_9 = 1.8, x_{10} = 2$$

To utilise (3.4.1) it is seen that seven previous values are required. For the purpose Taylor series can be used. The Tailor series up to 8th order is

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{iv} + \frac{h^5}{5!} y_n^{v} + \frac{h^6}{6!} y_n^{vi} + \frac{h^7}{7!} y_n^{vii} + \frac{h^8}{8!} y_n^{viii} + \ldots\ldots \tag{3.4.2}$$

In this problem

$$y_n' = -y_n^2,\quad y_n'' = -2y_n.y_n' = 2y_n^3,\quad y_n''' = -6y_n^2.y_n^2 = -6y_n^4,\quad y_n^{iv} = -24y_n^3.y_n' = 24y_n^5,$$

Similarly $y_n^{v} = -120y_n^6$, $y_n^{vi} = 720y_n^7$, $y_n^{vii} = -5040y_n^8$ and $y_n^{viii} = 40320y_n^9$

Putting these values in (3.4.2) we have,

$$y_{n+1} = y_n - hy_n^2 + h^2 y_n^3 - h^3 y_n^4 + h^4 y_n^5 - h^5 y_n^6 + h^6 y_n^7 - h^7 y_n^8 + h^8 y_n^9 \tag{3.4.3}$$

which is valid for $n = 0,1,2,3,4,$................

Repeated use of (3.4.3), for different values of n, will provide us y_1, y_2, y_3, etc and $y_n' = -y_n^2$ so we will have

$y_1 = y(0.2) = 0.8333$, $y_1' = -0.6944$; $\quad y_2 = y(0.4) = 0.7143$, $y_2' = -0.5102$;

$y_3 = y(0.6) = 0.6250$, $y_3' = -0.3906$; $\quad y_4 = y(0.8)0.5556$, $y_4' = -0.3086$;

$y_5 = y(1) = 0.5$, $y_5' = -0.25$; $\quad y_6 = y(1.2) = 0.4545$, $y_6' = -0.2066$;

$y_7 = y(1.4) = 0.4166$, $y_7' = -0.1736$

Putting these values along with $n = 7$ in (3.4.1) we will get the value of y_8. The value thus obtained is $\quad y_8 = 0.3847$

And thus $\quad y_8 = y(1.6) = 0.3847, \quad y_8' = -0.1480$

The eight order Adams-Moulton Corrector formula is given by,

$$y_{n+1} = y_n + h\sum_{m=0}^{8} \delta_m^{*(0)} f_{n+1-m}$$

$$or,\ y_{n+1} = y_n + h\left[\begin{matrix}\delta_0^{*(0)} f_{n+1} + \delta_1^{*(0)} f_n + \delta_2^{*(0)} f_{n-1} + \delta_3^{*(0)} f_{n-2} + \delta_4^{*(0)} f_{n-3} + \delta_5^{*(0)} f_{n-4} + \delta_6^{*(0)} f_{n-5} + \\ \delta_7^{*(0)} f_{n-6} + \delta_8^{*(0)} f_{n-7}\end{matrix}\right]$$

With the values of $\delta_0^{*(0)}, \delta_1^{*(0)}, \delta_2^{*(0)}$,......................... from Table (3.3.2) we get

$$y_{n+1} = y_n + 0.2\left[\begin{matrix}\frac{1074217}{3628800} f_{n+1} + \frac{4437694}{3628800} f_n - \frac{4516394}{3628800} f_{n-1} + \frac{5448358}{3628800} f_{n-2} - \frac{4886120}{3628800} f_{n-3} \\ + \frac{3058138}{3628800} f_{n-4} - \frac{1261814}{3628800} f_{n-5} + \frac{308674}{3628800} f_{n-6} - \frac{33953}{3628800} f_{n-7}\end{matrix}\right]$$

$$y_{n+1} = y_n + 0.2\left(\begin{matrix}0.296 y_{n+1}' + 1.2229 y_n' - 1.2446 y_{n-1}' + 1.5014 y_{n-2}' - 1.3465 y_{n-3}' + 0.8427 y_{n-4}' \\ - 0.3477 y_{n-5}' + 0.0851 y_{n-6}' - 0.0094 y_{n-7}'\end{matrix}\right) \tag{3.4.4}$$

and is valid for $n \geq 7$. Putting $n = 7$ the 8th order Adams-Moulton Corrector formula becomes,

$$y_8 = y_7 + 0.2\left(\begin{matrix}0.296 y_8' + 1.2229 y_7' - 1.2446 y_6' + 1.5014 y_5' - 1.3465 y_4' + 0.8427 y_3' - 0.3477 y_2' + \\ 0.0851 y_1' - 0.0094 y_0'\end{matrix}\right)$$

And thus $\quad y_8 = y(1.6) = 0.3846, \quad y_8' = -0.1479$

Putting $n = 8$ in (3.4.1) we get

$$y_9 = y_8 + 0.2\left(\begin{matrix}3.5899 y_8' - 9.5252 y_7' + 18.0545 y_6' - 22.0278 y_5' + 17.3797 y_4' - 8.6122 y_3' + \\ 2.4452 y_2' - 0.3042 y_1'\end{matrix}\right)$$

which provides $y_9 = 0.3574$ and so $y'_9 = -0.1277$

Again putting $n = 8$ in (3.4.4) the corrected value of y_9 is given by,

$$y_9 = y_8 + 0.2\begin{pmatrix} 0.296y'_9 + 1.2229y'_8 - 1.2446y'_7 + 1.5014y'_6 - 1.3465y'_5 + 0.8427y'_4 - 0.3477y'_3 + \\ 0.0851y'_2 - 0.0094y'_1 \end{pmatrix}$$

So $y_9 = y(1.8) = 0.3571$ and $y'_9 = -0.1275$

Putting $n = 9$ in (3.4.1) then eight order Predictor value is

$$y_{10} = y_9 + 0.2\begin{pmatrix} 3.5899y'_9 - 9.5252y'_8 + 18.0545y'_7 - 22.0278y'_6 + 17.3797y'_5 - 8.6122y'_4 + \\ 2.4452y'_3 - 0.3042y'_2 \end{pmatrix}$$

So $y_{10} = y(2) = 0.3332$ $y'_{10} = -0.111$

Again putting $n = 9$ in (3.4.4) then by 8th order Adams-Moulton Corrector formula the value is,

$$y_{10} = y_9 + 0.2\begin{pmatrix} 0.296y'_{10} + 1.2229y'_9 - 1.2446y'_8 + 1.5014y'_7 - 1.3465y'_6 + 0.8427y'_5 - 0.3477y'_4 + \\ 0.0851y'_3 - 0.0094y'_2 \end{pmatrix}$$

So $y_{10} = y(2) = 0.3333$

The exact solution of the problem is $y = 1/(x+1)$

Hence the percentage error is $(0.3333 - 0.3333)/0.3333 = 0\,\%$ i.e. the result is 100% correct up to four decimal places.

Chapter 4

In Chapter 2 some new members of the Runge-Kutta family i.e. some new formula of the same type have been proposed (two fifth, two sixth and one seventh order). There how they can be used also has been demonstrated. In this chapter comparison will be drawn between proposed formula with the existing formula of the same order. For the purpose a selected problem will be solved by all of the proposed (except 7th order) and known same order RK methods and the result obtained through them will be compared by calculating the percentage error. For the purpose a simple problem $y' = x + y$ is chosen. Value of y at $x = 0.1$ and $x = 0.2$ will be estimated taking $h = 0.1$, considering that $y = 1$ when $x = 0$. The exact values, correct up to four decimal places, are $y_1 = y(0.1) = 1.1103$ and $y_2 = y(0.2) = 1.2428$.

a) Solution through 1st fifth order (proposed) method

The coefficients and the weight factors for this form is tabulated below-

$\frac{1}{2}$	$\frac{1}{2}$					
$\frac{1}{3}$	0	$\frac{1}{3}$				
$\frac{1}{2}$	0	0	$\frac{1}{2}$			
$\frac{2}{3}$	0	0	0	$\frac{2}{3}$		
1	0	0	0	0	1	
	$\frac{11}{120}$	$\frac{-4}{15}$	$\frac{27}{40}$	$\frac{-4}{15}$	$\frac{27}{40}$	$\frac{11}{120}$

Here $f(x, y) = \frac{dy}{dx} = x + y.$

$$k_1 = hf(x_0, y_0) = 0.1$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1f(0.05, 1.05) = 0.11$$

$$k_3 = hf(x_0 + \frac{h}{3}, y_0 + \frac{k_2}{3}) = 0.1f(0.0333,1.037) = 0.1069$$

$$k_4 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}) = 0.1f(0.05, 1.053) = 0.1103$$

$$k_5 = hf(x_0 + \frac{2h}{3}, y_0 + \frac{2k_4}{3}) = 0.1f(0.06, 1.0736) = 0.1140$$

$$k_6 = hf(x_0 + h, y_0 + k_5) = 0.1f(0.1, 1.1140) = 0.1214$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + 81k_3 - 32k_4 + 81k_5 + 11k_6) = 0.1024$$

$$\therefore y_1 = y_0 + \Delta y = 1.1024$$

Percentage error: $(1.1103 - 1.1024)/1.1103 = 7.12 \times 10^{-3}\%$

For the second interval we have,

$$x_1 = 0.1, \quad y_1 = 1.1024 \quad and \quad f(x,y) = x + y$$

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1024) = 0.12024$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1f(0.15, 1.1625) = 0.1313$$

$$k_3 = hf(x_1 + \frac{h}{3}, y_1 + \frac{k_2}{3}) = 0.1f(0.13, 1.1462) = 0.1279$$

$$k_4 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_3}{2}) = 0.1f(0.15, 1.1664) = 0.1316$$

$$k_5 = hf(x_1 + \frac{2h}{3}, y_1 + \frac{2k_4}{3}) = 0.1f(0.1667, 1.1901) = 0.1357$$

$$k_6 = hf(x_1 + h, y_1 + k_5) = 0.1f(0.2, 1.2381) = 0.1438$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + 81k_3 - 32k_4 + 81k_5 + 11k_6) = 0.1315$$

$$\therefore y_2 = y_1 + \Delta y = 1.2339$$

Percentage error: (1.2428-1.2339)/1.2428 = $7.16 \times 10^{-3}\%$

b) Solution through 2nd fifth order (proposed) method

The coefficients and the weights are presented below

$\frac{1}{2}$	$\frac{1}{2}$					
$\frac{1}{2}$	$\frac{13}{48}$	$\frac{11}{48}$				
1	$\frac{-5}{33}$	0	$\frac{38}{33}$			
$\frac{1}{3}$	$\frac{20}{81}$	0	$\frac{10}{81}$	$\frac{-3}{81}$		
$\frac{2}{3}$	$\frac{25}{81}$	0	$\frac{-4}{81}$	$\frac{6}{81}$	$\frac{27}{81}$	
	$\frac{11}{120}$	0	$\frac{-64}{120}$	$\frac{11}{120}$	$\frac{81}{120}$	$\frac{81}{120}$

Here we have

$$k_1 = hf(x_0, y_0) = \cdot 1(0+1) = \cdot 1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = \cdot 1\left(\frac{\cdot 1}{2} + 1 + \frac{\cdot 1}{2}\right) = \cdot 11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{13k_1}{48} + \frac{11k_2}{48}\right) = \cdot 1\left[\frac{\cdot 1}{2} + 1 + \frac{1}{48}(13 \times \cdot 1 + 11 \times \cdot 11)\right] = \cdot 10779$$

$$k_4 = hf\left(x_0 + h, y_0 - \frac{5}{33}k_1 + \frac{38}{33}k_3\right) = \cdot 1\left(\cdot 1 + 1 - \frac{5 \times \cdot 1}{33} + \frac{38 \times \cdot 10779}{33}\right) = \cdot 1209$$

$$k_5 = hf\left(x_0 + \frac{h}{3}, y_0 + \frac{20}{81}k_1 + \frac{10}{81}k_3 - \frac{3}{81}k_4\right)$$

$$k_6 = hf\left(x_0 + \frac{2h}{3}, y_0 + \frac{25}{81}k_1 - \frac{4}{81}k_3 + \frac{6}{81}k_4 + \frac{27}{81}k_5\right)$$

$$= \cdot 1\left[\frac{\cdot 2}{3} + 1 + \frac{1}{81}(25 \times .11998 - 4 \times \cdot 10779 + 6 \times .1209 + 27 \times \cdot 10446)\right] = \cdot 11082$$

$$\therefore \Delta y = \frac{1}{120}(11k_1 - 64k_3 + 11k_4 + 81k_5 + 81k_6) = \cdot 09983$$

Thus $x_1 = x_0 + h = \cdot 1$ and $y_1 = y_0 + \Delta y = 1 + \cdot 09983 = 1 \cdot 09983$

Percentage error: $(1.1103 - 1.0998)/1.1103 = 9.46 \times 10^{-3}\%$

For the 2nd interval we have,

$$k_1 = hf(x_1, y_1) = \cdot 1(x_1 + y_1) = \cdot 1(\cdot 1 + 1 \cdot 0998) = \cdot 11998$$

$$k_2 = h\left(x_1 + \frac{h}{2} + y_1 + \frac{k_1}{2}\right) = \cdot 1\left(\cdot 1 + \frac{\cdot 1}{2} + 1 \cdot 09983 + \frac{\cdot 11998}{2}\right) = \cdot 13098$$

$$k_3 = h\left(x_1 + \frac{h}{2} + y_1 + \frac{13k_1}{48} + \frac{11k_2}{48}\right) = \cdot 1\left[\cdot 1 + \frac{\cdot 1}{2} + 1 \cdot 09983 + \frac{13 \times \cdot 11998}{48} + \frac{11 \times \cdot 13098}{48}\right] = \cdot 13123$$

$$k_4 = h\left(x_1 + h + y_1 - \frac{5}{33}k_1 + \frac{38}{33}k_3\right) = \cdot 1\left(\cdot 1 + \cdot 1 + 1 \cdot 009983 - \frac{5 \times \cdot 11998}{33} + \frac{38 \times \cdot 13123}{33}\right) = \cdot 14328$$

$$k_5 = h\left(x_1 + \frac{h}{3} + y_1 + \frac{20}{81}k_1 + \frac{10}{81}k_3 - \frac{3}{81}k_4\right)$$

$$= \cdot 1\left[\cdot 1 + \frac{\cdot 1}{3} + 1 \cdot 09983 + \frac{1}{81}(20 \times \cdot 11998 + 10 \times \cdot 13123 - 3 \times \cdot 14328)\right] = 0 \cdot 12737$$

$$k_6 = h\left(x_1 + \frac{2h}{3} + y_1 + \frac{25}{81}k_1 - \frac{4}{81}k_3 + \frac{6}{81}k_4 + \frac{27}{81}k_5\right)$$

$$= \cdot 1\left[\cdot 1 + \frac{\cdot 2}{3} + 1 \cdot 09983 + \frac{1}{81}(25 \times \cdot 11998 - 4 \times \cdot 12123 + 6 \times \cdot 14328 + 27 \times \cdot 12737)\right] = 0 \cdot 13501$$

$$\Delta y = \frac{1}{120}(11k_1 - 64k_3 + 11k_4 + 81k_5 + 81k_6) = 0 \cdot 13125$$

Thus $x_2 = x_1 + h = 0 \cdot 1 + 0 \cdot 1 = 0 \cdot 2$ and

$$y_2 = y_1 + \Delta y = 1 \cdot 09983 + 0 \cdot 13125 = 1 \cdot 23108$$

Percentage error: $(1.2428 - 1.23108)/1.2428 = 9 \times 10^{-3}\%$

c) Solution through Lawson method (fifth order)

The coefficients and the weights are presented in the following table:

$\frac{1}{2}$	$\frac{1}{2}$					
$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$				
$\frac{1}{2}$	0	0	$\frac{1}{2}$			
$\frac{3}{4}$	0	$\frac{-3}{16}$	$\frac{6}{16}$	$\frac{9}{16}$		
1	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{-12}{7}$	$\frac{8}{7}$	
	$\frac{7}{90}$	0	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$

Here

$$k_1 = hf(x_0, y_0) = h(x_0 + y_0) = 0\cdot 1(0+1) = 0.1$$

$$k_2 = hf\left(x_o + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1\left(0 + \frac{.1}{2} + 1 + \frac{.1}{2}\right) = 0.11$$

$$k_3 = hf\left(x_o + \frac{1}{4}h, y_0 + \frac{3}{16}k_1 + \frac{1}{16}k_2\right) = 0.1\left(\frac{.1}{4} + 1 + \frac{.3}{16} + \frac{.11}{16}\right) = .10506$$

$$k_4 = hf\left(x_o + \frac{1}{2}h, y_0 + \frac{1}{2}k_3\right) = 0.1\left(\frac{.1}{2} + 1 + \frac{.10506}{2}\right) = 0.11025$$

$$k_5 = hf\left(x_0 + \frac{3}{4}h, y_0 - \frac{3}{16}k_2 + \frac{6}{16}k_3 + \frac{9}{16}k_4\right) = .1\left[\frac{.3}{4} + 1 + \frac{1}{16}(-3k_2 + 6k_3 + 9k_4)\right]$$

$$= .1\left[\frac{.3}{4} + 1 + \frac{1}{16}(-3\times.11 + 6\times.10506 + 9\times.11025)\right] = 0.11558$$

$$k_6 = hf\left(x_0 + h, y_0 + \frac{1}{7}k_1 + \frac{4}{7}k_2 + \frac{6}{7}k_3 - \frac{12}{7}k_4 + \frac{8}{7}k_5\right)$$

$$= .1\left[.1 + 1 + \frac{1}{7}(.1 + 4\times.11 + 6\times.10506 - 12\times0.11025 + 8\times.11558)\right] = 0.12103$$

$$\Delta y = \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6) = 0.1103$$

Thus $x_1 = x_0 + h = .1$ and $y_1 = y_0 + \Delta y = 1 + 0.1103 = 1.1103$,

which matches exactly with the exact value up to four decimal places.

For 2nd interval we have,

$$k_1 = hf(x_1 y_1) = .1(.1 + 1.11034) = 0.12103$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1\left(.1 + \frac{.1}{2} + 1.11034 + \frac{.1}{2}\right) = 0.13103$$

$$k_3 = .1\left[.1 + \frac{.1}{4} + 1.11034 + \frac{1}{16}(3 \times .12103 + .13103)\right] = 0.12662$$

$$k_4 = .1\left(.1 + \frac{.1}{2} + 1.11034 + \frac{.12662}{2}\right) = 0.13237$$

$$k_5 = .1\left[.1 + \frac{.3}{4} + 1.11034 + \frac{1}{16}(-3 \times .13103 + 6 \times .12662 + 9 \times .13237)\right]$$

$$k_6 = .1\left[.1 + .1 + 1.11034 + \frac{1}{7}(.12103 + 4 \times .13103 + 6 \times .12662 - 12 \times .13237 + 8 \times .13827)\right] = 0.1442$$

$$\Delta y = \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6) = 0.13246$$

Thus $x_2 = x_1 + h = .2$ and $y_2 = y_1 + \Delta y = 1.11034 + 0.13246 = 1.2428$,

which again matches with the exact value up to four decimal places.

d) Solution through Nystrom method (fifth order)

As before, the following table contains the coefficients and the weights.

$\frac{1}{3}$	$\frac{1}{3}$					
$\frac{2}{5}$	$\frac{4}{25}$	$\frac{6}{25}$				
1	$\frac{1}{4}$	$\frac{-12}{4}$	$\frac{15}{4}$			
$\frac{2}{3}$	$\frac{6}{81}$	$\frac{90}{81}$	$\frac{-50}{81}$	$\frac{8}{81}$		
$\frac{4}{5}$	$\frac{6}{75}$	$\frac{36}{75}$	$\frac{10}{75}$	$\frac{8}{75}$	0	
	$\frac{23}{192}$	0	$\frac{125}{192}$	0	$\frac{-81}{192}$	$\frac{125}{192}$

Here we have,

$$k_1 = hf(x_o, y_o) = h(x_0 + y_o) = .1(0 + 1) = .1$$

$$k_2 = hf(x_n + c_2 h, y_n + a_{21}k_1) = hf\left(x_0 + \frac{1}{3}h, y_0 + \frac{1}{3}k_1\right) = .1\left(0 + \frac{.1}{3} + 1 + \frac{1}{3}\right) = 0.10667$$

$$k_3 = hf(x_n + c_3h, y_n + a_{31}k_1 + a_{32}k_2) = h\left[x_0 + \frac{2}{5}h, y_0 + \frac{4}{25}k_1 + \frac{6}{25}k_2\right]$$

$$= .1\left(0 + \frac{.2}{3} + 1 + \frac{.4}{25} + \frac{6 \times .10667}{25}\right) = 0.10816$$

$$k_4 = hf(x_n + c_4h, y_n + a_{41}k_1 + a_{42}k_2 + a_{43}k_3)$$

$$= hf\left(x_0 + h, y_n + \frac{1}{4}k_1 - \frac{12}{4}k_2 + \frac{15}{4}k_3\right) = .1\left[.1 + 1 + \frac{.1}{4} + \frac{1}{4}(-12 \times .10667 + 15 \times .10816)\right] = 0.12106$$

$$k_5 = hf\left(x_0 + \frac{2}{3}h, y_0 + \frac{6}{81}k_1 + \frac{90}{81}k_2 - \frac{50}{81}k_3 + \frac{8}{81}k_4\right)$$

$$= .1\left[\frac{.2}{3} + 1 + \frac{1}{81}(.6 + 90 \times .10667 - 50 \times .10816 + 8 \times .121)\right] = .11378$$

$$k_6 = hf\left(x_0 + \frac{4}{5}h, y_0 + \frac{6}{75}k_1 + \frac{36}{75}k_2 + \frac{10}{75}k_3 + \frac{8}{75}k_4\right)$$

$$= .1\left[\frac{.4}{5} + 1 + \frac{1}{75}(.6 + 36 \times .10667 + 10 \times .10816 + 8 \times .121)\right] = .11665$$

$$\Delta y = w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4 + w_5k_5 + w_6k_6 = \frac{1}{192}(23k_1 + 125k_3 - 81k_5 + 125k_6) = 0.09956$$

Thus $x_1 = x_0 + h = .1$ and $y_1 = y_0 + \Delta y = 1 + .09956 = 1.09956$

The percentage error is $(1.1103 - 1.0996)/1.1103 = 9.64 \times 10^{-3}\,\%$

For the 2nd interval we have,

$$k_1 = hf(x_1 y_1) = .1(x_1 + y_1) = .1(.1 + 1.09956) = 0.11996$$

$$k_2 = h\left(x_1 + \frac{h}{3} + y_1 + \frac{k_1}{3}\right) = .1\left(.1 + \frac{.1}{3} + 1.09956 + \frac{.1}{3}\right) = .12662$$

$$k_3 = h\left(x_1 + \frac{2h}{5} + y_1 + \frac{4k_1}{25} + \frac{6k_2}{25}\right)$$

$$= .1\left[.1 + \frac{.2}{5} + 1.09956 + \frac{1}{25}(4 \times .11996 + 6 \times .12662)\right] = .12891$$

$$k_4 = h\left[x_1 + h + y_1 + \frac{1}{4}(k_1 - 12k_2 + 15k_3)\right]$$

$$= .1\left[.1 + .1 + 1.09956 + \frac{1}{4}(.11996 - 12 \times .12662 + 15 \times .12891)\right] = .14331$$

$$k_5 = h\left[x_1 + \frac{2h}{3} + y_1 + \frac{1}{81}(6k_1 + 90k_2 - 50k_3 + 8k_4)\right]$$

$$=.1\left[.1+\frac{.2}{3}+1.09956+\frac{1}{81}(6\times.11996+90\times.12891-50\times.12891+8\times.14331)\right]=.13504$$

$$k_6=h\left[x_1+\frac{4h}{5}+y_1+\frac{1}{75}(6k_1+36k_2+10k_3+8k_4)\right]$$

$$=.1\left[.1+\frac{.4}{5}+1.09956+\frac{1}{75}(6\times.11996+36\times.12662+10\times.12891+8\times.14331)\right]=.13824$$

$$\Delta y=\frac{1}{192}(23k_1+125k_3-81k_5+125k_6)=0.13133$$

Thus $x_2=x_1+h=.1+.1=.2$

and $y_2=y_1+\Delta y=1.09956+.13133=1.23089$

Hence the percentage error is $(1.2428-1.2309)/1.2428=9.58\times10^{-3}\%$

e) Solution through 1st sixth order (proposed) method

The proposed listing for the coefficients and the weights is listed below

$\frac{1}{2}$	$\frac{1}{2}$						
$\frac{1}{3}$	0	$\frac{1}{3}$					
$\frac{2}{3}$	0	0	$\frac{2}{3}$				
$\frac{1}{3}$	0	0	0	$\frac{1}{3}$			
$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$		
1	0	0	0	0	0	1	
	$\frac{11}{120}$	$\frac{-4}{15}$	$\frac{27}{80}$	$\frac{27}{40}$	$\frac{27}{80}$	$\frac{-4}{15}$	$\frac{11}{120}$

Here

$$k_1=hf(x_0,y_0)=0.1$$
$$k_2=hf(x_0+\frac{h}{2},y_0+\frac{k_1}{2})=0.1f(0.05,1.05)=0.11$$
$$k_3=hf(x_0+\frac{h}{3},y_0+\frac{k_2}{3})=0.1f(0.0333,1.0367)=0.1070$$
$$k_4=hf(x_0+\frac{2h}{3},y_0+\frac{2k_3}{3})=0.1f(0.0667,1.0713)=0.1138$$
$$k_5=hf(x_0+\frac{h}{3},y_0+\frac{k_4}{3})=.01f(0.0333,1.0379)=0.1071$$
$$k_6=hf(x_0+\frac{h}{2},y_0+\frac{k_5}{2})=0.1f(0.05,1.0536)=0.1104$$

$$k_7 = hf(x_0 + h,\ y_0 + k_6) = 0.1f(0.1, 1.1104) = 0.1210$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + \frac{81}{2}k_3 + 81k_4 + \frac{81}{2}k_5 - 32k_6 + 11k_7) = 0.1023$$

$$y_1 = y_0 + \Delta y = 1.1023$$

Percentage error: $(1.1103 - 1.1023)/1.1103 = 7.21\times10^{-3}\,\%$

For the second interval we have,

$$x_1 = 0.1,\ y_1 = 1.1023$$

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1023) = 0.1202$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1f(0.15, 1.1624) = 0.1312$$

$$k_3 = hf(x_1 + \frac{h}{3}, y_1 + \frac{k_2}{3}) = 0.1f(0.1333, 1.1460) = 0.1279$$

$$k_4 = hf(x_1 + \frac{2h}{3}, y_1 + \frac{2k_3}{3}) = 0.1f(0.1667, 1.1876) = 0.1354$$

$$k_5 = hf(x_1 + \frac{h}{3}, y_1 + \frac{k_4}{3}) = 0.1f(0.1333, 1.1474) = 0.1281$$

$$k_6 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_5}{2}) = 0.1f(0.15, 1.1664) = 0.1316$$

$$k_7 = hf(x_1 + h, y_1 + k_6) = 0.1f(0.2, 1.2339) = 0.1434$$

$$\Delta y = \frac{1}{120}(11k_1 - 32k_2 + \frac{81}{2}k_3 + 81k_4 + \frac{81}{2}k_5 - 32k_6 + 11k_7) = 0.1319$$

$$\therefore y_2 = y_1 + \Delta y = 1.2342$$

Percentage error: $(1.2428 - 1.2342)/1.2428 = 6.92\times10^{-3}\,\%$

f) Solution through 2nd sixth order (proposed) method

$\frac{1}{2}$	$\frac{1}{2}$						
$\frac{2}{3}$	$\frac{155}{216}$	$\frac{-11}{216}$					
$\frac{1}{2}$	$\frac{191}{324}$	$\frac{-14}{81}$	$\frac{1}{12}$				
$\frac{1}{3}$	$\frac{1}{3}$	0	0	0			
$\frac{1}{3}$	$\frac{67}{192}$	0	0	$\frac{1}{8}$	$\frac{-9}{64}$		
1	$\frac{17}{6}$	0	$\frac{-27}{8}$	$\frac{17}{3}$	0	$\frac{-33}{8}$	
	$\frac{-1}{10}$	0	$\frac{24}{10}$	$\frac{-36}{10}$	0	$\frac{24}{10}$	$\frac{-1}{10}$

The above table is the listing of the proposed coefficients and weights.

In this case we will have

$$k_1 = hf(x_0, y_0) = 0.1$$

$$k_2 = 0.1\left(x_0 + \frac{1}{2}h + y_0 + \frac{1}{2}k_1\right) = 0.11$$

$$k_3 = 0.1\left(x_0 + \frac{2}{3}h + y_0 + \frac{155}{216}k_1 - \frac{11}{216}k_2\right) = 0.1\left(\frac{0.2}{3} + 1 + \frac{0.155}{216} - \frac{11\times 0.11}{216}\right) = 0.106178$$

$$k_4 = 0.1\left(x_0 + \frac{1}{2}h + y_0 + \frac{191}{324}k_1 - \frac{14}{81}k_2 + \frac{1}{12}k_3\right)$$

$$= 0.1\left(\frac{0.1}{2} + 1 + \frac{0.191}{324} - \frac{14\times 0.11}{81} + \frac{0.106178}{12}\right) = 0.10404$$

$$k_5 = 0.1\left(x_0 + \frac{1}{3}h + y_0 + \frac{1}{3}k_1\right) = 0.1\left(\frac{0.1}{3} + 1 + \frac{0.1}{3}\right) = 0.106666$$

$$k_6 = 0.1\left(x_0 + \frac{1}{3}h + y_0 + \frac{67}{192}k_1 + \frac{1}{8}k_4 - \frac{9}{64}k_5\right)$$

$$= 0.1\left(\frac{0.1}{3} + 1 + \frac{0.67}{192} + \frac{0.10404}{8} - \frac{9\times 0.106666}{64}\right) = 0.103482$$

$$k_7 = 0.1\left(x_0 + h + y_0 + \frac{17}{6}k_1 - \frac{27}{8}k_3 + \frac{17}{3}k_4 - \frac{33}{8}k_6\right)$$

$$= 0.1\left(0.1 + 1 + \frac{0.17}{6} - \frac{27\times 0.106178}{8} + \frac{17\times 0.10404}{3} - \frac{33\times 0.103482}{8}\right) = 0.09326$$

$$\Delta y = w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4 + w_5k_5 + w_6k_6 + w_7k_7 = \frac{1}{10}\left(-k_1 + 24k_3 - 36k_4 + 24k_6 - k_7\right)$$

$$=0.109314$$

Thus $x_1 = x_0 + h = 0.1$ and $y_1 = y_0 + \Delta y = 1.1093$

Hence the percentage error is $(1.1103 - 1.1093)/1.1103 = 9\times 10^{-4}\,\%$

For the second interval, we have

$$k_1 = hf(x_1, y_1) = 0.1(0.1 + 1.109314) = 0.120931$$

$$k_2 = 0.1\left(x_1 + \frac{1}{2}h + y_1 + \frac{1}{2}k_1\right) = 0.1\left(0.1 + \frac{0.1}{2} + 1.109314 + \frac{0.120931}{2}\right) = 0.131978$$

$$k_3 = 0.1\left(x_1 + \frac{2}{3}h + y_1 + \frac{155}{216}k_1 - \frac{11}{216}k_2\right)$$

$$=0.1\left(0.1 + \frac{0.2}{3} + 1.109314 + \frac{155\times 0.120931}{216} - \frac{11\times 0.131978}{216}\right) = 0.135604$$

$$k_4 = 0.1\left(x_1 + \frac{1}{2}h + y_1 + \frac{191}{324}k_1 - \frac{14}{81}k_2 + \frac{1}{12}k_3\right)$$

$$= 0.1\left(0.1 + \frac{0.1}{2} + 1.109314 + \frac{191 \times 0.120931}{324} - \frac{14 \times 0.131978}{81} + \frac{0.135604}{12}\right) = 0.131909$$

$$k_5 = 0.1\left(x_1 + \frac{1}{3}h + y_1 + \frac{1}{3}k_1\right) = 0.1\left(0.1 + \frac{0.1}{3} + 1.109314 + \frac{0.120931}{3}\right) = 0.128296$$

$$k_6 = 0.1\left(x_1 + \frac{1}{3}h + y_1 + \frac{67}{192}k_1 + \frac{1}{8}k_4 - \frac{9}{64}k_5\right)$$

$$= 0.1\left(0.1 + \frac{0.1}{3} + 1.109314 + \frac{67 \times 0.120931}{192} + \frac{0.131909}{8} - \frac{9 \times 0.128296}{64}\right) = 0.128329$$

$$k_7 = 0.1\left(x_1 + h + y_1 + \frac{17}{6}k_1 - \frac{27}{8}k_3 + \frac{17}{3}k_4 - \frac{33}{8}k_6\right)$$

$$= 0.1\left(\begin{matrix} 0.1 + 0.1 + 1.109314 + \dfrac{17 \times 0.120931}{6} - \dfrac{27 \times 0.135604}{8} + \dfrac{17 \times 0.131909}{3} - \\ \dfrac{33 \times 0.128329}{8} \end{matrix}\right) = 0.141242$$

$$\Delta y = \frac{1}{10}\left(-k_1 + 24k_3 - 36k_4 + 24k_6 - k_7\right) = 0.132349$$

Thus $x_2 = x_1 + h = 0.2$ and $y_2 = y_1 + \Delta y = 1.2417$

The percentage error is $(1.2428 - 1.2417)/1.2428 = 8.85 \times 10^{-4}\,\%$

g) Solution through Butcher method (sixth order)

In the following listing we have presented the coefficients and weights of the method.

$$\begin{array}{c|ccccccc}
\frac{1}{3} & \frac{1}{3} & & & & & & \\
\frac{2}{3} & 0 & \frac{2}{3} & & & & & \\
\frac{1}{3} & \frac{1}{12} & \frac{1}{3} & \frac{-1}{12} & & & & \\
\frac{1}{2} & \frac{-1}{16} & \frac{9}{8} & \frac{-3}{16} & \frac{-3}{8} & & & \\
\frac{1}{2} & 0 & \frac{9}{8} & \frac{-3}{8} & \frac{-3}{4} & \frac{1}{2} & & \\
1 & \frac{9}{44} & \frac{-9}{11} & \frac{63}{44} & \frac{18}{11} & 0 & \frac{-16}{11} & \\
\hline
 & \frac{11}{120} & 0 & \frac{27}{40} & \frac{27}{40} & \frac{-4}{15} & \frac{-4}{15} & \frac{11}{120}
\end{array}$$

In this case

$k_1 = hf(x_0, y_0) = 0.1(0 + 1) = 0.1$

$$k_2 = hf(x_n + c_2h, y_n + a_{21}k_1) = hf(x_0 + \frac{1}{3}h + y_0 + \frac{1}{3}k_1) = 0.1\left(0 + \frac{0.1}{3} + 1 + \frac{0.1}{3}\right) = 0.10667$$

$$k_3 = hf(x_n + c_3h, y_n + a_{31}k_1 + a_{32}k_2) = h\left(x_0 + \frac{2}{3}h + y_0 + \frac{2}{3}k_2\right)$$

$$= 0.1\left(0 + \frac{0.2}{3} + 1 + \frac{2 \times 0.10667}{3}\right) = 0.113778$$

$$k_4 = h\left(x_0 + \frac{1}{3}h + y_0 + \frac{1}{12}k_1 + \frac{1}{3}k_2 - \frac{1}{12}k_3\right)$$

$$= 0.1\left(\frac{0.1}{3} + 1 + \frac{0.1}{12} + \frac{0.10667}{3} - \frac{0.113778}{12}\right) = 0.10677$$

$$k_5 = h\left(x_0 + \frac{1}{2}h + y_0 - \frac{1}{16}k_1 + \frac{9}{8}k_2 - \frac{3}{16}k_3 - \frac{3}{8}k_4\right)$$

$$= 0.1\left(\frac{0.1}{2} + 1 - \frac{0.1}{16} + \frac{9 \times 0.10667}{8} - \frac{3 \times 0.113778}{16} - \frac{3 \times 0.10677}{8}\right) = 0.110238$$

$$k_6 = h\left(x_0 + \frac{1}{2}h + y_0 + \frac{9}{8}k_2 - \frac{3}{8}k_3 - \frac{3}{4}k_4 + \frac{1}{2}k_5\right)$$

$$= 0.1\left(\frac{0.1}{2} + 1 + \frac{9 \times 0.10667}{8} - \frac{3 \times 0.113778}{8} - \frac{3 \times 0.10677}{4} + \frac{0.110238}{2}\right) = 0.110238$$

$$k_7 = h\left[x_0 + h + y_0 + \frac{1}{44}(9k_1 - 36k_2 + 63k_3 + 72k_4 - 64k_6)\right]$$

$$= 0.1\left[0.1 + 1 + \frac{1}{44}(0.9 - 36 \times 0.10667 + 63 \times 0.113778 + 72 \times 0.10677 - 64 \times 0.110238)\right]$$

$=0.121045$

$$\Delta y = w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4 + w_5k_5 + w_6k_6 + w_7k_7$$

$$= \frac{1}{120}(11k_1 + 81k_3 + 81k_4 - 32k_5 - 32k_6 + 11k_7) = 0.102088$$

Thus $x_1 = x_0 + h = 0.1$ and $y_1 = y_0 + \Delta y = 1.1021$

and the percentage error is $(1.1103 - 1.1021)/1.1103 = 7.39 \times 10^{-3}\%$

For the second interval, we have

$$k_1 = hf(x_1, y_1) = 0.1(0.1 + 1.102088) = 0.1202088$$

$$k_2 = h\left(x_1 + \frac{1}{3}h + y_1 + \frac{1}{3}k_1\right) = 0.1\left(0.1 + \frac{0.1}{3} + 1.102088 + \frac{0.1202088}{3}\right) = 0.127549$$

$$k_3 = h\left(x_1 + \frac{2}{3}h + y_1 + \frac{2}{3}k_2\right) = 0.1\left(0.1 + \frac{0.2}{3} + 1.102088 + \frac{2 \times 0.12749}{3}\right) = 0.135378$$

$$k_4 = h\left(x_1 + \frac{1}{3}h + y_1 + \frac{1}{12}k_1 + \frac{1}{3}k_2 - \frac{1}{12}k_3\right)$$

$$= 0.1\left(0.1 + \frac{0.1}{3} + 1.102088 + \frac{0.1202088}{12} + \frac{0.127549}{3} - \frac{0.135378}{12}\right) = 0.127667$$

$$k_5 = h\left(x_1 + \frac{1}{2}h + y_1 - \frac{1}{16}k_1 + \frac{9}{8}k_2 - \frac{3}{16}k_3 - \frac{3}{8}k_4\right)$$

$$= 0.1\left(0.1 + \frac{0.1}{2} + 1.102088 - \frac{0.1202088}{16} + \frac{9 \times 0.127549}{8} - \frac{3 \times 0.135378}{16} - \frac{3 \times 0.127667}{8}\right)$$

$=0.1314809$

$$k_6 = h\left(x_1 + \frac{1}{2}h + y_1 + \frac{9}{8}k_2 - \frac{3}{8}k_3 - \frac{3}{4}k_4 + \frac{1}{2}k_5\right)$$

$$= 0.1\left(0. + \frac{0.1}{2} + 1.102088 + \frac{9 \times 0.127549}{8} - \frac{3 \times 0.135378}{8} - \frac{3 \times 0.12766}{4} + \frac{0.1314809}{2}\right)$$

$=0.1314804$

$$k_7 = h\left[x_1 + h + y_1 + \frac{1}{44}(9k_1 - 36k_2 + 63k_3 + 72k_4 - 64k_6)\right]$$

$$= 0.1\left[0.1 + 0.1 + 1.102088 + \frac{1}{44}\begin{pmatrix}9 \times 0.1202088 - 36 \times 0.127549 + 63 \times 0.135378 + \\ 72 \times 0.127667 - 64 \times 0.1314804\end{pmatrix}\right]$$

$=0.143382$

$$\Delta y = \frac{1}{120}(11k_1 + 81k_3 + 81k_4 - 32k_5 - 32k_6 + 11k_7) = 0.1316$$

Thus $x_2 = x_1 + h = 0.2$ and $y_2 = y_1 + \Delta y = 1.2337$

and the percentage error is $(1.2428 - 1.2337)/1.2428 = 7.32 \times 10^{-3}\,\%$

Conclusion

It was a study of single and multi step methods to solve differential equations. During the study it was found that, especially for the Runge-Kutta families, there are options to choose arbitrary constant and as a result new formula can be put forward. The opportunity was taken and five new formulas of different orders have been proposed. The performances were compared with the available methods of same order. It is found that the proposed formulas are comparable with the existing formulas in terms of the percentage error.

Extension in the order of Adams-Bashforth and Adams-Moulton formulas up to tenth order is also done and tables for the coefficients required are also provided. Use of the new extended methods of the predictor-corrector formula indicates that the error is reduced.

APPENDIX-A

The Tailor series is

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \frac{h^4}{4!}y^{iv}_n + \frac{h^5}{5!}y^{v}_n + \frac{h^6}{6!}y^{vi}_n + \cdots\cdots \tag{1}$$

where $y' = f$

$$y'' = f_x + ff_y$$

$$y''' = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + ff_y^2,$$

$$y^{iv} = f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy} + f_{xx} f_y + 5ff_y f_{xy} + 4f^2 f_y f_{yy} + 3f_x f_{xy} + \\ 3ff_x f_{yy} + f_x f_y^2 + ff_y^3$$

$$y^{v} = f_{xxxx} + 4ff_{xxxy} + 6f_x f_{xxy} + 9ff_y f_{xxy} + 6f^2 f_{xxyy} + 4f^3 f_{xyyy} + 12ff_x f_{xyy} + 15f^2 f_y f_{xyy} + \\ f^4 f_{yyyy} + 7f^3 f_y f_{yyy} + 4f_{xx} f_{xy} + 4ff_{xx} f_{yy} + f_{xxx} f_y + 7f_x f_y f_{xy} + 8ff_y^2 f_{xy} + 8ff_{xy}^2 + 12f^2 f_{xy} f_{yy} + \\ 13ff_x f_y f_{yy} + 11f^2 f_y^2 f_{yy} + 4f^3 f_{yy}^2 + 6f^2 f_x f_{yyy} + 3f_x^2 f_{yy} + f_{xx} f_y^2 + ff_{xy} f_y^2 + f_x f_y^3 + ff_y^4$$

$$y^{vi} = f_{xxxxx} + 5ff_{xxxxy} + 10f_x f_{xxxy} + 14ff_y f_{xxxy} + 10f^2 f_{xxxyy} + 10f_{xx} f_{xxy} + 41ff_{xy} f_{xxy} + 30ff_x f_{xxyy} + \\ 36f^2 f_y f_{xxyy} + 16f_x f_y f_{xxy} + 19ff_y^2 f_{xxy} + 25f^2 f_{yy} f_{xxy} + 10f^3 f_{xxyyy} + 5f^4 f_{xyyyy} + 30f^2 f_x f_{xyyy} + \\ 34f^3 f_y f_{xyyy} + 15f_x^2 f_{xyy} + 62ff_x f_y f_{xyy} + 20ff_{xx} f_{xyy} + 55f^2 f_{xy} f_{xyy} + 35f^3 f_{yy} f_{xyy} + \\ 50f^2 f_y^2 f_{xyy} + f^5 f_{yyyyy} + 16f^3 f_x f_{yyyy} + 7f^3 f_y f_{yyyy} + 19f^3 f_{xy} f_{yyy} + 7f^3 f_{yy} f_{yyy} + \\ 46f^2 f_x f_y f_{yyy} + 4f^4 f_y f_{yyyy} + 32f^3 f_y^2 f_{yyy} + 5f_{xy} f_{xxx} + 10f_x f_{xx} f_{yy} + 19ff_y f_{xx} f_{yy} + 5ff_{yy} f_{xxx} + \\ f_y f_{xxxx} + 15f_x f_{xy}^2 + 50ff_x f_{xy} f_{yy} + 9f_y f_{xx} f_{xy} + 33ff_y f_{xy}^2 + 12f_x f_y^2 f_{xy} + 14ff_y^3 f_{xy} + \\ 77f^2 f_y f_{xy} f_{yy} + 13f_x^2 f_y f_{yy} + 38ff_x f_y^2 f_{yy} + 25f^2 f_x f_{yy}^2 + 26f^2 f_y^3 f_{yy} + 34f^3 f_y f_{yy}^2 + \\ 8f^4 f_{yy} f_{yyy} + 10f^2 f_{xx} f_{yyy} + 15ff_x^2 f_{yyy} + f_y^2 f_{xxx} + f_{xx} f_y^3 + f_x f_y^4 + ff_y^5$$

Also we will have

$$y_{n+1} = y_n + w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4 + w_5 k_5 + w_6 k_6 + w_7 k_7 \tag{2}$$

Now

$$k_1 = hf$$

$$k_2 = hf + h^2 c_2 f_x + \frac{h^3}{2} c_2^2 f_{xx} + \frac{h^4}{6} c_2^3 f_{xxx} + \frac{h^5}{24} c_2^4 f_{xxxx} + \frac{h^6}{120} c_2^5 f_{xxxxx} + h^2 a_{21} ff_y +$$

$$h^3 a_{21} c_2 ff_{xy} + \frac{h^4}{2} a_{21} c_2^2 ff_{xxy} + \frac{h^5}{6} a_{21} c_2^3 ff_{xxxy} + \frac{h^6}{24} a_{21} c_2^4 ff_{xxxxy} + \frac{h^3}{2} a_{21}^2 f^2 f_{yy} +$$

$$\frac{h^4}{2} c_2 a_{21}^2 f^2 f_{xyy} + \frac{h^5}{4} a_{21}^2 c_2^2 f^2 f_{xxyy} + \frac{h^6}{12} a_{21}^2 c_2^3 f^2 f_{xxxyy} + \frac{h^4}{6} a_{21}^3 f^3 f_{yyy} + \frac{h^5}{6} a_{21}^3 c_2 f^3 f_{xyyy} +$$

$$\frac{h^6}{12} a_{21}^3 c_2^2 f^3 f_{xxyyy} + \frac{h^5}{24} a_{21}^4 f^4 f_{yyyy} + \frac{h^6}{24} a_{21}^4 c_2 f^4 f_{xyyyy} + \frac{h^6}{120} a_{21}^5 f^5 f_{yyyyy}$$

$$k_3 = hf + h^2 c_3 f_x + \frac{h^3}{2} c_3^2 f_{xx} + \frac{h^4}{6} c_3^3 f_{xxx} + \frac{h^5}{24} c_3^4 f_{xxxx} + \frac{h^6}{120} c_3^5 f_{xxxxx} + h^2 (a_{31} + a_{32}) ff_y +$$

$$h^3 a_{32} c_2 f_x f_y + \frac{h^4}{2} a_{32} c_2^2 f_{xx} f_y + \frac{h^5}{6} a_{32} c_2^3 f_{xxx} f_y + \frac{h^6}{24} a_{32} c_2^4 f_{xxxx} f_y + h^3 (a_{31} + a_{32}) c_3 ff_{xy} +$$

$$h^4 a_{32} c_2 c_3 f_x f_{xy} + \frac{h^5}{2} a_{32} c_2^2 c_3 f_{xx} f_{xy} + \frac{h^6}{6} a_{32} c_2^3 c_3 f_{xxx} f_{xy} + \frac{h^4}{2} (a_{31} + a_{32}) c_3^2 ff_{xxy} + \frac{h^5}{2} a_{32} c_2 c_3^2 f_x f_{xxy} +$$

$$\frac{h^6}{4} a_{32} c_2^2 c_3^2 f_{xx} f_{xxy} + \frac{h^5}{6} (a_{31} + a_{32}) c_3^3 ff_{xxxy} + \frac{h^6}{6} a_{32} c_2 c_3^3 f_x f_{xxxy} + \frac{h^6}{24} (a_{31} + a_{32}) c_3^4 ff_{xxxxy} +$$

$$\frac{h^3}{2} (a_{31} + a_{32})^2 f^2 f_{yy} + h^4 (a_{31} + a_{32}) a_{32} c_2 ff_x f_{yy} + \frac{h^5}{2} (a_{31} + a_{32}) a_{32} c_2^2 ff_{xx} f_{yy} +$$

$$\frac{h^6}{6} (a_{31} + a_{32}) a_{32} c_2^3 ff_{xxx} f_{yy} + \frac{h^5}{2} a_{32}^2 c_2^2 f_x^2 f_{yy} + \frac{h^6}{2} a_{32}^2 c_2^3 f_x f_{xx} f_{yy} + \frac{h^4}{2} (a_{31} + a_{32})^2 c_3 f^2 f_{xyy} +$$

$$h^5 (a_{31} + a_{32}) a_{32} c_2 c_3 ff_x f_{xyy} + \frac{h^6}{2} (a_{31} + a_{32}) a_{32} c_2^2 c_3 ff_{xx} f_{xyy} + \frac{h^6}{2} a_{32}^2 c_2^2 c_3 f_x^2 f_{xyy} +$$

$$\frac{h^5}{4} c_3^2 (a_{31} + a_{32})^2 f^2 f_{xxyy} + \frac{h^5}{2} (a_{31} + a_{32}) a_{32} c_2 c_3^2 ff_x f_{xxyy} + \frac{h^6}{12} (a_{31} + a_{32})^2 c_3^3 f^2 f_{xxxyy} +$$

$$\frac{h^6}{6} (a_{31} + a_{32}) a_{32} c_2 c_3^3 ff_x f_{xxxyy} + \frac{h^4}{6} (a_{31} + a_{32})^3 f^3 f_{yyy} + \frac{h^5}{2} (a_{31} + a_{32})^2 a_{32} c_2 f^2 f_x f_{yyy} +$$

$$\frac{h^6}{2} (a_{31} + a_{32}) a_{32}^2 c_2^2 ff_x^2 f_{yyy} + \frac{h^5}{6} (a_{31} + a_{32})^3 c_3 f^3 f_{xyyy} + \frac{h^6}{2} (a_{31} + a_{32})^2 a_{32} c_2 c_3 f^2 f_x f_{xyyy} +$$

$$\frac{h^6}{12} c_3^2 (a_{31} + a_{32})^3 f^3 f_{xxyyy} + \frac{h^5}{24} (a_{31} + a_{32})^4 f^4 f_{yyyy} + \frac{h^6}{6} (a_{31} + a_{32})^3 a_{32} c_2 f^3 f_x f_{yyyy} +$$

$$\frac{h^6}{24} (a_{31} + a_{32})^4 c_3 f^4 f_{xyyyy} + \frac{h^6}{120} (a_{31} + a_{32})^5 f^5 f_{yyyyy}$$

$$k_4 = hf + h^2 c_4 f_x + \frac{h^3}{2} c_4^2 f_{xx} + \frac{h^4}{6} c_4^3 f_{xxx} + \frac{h^5}{24} c_4^4 f_{xxxx} + \frac{h^6}{120} c_4^5 f_{xxxxx} + h^2 (a_{41} + a_{42} + a_{43}) ff_y +$$

$$h^3 (a_{42} c_2 + a_{43} c_3) f_x f_y + \frac{h^4}{2} (a_{32} c_2^2 + a_{43} c_3^2) f_{xx} f_y + \frac{h^5}{6} (a_{42} c_2^3 + a_{43} c_3^3) f_{xxx} f_y +$$

$$\frac{h^6}{24} (a_{32} c_2^4 + a_{43} c_3^4) f_{xxxx} f_y + h^3 (a_{41} + a_{42} + a_{43}) c_4 ff_{xy} + h^4 (a_{42} c_2 + a_{43} c_3) c_4 f_x f_{xy} +$$

$$\frac{h^5}{2}(a_{42}c_2^2+a_{43}c_3^2)c_4f_{xx}f_{xy}+\frac{h^6}{6}(a_{42}c_2^3+a_{43}c_3^3)c_4f_{xxx}f_{xy}+\frac{h^4}{2}(a_{41}+a_{42}+a_{43})c_4^2ff_{xxy}+$$

$$\frac{h^5}{2}(a_{42}c_2+a_{43}c_3)c_4^2f_xf_{xxy}+\frac{h^6}{4}(a_{42}c_2^2+a_{43}c_3^2)c_4^2f_{xx}f_{xxy}+\frac{h^5}{6}(a_{41}+a_{42}+a_{43})c_4^3ff_{xxxy}+$$

$$\frac{h^6}{6}(a_{42}c_2+a_{43}c_3)c_4^3f_xf_{xxxy}+\frac{h^6}{24}(a_{41}+a_{42}+a_{43})c_4^4ff_{xxxxy}+\frac{h^3}{2}(a_{41}+a_{42}+a_{43})^2f^2f_{yy}+$$

$$h^4(a_{41}+a_{42}+a_{43})(a_{42}c_2+a_{43}c_3)ff_xf_{yy}+\frac{h^5}{2}(a_{41}+a_{42}+a_{43})(a_{42}c_2^2+a_{43}c_3^2)ff_{xx}f_{yy}+$$

$$\frac{h^6}{6}(a_{41}+a_{42}+a_{43})(a_{42}c_2^3+a_{43}c_3^3)ff_{xxx}f_{yy}+\frac{h^5}{2}(a_{42}c_2+a_{43}c_3)^2f_x^2f_{yy}+$$

$$\frac{h^6}{2}(a_{42}c_2+a_{43})(a_{42}c_2^2+a_{43}c_3^2)f_xf_{xx}f_{yy}+$$

$$\frac{h^4}{2}(a_{41}+a_{42}+a_{43})^2c_4f^2f_{xyy}+h^5c_4(a_{41}+a_{42}+a_{43})(a_{42}c_2+a_{43}c_3)ff_xf_{xyy}+$$

$$\frac{h^6}{2}(a_{41}+a_{42}+a_{43})(a_{42}c_2^2+a_{43}c_3^2)c_4ff_{xx}f_{xyy}+\frac{h^6}{2}(a_{42}c_2+a_{43}c_3)^2c_4f_x^2f_{xyy}+$$

$$\frac{h^5}{4}(a_{41}+a_{42}+a_{43})^2c_4^2f^2f_{xxyy}+\frac{h^5}{2}c_4^2(a_{41}+a_{42}+a_{43})(a_{42}c_2+a_{43}c_3)ff_xf_{xxyy}+$$

$$\frac{h^6}{12}(a_{41}+a_{42}+a_{43})^2c_4^3f^2f_{xxxyy}+\frac{h^6}{6}(a_{41}+a_{42}+a_{43})(a_{42}c_2+a_{43}c_3)c_4^3ff_xf_{xxxyy}+$$

$$\frac{h^4}{6}(a_{41}+a_{42}+a_{43})^3f^3f_{yyy}+\frac{h^5}{2}(a_{41}+a_{42}+a_{43})^2(a_{42}c_2+a_{43}c_3)f^2f_xf_{yyy}+$$

$$\frac{h^6}{2}(a_{41}+a_{42}+a_{43})(a_{42}c_2+a_{43}c_3)^2ff_x^2f_{yyy}+\frac{h^6}{4}(a_{41}+a_{42}+a_{43})^2(a_{42}c_2^2+a_{43}c_3^2)f^2f_{xx}f_{yyy}+$$

$$\frac{h^5}{6}(a_{41}+a_{42}+a_{43})^3c_4f^3f_{xyyy}+\frac{h^6}{2}(a_{41}+a_{42}+a_{43})^2(a_{42}c_2+a_{43}c_3)c_4f^2f_xf_{xyyy}+$$

$$\frac{h^6}{12}(a_{41}+a_{42}+a_{43})^3c_4^2f^3f_{xxyyy}+\frac{h^5}{24}(a_{41}+a_{42}+a_{43})^4f^4f_{yyyy}+$$

$$\frac{h^6}{6}(a_{41}+a_{42}+a_{43})^3(a_{42}c_2+a_{43}c_3)f^3f_xf_{yyyy}+\frac{h^6}{24}(a_{41}+a_{42}+a_{43})^4c_4f^4f_{xyyyy}+$$

$$\frac{h^6}{120}(a_{41}+a_{42}+a_{43})^5f^5f_{yyyyy}$$

$$k_5=hf+h^2c_5f_x+\frac{h^3}{2}c_5^2f_{xx}+\frac{h^4}{6}c_5^3f_{xxx}+\frac{h^5}{24}c_5^4f_{xxxx}+\frac{h^6}{120}c_5^5f_{xxxxx}+h^2(a_{51}+a_{52}+a_{53}+a_{54})ff_y$$

$$+h^3(a_{52}c_2+a_{53}c_3+a_{54}c_4)f_xf_y+\frac{h^4}{2}(a_{52}c_2^2+a_{53}c_3^2+a_{54}c_4^2)f_{xx}f_y+$$

$$\frac{h^5}{6}(a_{52}c_2^3+a_{53}c_3^3+a_{54}c_4^3)f_{xxx}f_y+\frac{h^6}{24}(a_{52}c_2^4+a_{53}c_3^4+a_{54}c_4^4)f_{xxxx}f_y+$$

$$h^3(a_{51}+a_{52}+a_{53}+a_{54})c_5 ff_{xy}+h^4(a_{52}c_2+a_{53}c_3+a_{54}c_4)c_5 f_x f_{xy}+$$

$$\frac{h^5}{2}(a_{52}c_2^2+a_{53}c_3^2+a_{54}c_4^2)c_5 f_{xx} f_{xy}+\frac{h^6}{6}(a_{52}c_2^3+a_{53}c_3^3+a_{54}c_4^3)c_5 f_{xxx} f_{xy}+$$

$$\frac{h^4}{2}(a_{51}+a_{52}+a_{53}+a_{54})c_5^2 ff_{xxy}+\frac{h^5}{2}(a_{52}c_2+a_{53}c_3+a_{54}c_4)c_5^2 f_x f_{xxy}+$$

$$\frac{h^6}{4}(a_{52}c_2^2+a_{53}c_3^2+a_{54}c_4^2)c_5^2 f_{xx} f_{xxy}+\frac{h^5}{6}(a_{51}+a_{52}+a_{53}+a_{54})c_5^3 ff_{xxxy}+$$

$$\frac{h^6}{6}(a_{52}c_2+a_{53}c_3+a_{54}c_4)c_5^3 f_x f_{xxxy}+\frac{h^6}{24}(a_{51}+a_{52}+a_{53}+a_{54})c_5^4 ff_{xxxxy}+$$

$$\frac{h^3}{2}(a_{51}+a_{52}+a_{53}+a_{54})^2 f^2 f_{yy}+h^4(a_{51}+a_{52}+a_{53}+a_{54})(a_{52}c_2+a_{53}c_3+a_{54}c_4)ff_x f_{yy}+$$

$$\frac{h^5}{2}(a_{51}+a_{52}+a_{53}+a_{54})(a_{52}c_2^2+a_{53}c_3^2+a_{54}c_4^2)ff_{xx} f_{yy}+$$

$$\frac{h^6}{6}(a_{51}+a_{52}+a_{53})(a_{52}c_2^3+a_{53}c_3^3+a_{54}c_4^3)ff_{xxx} f_{yy}+\frac{h^5}{2}(a_{52}c_2+a_{53}c_3+a_{54}c_4)^2 f_x^2 f_{yy}+$$

$$\frac{h^6}{2}(a_{52}c_2+a_{53}c_3+a_{54}c_4)(a_{52}c_2^2+a_{53}c_3^2+a_{54}c_4^2)f_x f_{xx} f_{yy}+\frac{h^4}{2}(a_{51}+a_{52}+a_{53}+a_{54})^2 c_5 f^2 f_{xyy}+$$

$$h^5 c_5(a_{51}+a_{52}+a_{53}+a_{54})(a_{52}c_2+a_{53}c_3+a_{54}c_4)ff_x f_{xyy}+$$

$$\frac{h^6}{2}(a_{51}+a_{52}+a_{53}+a_{54})(a_{52}c_2^2+a_{53}c_3^2+a_{54}c_4^2)c_5 ff_{xx} f_{xyy}+\frac{h^6}{2}(a_{52}c_2+a_{53}c_3+a_{54}c_4)^2 c_5 f_x^2 f_{xyy}+$$

$$\frac{h^5}{4}(a_{51}+a_{52}+a_{53}+a_{54})^2 c_5^2 f^2 f_{xxyy}+\frac{h^5}{2}(a_{51}+a_{52}+a_{53}+a_{54})(a_{52}c_2+a_{53}c_3+a_{54}c_4)c_5^2 ff_x f_{xxyy}+$$

$$\frac{h^6}{12}(a_{51}+a_{52}+a_{53}+a_{54})^2 c_5^3 f^2 f_{xxxyy}+\frac{h^6}{6}(a_{51}+a_{52}+a_{53}+a_{54})(a_{52}c_2+a_{53}c_3+a_{54}c_4)c_5^3 ff_x f_{xxxyy}+$$

$$\frac{h^4}{6}(a_{51}+a_{52}+a_{53}+a_{54})^3 f^3 f_{yyy}+\frac{h^5}{2}(a_{51}+a_{52}+a_{53}+a_{54})^2(a_{52}c_2+a_{53}c_3+a_{54}c_4)f^2 f_x f_{yyy}+$$

$$\frac{h^6}{2}(a_{51}+a_{52}+a_{53}+a_{54})(a_{52}c_2+a_{53}c_3+a_{54}c_4)^2 ff_x^2 f_{yyy}+\frac{h^5}{6}(a_{51}+a_{52}+a_{53}+a_{54})^3 c_5 f^3 f_{xyyy}+$$

$$\frac{h^6}{4}(a_{51}+a_{52}+a_{53}+a_{54})^2(a_{52}c_2+a_{53}c_3+a_{54}c_4)c_5 f^2 f_x f_{xyyy}+$$

$$\frac{h^6}{12}(a_{51}+a_{52}+a_{53}+a_{54})^3 c_5^2 f^3 f_{xxyyy}+\frac{h^5}{24}(a_{51}+a_{52}+a_{53}+a_{54})^4 f^4 f_{yyyy}+$$

$$\frac{h^6}{6}(a_{51}+a_{52}+a_{53}+a_{54})^3(a_{52}c_2+a_{53}c_3+a_{54}c_4)f^3 f_x f_{yyyy}+$$

$$\frac{h^6}{4}(a_{51}+a_{52}+a_{53}+a_{54})^2(a_{52}c_2^2+a_{53}c_3^2+a_{54}c_4^2)f^2 f_{xx} f_{yyy}+$$

$$\frac{h^6}{24}(a_{51}+a_{52}+a_{53}+a_{54})^4 c_5 f^4 f_{xyyyy}+\frac{h^6}{120}(a_{51}+a_{52}+a_{53}+a_{54})^5 f^5 f_{yyyyy}$$

$$
\begin{aligned}
k_6 = {} & hf + h^2 c_6 f_x + \frac{h^3}{2} c_6^2 f_{xx} + \frac{h^4}{6} c_6^3 f_{xxx} + \frac{h^5}{24} c_6^4 f_{xxxx} + \\
& \frac{h^6}{120} c_6^5 f_{xxxxx} + h^2 (a_{61} + a_{62} + a_{63} + a_{64} + a_{65}) ff_y + \\
& h^3 (a_{62}c_2 + a_{63}c_3 + a_{64} + a_{65}) f_x f_y + \frac{h^4}{2} (a_{62}c_2^2 + a_{63}c_3^2 + a_{64}c_4^2 + a_{65}c_5^2) f_{xx} f_y + \\
& \frac{h^5}{6} (a_{62}c_2^3 + a_{63}c_3^3 + a_{64}c_4^3 + a_{65}c_5^3) f_{xxx} f_y + \frac{h^6}{24} (a_{62}c_2^4 + a_{63}c_3^4 + a_{64}c_4^4 + a_{65}c_5^4) f_{xxxx} f_y + \\
& h^3 (a_{61} + a_{62} + a_{63} + a_{64} + a_{65}) c_6 ff_{xy} + h^4 (a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5) c_6 f_x f_{xy} + \\
& \frac{h^5}{2} (a_{62}c_2^2 + a_{63}c_3^2 + a_{64}c_4^2 + a_{65}c_5^2) c_6 f_{xx} f_{xy} + \frac{h^6}{6} (a_{62}c_2^3 + a_{63}c_3^3 + a_{64}c_4^3 + a_{65}c_5^3) c_6 f_{xxx} f_{xy} + \\
& \frac{h^4}{2} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65}) c_6^2 ff_{xxy} + \frac{h^5}{2} (a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5) c_6^2 f_x f_{xxy} + \\
& \frac{h^6}{4} (a_{62}c_2^2 + a_{63}c_3^2 + a_{64}c_4^2 + a_{65}c_5^2) c_6^2 f_{xx} f_{xxy} + \frac{h^5}{6} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65}) c_6^3 ff_{xxxy} + \\
& \frac{h^6}{6} (a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5) c_6^3 f_x f_{xxxy} + \frac{h^6}{24} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65}) c_6^4 ff_{xxxxy} + \\
& h^4 (a +_{61} a_{62} + a_{63} + a_{64} + a_{65})(a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5) ff_x f_{yy} + \\
& \frac{h^5}{2} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})(a_{62}c_2^2 + a_{63}c_3^2 + a_{64}c_4^2 + a_{65}c_5^2) ff_{xx} f_{yy} + \\
& \frac{h^6}{6} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})(a_{62}c_2^3 + a_{63}c_3^3 + a_{64}c_4^3 + a_{65}c_5^3) ff_{xxx} f_{yy} + \\
& \frac{h^5}{2} (a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5)^2 f_x^2 f_{yy} + \frac{h^3}{2} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})^2 f^2 f_{yy} + \\
& h^6 (a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5)(a_{62}c_2^2 + a_{63}c_3^2 + a_{64}c_4^2 + a_{65}c_5^2) f_x f_{xx} f_{yy} + \\
& \frac{h^4}{2} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})^2 c_6 f^2 f_{xyy} + h^4 c_6 (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})(a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5) ff_x f_{xyy} + \\
& \frac{h^6}{2} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})(a_{62}c_2^2 + a_{63}c_3^2 + a_{64}c_4^2 + a_{65}c_5^2) c_6 ff_{xx} f_{xyy} + \frac{h^6}{2} (a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5)^2 c_6 f_x^2 f_{xyy} + \\
& \frac{h^5}{4} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})^2 c_6^2 f^2 f_{xxyy} + \frac{h^5}{2} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})(a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5) c_6^2 ff_x f_{xxyy} + \\
& \frac{h^6}{12} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})^2 c_6^3 f^2 f_{xxxyy} + \frac{h^6}{6} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65}c_5)(a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5) c_6^2 ff_x f_{xxxyy} + \\
& \frac{h^4}{6} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})^3 f^3 f_{yyy} + \frac{h^5}{2} (a_{61} + a_{62} + a_{63} + a_{64} + a_{65})^2 (a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5) f^2 f_x f_{yyy} +
\end{aligned}
$$

$$\frac{h^6}{2}(a_{61}+a_{62}+a_{63}+a_{64}+a_{65})(a_{62}c_2+a_{63}c_3+a_{64}c_4+a_{65}c_5)^2 ff_x^2 f_{yyy}+\frac{h^5}{6}(a_{61}+a_{62}+a_{63}+a_{64}+a_{65})^3 c_6 f^3 f_{yyy}+$$

$$\frac{h^6}{4}(a_{61}+a_{62}+a_{63}+a_{64}+a_{65})^2(a_{62}c_2+a_{63}c_3+a_{64}c_4+a_{65}c_5)c_6 f^2 f_x f_{xyy}+$$

$$\frac{h^6}{12}(a_{61}+a_{62}+a_{63}+a_{64}+a_{65})^3 c_6^2 f^3 f_{xxyyy}+\frac{h^5}{24}(a_{61}+a_{62}+a_{63}+a_{64}+a_{65})^4 f^4 f_{yyyy}+$$

$$\frac{h^6}{6}(a_{61}+a_{62}+a_{63}+a_{64}+a_{65})^3(a_{62}c_2+a_{63}c_3+a_{64}c_4)f^3 f_x f_{yyyy}+$$

$$\frac{h^6}{24}(a_{61}+a_{62}+a_{63}+a_{64}+a_{65})^4 c_6 f^4 f_{xyyyy}+$$

$$\frac{h^6}{4}(a_{61}+a_{62}+a_{63}+a_{64}+a_{65})^2(a_{62}c_2^2+a_{63}c_3^2+a_{64}c_4^2+a_{65}c_5^2)f^2 f_{xx} f_{yyy}+$$

$$\frac{h^6}{120}(a_{61}+a_{62}+a_{63}+a_{64}+a_{65})^5 f^5 f_{yyyyy}$$

$$k_7 = hf + h^2 c_7 f_x + \frac{h^3}{2}c_7^2 f_{xx} + \frac{h^4}{6}c_7^3 f_{xxx} + \frac{h^5}{24}c_7^4 f_{xxxx} + \frac{h^6}{120}c_7^5 f_{xxxxx} +$$

$$h^2(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})ff_y + h^3(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)f_x f_y +$$

$$\frac{h^4}{2}(a_{72}c_2^2+a_{73}c_3^2+a_{74}c_4^2+a_{75}c_5^2+a_{76}c_6^2)f_{xx} f_y +$$

$$\frac{h^5}{6}(a_{72}c_2^3+a_{73}c_3^3+a_{74}c_4^3+a_{75}c_5^3+a_{76}c_6^3)f_{xxx} f_y +$$

$$\frac{h^6}{24}(a_{72}c_2^4+a_{73}c_3^4+a_{74}c_4^4+a_{75}c_5^4+a_{76}c_6^4)f_{xxxx} f_y +$$

$$h^3(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})c_7 ff_{xy} + h^4(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)c_7 f_x f_{xy} +$$

$$\frac{h^5}{2}(a_{72}c_2^2+a_{73}c_3^2+a_{74}c_4^2+a_{75}c_5^2+a_{76}c_6^2)c_7 f_{xx} f_{xy} + \frac{h^6}{6}(a_{72}c_2^3+a_{73}c_3^3+a_{74}c_4^3+a_{75}c_5^3+a_{76}c_6^3)c_7 f_{xxx} f_{xy} +$$

$$\frac{h^4}{2}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})c_7^2 ff_{xxy} + \frac{h^5}{2}(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)c_7^2 f_x f_{xxy} +$$

$$\frac{h^6}{4}(a_{72}c_2^2+a_{73}c_3^2+a_{74}c_4^2+a_{75}c_5^2+a_{76}c_6^2)c_7^2 f_{xx} f_{xxy} + \frac{h^5}{6}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75})c_7^3 ff_{xxxy} +$$

$$\frac{h^6}{6}(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)c_7^3 f_x f_{xxxy} + \frac{h^6}{24}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})c_7^4 ff_{xxxxy} +$$

$$\frac{h^3}{2}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^2 f^2 f_{yy} +$$

$$h^4(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)ff_x f_{yy} +$$

$$\frac{h^5}{2}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})(a_{72}c_2^2+a_{73}c_3^2+a_{74}c_4^2+a_{75}c_5^2+a_{76}c_6^2)ff_{xx} f_{yy} +$$

$$\frac{h^6}{6}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})(a_{72}c_2^3+a_{73}c_3^3+a_{74}c_4^3+a_{75}c_5^3+a_{76}c_6^2)ff_{xxx} f_{yy} +$$

$$\frac{h^5}{2}\left(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6\right)^2 f_x^2 f_{yy}+$$
$$h^6(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)(a_{72}c_2^2+a_{73}c_3^2+a_{74}c_4^2+a_{75}c_5^2+a_{76}c_6^2)f_x f_{xx} f_{yy}+$$
$$\frac{h^4}{2}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^2 c_7 f^2 f_{xyy}+$$
$$h^4 c_7(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)ff_x f_{xyy}+$$
$$\frac{h^6}{2}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})(a_{72}c_2^2+a_{73}c_3^2+a_{74}c_4^2+a_{75}c_5^2+a_{76}c_6^2)c_7 ff_{xx} f_{xyy}+$$
$$\frac{h^6}{2}(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5)^2 c_7 f_x^2 f_{xyy}+\frac{h^5}{4}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^2 c_7^2 f^2 f_{xxyy}+$$
$$\frac{h^5}{2}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)c_7^2 ff_x f_{xxyy}+$$
$$\frac{h^6}{12}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^2 c_7^3 f^2 f_{xxxyy}+$$
$$\frac{h^6}{6}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)c_7^2 ff_x f_{xxxyy}+$$
$$\frac{h^4}{6}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75})^3 f^3 f_{yyy}+$$
$$\frac{h^5}{2}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^2(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)f^2 f_x f_{yyy}+$$
$$\frac{h^6}{2}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)^2 ff_x^2 f_{yyy}+$$
$$\frac{h^5}{6}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^3 c_7 f^3 f_{xyyy}+$$
$$\frac{h^6}{4}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^2(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)c_7 f^2 f_x f_{xyyy}+$$
$$\frac{h^6}{12}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^3 c_7^2 f^3 f_{xxyyy}+\frac{h^5}{24}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^4 f^4 f_{yyyy}+$$
$$\frac{h^6}{6}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^3(a_{72}c_2+a_{73}c_3+a_{74}c_4+a_{75}c_5+a_{76}c_6)f^3 f_x f_{yyyy}+$$
$$\frac{h^6}{24}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^4 c_7 f^4 f_{xyyyy}+$$
$$\frac{h^6}{4}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^2(a_{72}c_2^2+a_{73}c_3^2+a_{74}c_4^2+a_{75}c_5^2+a_{76}c_6^2)f^2 f_{xx} f_{yyy}+$$
$$\frac{h^6}{120}(a_{71}+a_{72}+a_{73}+a_{74}+a_{75}+a_{76})^5 f^5 f_{yyyyy}$$

Substituting the values of k's in (2) and then comparing with (1) we will get 24 equations with 34 unknowns. So we require assigning 10 constants. The equations are as follows:

$a_{21}=c_2$

$a_{31}+a_{32}=c_3$

$$a_{41} + a_{42} + a_{43} = c_4$$
$$a_{51} + a_{52} + a_{53} + a_{54} = c_5$$
$$a_{61} + a_{62} + a_{63} + a_{64} + a_{65} = c_6$$
$$a_{71} + a_{72} + a_{73} + a_{74} + a_{75} + a_{76} = c_7$$
$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 = 1$$
$$w_2c_2 + w_3c_3 + w_4c_4 + w_5c_5 + w_6c_6 + w_7c_7 = \frac{1}{2}$$
$$w_2c_2^2 + w_3c_3^2 + w_4c_4^2 + w_5c_5^2 + w_6c_6^2 + w_7c_7^2 = \frac{1}{3}$$
$$w_2c_2^3 + w_3c_3^3 + w_4c_4^3 + w_5c_5^3 + w_6c_6^3 + w_7c_7^3 = \frac{1}{4}$$
$$w_2c_2^4 + w_3c_3^4 + w_4c_4^4 + w_5c_5^4 + w_6c_6^4 + w_7c_7^4 = \frac{1}{5}$$
$$w_2c_2^5 + w_3c_3^5 + w_4c_4^5 + w_5c_5^5 + w_6c_6^5 + w_7c_7^5 = \frac{1}{6}$$
$$w_3c_2a_{32} + w_4(a_{42}c_2 + a_{43}c_3) + w_5(a_{52}c_2 + a_{53}c_3 + a_{54}c_4) + w_6(a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5) +$$
$$w_7(a_{72}c_2 + a_{73}c_3 + a_{74}c_4 + a_{75}c_5 + a_{76}c_6) = \frac{1}{6}$$
$$w_3a_{32}c_2^2 + w_4(a_{42}c_2^2 + a_{43}c_3^2) + w_5(a_{52}c_2^2 + a_{53}c_3^2 + a_{54}c_4^2) + w_6(a_{62}c_2^2 + a_{63}c_3^2 + a_{64}c_4^2 + a_{65}c_5^2) +$$
$$w_7(a_{72}c_2^2 + a_{73}c_3^2 + a_{74}c_4^2 + a_{75}c_5^2 + a_{76}c_7^2) = \frac{1}{12}$$
$$w_3a_{32}c_2c_3 + w_4c_4(a_{42}c_2 + a_{43}c_3) + w_5c_5(a_{52}c_2 + a_{53}c_3 + a_{54}c_4) + w_6c_6(a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5)$$
$$+ w_7c_7(a_{72}c_2 + a_{73}c_3 + a_{74}c_4 + a_{75}c_5 + a_{76}c_6) = \frac{1}{8}$$
$$w_3a_{32}c_2c_3^3 + w_4c_4^2(a_{42}c_2 + a_{43}c_3) + w_5c_5^2(a_{52}c_2 + a_{53}c_3 + a_{54}c_4) + w_6c_6^2(a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5)$$
$$+ w_7c_7^2(a_{72}c_2 + a_{73}c_3 + a_{74}c_4 + a_{75}c_5 + a_{76}c_6) = \frac{1}{10}$$
$$w_3a_{32}c_2^2c_3 + w_4c_4(a_{42}c_2^2 + a_{43}c_3^2) + w_5c_5(a_{52}c_2^2 + a_{53}c_3^2 + a_{54}c_4^2) + w_6c_6(a_{62}c_2^2 + a_{63}c_3^2 + a_{64}c_4^2 +$$
$$a_{65}c_5^2) + w_7c_7(a_{72}c_2^2 + a_{73}c_3^2 + a_{74}c_4^2 + a_{75}c_5^2 + a_{76}c_6^2) = \frac{1}{15}$$
$$w_3a_{32}c_2^3 + w_4(a_{42}c_2^3 + a_{43}c_3^3) + w_5(a_{52}c_2^3 + a_{53}c_3^3 + a_{54}c_4^3) + w_6(a_{62}c_2^3 + a_{63}c_3^3 + a_{64}c_4^3 + a_{65}c_5^3) +$$
$$w_7(a_{72}c_2^3 + a_{73}c_3^3 + a_{74}c_4^3 + a_{75}c_5^3 + a_{76}c_6^3) = \frac{1}{20}$$
$$w_3a_{32}^2c_2^2 + w_4(a_{42}c_2 + a_{43}c_3)^2 + w_5(a_{52}c_2 + a_{53}c_3 + a_{54}c_4)^2 + w_6(a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5)^2$$
$$+ w_7(a_{72}c_2 + a_{73}c_3 + a_{74}c_4 + a_{75}c_5 + a_{76}c_6)^2 = \frac{1}{20}$$
$$w_3a_{32}c_2c_3^3 + w_4c_4^3(a_{42}c_2 + a_{43}c_3) + w_5c_5^3(a_{52}c_2 + a_{53}c_3 + a_{54}c_4) + w_6c_6^3(a_{62}c_2 + a_{63}c_3 + a_{64}c_4 + a_{65}c_5)$$
$$+ w_7c_7^3(a_{72}c_2 + a_{73}c_3 + a_{74}c_4 + a_{75}c_5 + a_{76}c_6) = \frac{1}{12}$$

$$w_3 a_{32} c_2^2 c_3^2 + w_4 c_4^2 (a_{42} c_2^2 + a_{43} c_3^2) + w_5 c_5^2 (a_{52} c_2^2 + a_{53} c_3^2 + a_{54} c_4^2) + w_6 c_6^2 (a_{62} c_2^2 + a_{63} c_3^2 + a_{64} c_4^2 + a_{65} c_5^2) + w_7 c_7^2 (a_{72} c_2^2 + a_{73} c_3^2 + a_{74} c_4^2 + a_{75} c_5^2 + a_{76} c_6^2) = \frac{1}{18}$$

$$w_3 a_{32}^2 c_2^2 c_3 + w_4 c_4 (a_{42} c_2 + a_{43} c_3)^2 + w_5 c_5 (a_{52} c_2 + a_{53} c_3 + a_{54} c_4)^2 + w_6 c_6 (a_{62} c_2 + a_{63} c_3 + a_{64} c_4 + a_{65} c_5)^2 + w_7 c_7 (a_{72} c_2 + a_{73} c_3 + a_{74} c_4 + a_{75} c_5 + a_{76} c_6)^2 = \frac{1}{24}$$

$$w_3 a_{32} c_2 c_3^3 + w_4 c_4^3 (a_{42} c_2 + a_{43} c_3) + w_5 c_5^3 (a_{52} c_2 + a_{53} c_3 + a_{54} c_4) + w_6 c_6^3 (a_{62} c_2 + a_{63} c_3 + a_{64} c_4 + a_{65} c_5) + w_7 c_7^3 (a_{72} c_2 + a_{73} c_3 + a_{74} c_4 + a_{75} c_5 + a_{76} c_6) = \frac{2}{15}$$

$$w_3 a_{32} c_2^3 c_3 + w_4 c_4 (a_{42} c_2^3 + a_{43} c_3^3) + w_5 c_5 (a_{52} c_2^3 + a_{53} c_3^3 + a_{54} c_4^3) + w_6 c_6 (a_{62} c_2^3 + a_{63} c_3^3 + a_{64} c_4^3 + a_{65} c_5^3) + w_7 c_7 (a_{72} c_2^3 + a_{73} c_3^3 + a_{74} c_4^3 + a_{75} c_5^3 + a_{76} c_6^3) = \frac{1}{24}$$

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