STUDY OF PSEUDOCOMPLEMENTED LATTICE

A Thesis

Submitted for the partial fulfillment of the requirements for the Degree of

MASTER OF PHILOSOPHY

In

Mathematics

BY

KHAKI MASUDUR RAHMAN

Registration No.- 115

Roll No.-0051510,

Session: 2000-2001,

To

Department of Mathematics

Khulna University of Engineering & Technology.

Khulna, Bangladesh.

June, 2008

We hereby recommend the thesis prepared by **KHAKI MASUDUR RAHMAN,** Roll no. 0051510,Registration No. 115 , Session 2000-2001 Entitled "**STUDY OF PSEUDOCOMPLEMENTED LATTICE**" to be accepted as fulfilling this part of the requirements for the degree of Master of Philosophy in the department of Mathematics.

Committee of Examiners

1.	Dr. Md. Abul Kalam Azad Supervisor Department of Mathematics Khulna University of Engineering & Thchnology. Bangladesh	Chairman
2.	Professor Dr. Fouzia Rahman Department of Mathematics Khulna University of Engineering & Thchnology. Bangladesh	Member
3.	Professor Dr. Md. Bazlar Rahman Department of Mathematics Khulna University of Engineering & Thchnology. Bangladesh	Member 16.6.08
4.	Professor Dr. Mohammad Arif Hossain Head Department of Mathematics Khulna University of Engineering & Thchnology. Bangladesh	Member 16.6.08
5.	Professor Dr. M. A. Latif Department of Mathematics University of Rajshahi,Rajshahi	Member(External) 16.6.08

Dedicated

To My Parents

DECLARATION

I hereby declare that this thesis entitled "Study of Peudocomplemented Lattice" submitted for the partial fulfillment for the degree of Master of Philosophy is done by myself under the supervision of **Dr. Md. Abul Kalam Azad** and is not submitted elsewhere for any other degree or diploma.

M. Phill Research Fellow.

Khulna

Jun 01, 2008

(Khaki Masudur Rahman)

Couter signed by:

....

Supervisor

Dr. Md. Abul Kalam Azad

Associate Professor

Department of Mathematics

Khulna University of Engineering & Technology.

Khulna- 9230

ACKNOLEDGEMENT

It is my heartfelt pleasure expressing my gratitude to my supervisor ***Dr. Md. Abul Kalam Azad*** , Associate Professor Department of Mathematics, Khulna university of Engineering and Technology for his guidance, criticism and encouragement throughout my research work. I am obliged to express my heartiest thanks to my supervisor who taught the topic of the present thesis and provided me the way with his suggestions .Without his encouragement and sincere help, my thesis would never come to light, In particular I would like to extend my gratitude to ***Dr. Md. Bazlar Rahman***, Professor, Department of Mathematics, KUET for hisconstructive suggestions and kind discussion help to improve my thesis. I must also thank ***Dr. Mohammad Arif Hossain***, Professor, Department of Mathematics, KUET for his encouragement and caring concern.

I can not but express my sincere admiration, appreciation and gratitude to all the teachers of the Department of Mathematics, KUET for providing me with valuable suggestions during the period of research work, particularly Md. Zaidur Rahman, Assistant professor, Department of Mathematics, KUET for willing help to improve my thesis.

My family, inevitably, had to suffer a lot while this work was being done. This thesis owes a lot to the patience and co-operation shown to me by my wife Hazera and our three children Moury ,Saad and Fariha for their suffering while I remained absent from home for thesis purpose; I am deeply thankful to them.

Finally, I would like to shoulder upon all the errors and shortcoming in the study if there be any, I am extremely sorry for that.

(Khaki Masudur Rahman)

TABLE OF CONTENTS

SUMMARY

This thesis studies the nature of *Pseudocomplemented lattice*. We can define a lattice in two ways; (i) *Set theoretically* and (ii) *Algebraically.*

Set theoretically: A poset $< L; \leq >$ is a lattice if for every $a, b \in L$ both Sup{a,b} and Inf{a,b} exists in L.

Algebraically : A nonempty set L with two binary operations $\wedge$ and $\vee$ is called a lattice if $\forall \ a, b, c \in L$. The following conditions hold.

i) $a \wedge a = a, \ a \vee a = a$

ii) $a \wedge b = b \wedge a, \ a \vee b = b \vee a$.

iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \ a \vee (b \vee c) = (a \vee b) \vee c$,

iv) $a \wedge (a \vee b) = a, \ a \vee (a \wedge b) = a$.

In this thesis, we have studied several properties of *pseudocomplemented lattices*. Moreover, we give several results on *pseudocomplemented* lattices which certainly extend and generalize many results in lattice theory.

In Chapter one, we have discussed *posets*, *lattices* and *Ideals of a lattice* which are explain with some examples and generalized many theorems of them.

In chapter two, congruence *of lattices*, *distributive lattices, Complemented lattices* and *Boolean algebra* have been discussed, which are basic concept of this thesis.

In chapter three we give a description of *pseudocomplemented lattices.* We have also studied *distributive pseudocomplemented lattices* and *algebraic lattices. Pseudocomplemented* lattices have been studied by G. Gratzer [7] and many other authors. Here we extend several results of G. Gratzer [7] to lattices.

Chapter four introduces the concepts of *stone lattices*. Stone lattices have been studied by Gratzer [7], Katrinak [11] and many other authors. We have given a characterization of *minimal prime ideals* of *pseudocomplemented distributive lattices.*

Chapter five introduces the concept of *distributive and modular lattice* with *n-ideals*. Here we include several characterizations of *n-ideals*. We have proved some interesting result which are generalizes several results on *distributive* ,*modular* and *ideals of a lattices.* Latif [20] in his thesis has introduced the concept of *standard n-ideals of a lattice*. We conclude this thesis with some more properties of *standard* and *neutral n-ideals.*

CHAPTER ONE

LATTICES AND IDEALS

1. Lattices:

Introduction: The intention of this section is to outline and fix the notation for some of the concepts of *lattices* which are basic to this thesis. We also formulate some results on arbitrary *lattices* for later use. For the background material in lattice theory we refer the reader to the text of G. Birkhoff [l], G. Gratzer [7], [8], D.E. Rutherford [17] and vijay K. Khanna [18].

Definition (Poset): A nonempty set *P*, together with a binary relation ρ is said to form a partially ordered set or a *poset* of the following conditions hold: For all $a, b, c \in P$

i) Reflexivity : $a \rho a$

ii) Anti – symmetry: $a \rho b$ and $b \rho a$ imply that $a = b$

iii) Transitivity: $a \rho b$ and $b \rho a$ imply that $a \rho c$

We also use the partially ordering relation '$\leq$' in lieu of ρ.

Now we give an example of a *poset*.

Example 1.1.1 : The set *N of natural numbers* form *a poset* under the usual '$\leq$'. Similarly, *the set of integers Z, the set of rationals Q* and *the set of real numbers R* also form *posets* under usual '$\leq$'.

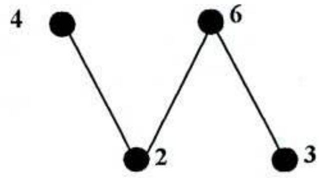


Figure 1.1

As a particular case, the *poset* $\{2,3,4,6\}$ under divisibility is represented by figure 1.1

Definition (Chain): If P is a *poset* in which every two members are comparable it is called a *totally ordered set* or *to set* or a *chain*. Thus if P is a *chain* and $x, y \in P$ then either $x \leq y$ or $y \leq x$. The *poset* in figure 1.2 is a *chain.*

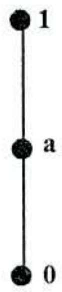


Figure 1.2

Let P be a *poset*. If there exists an element $a \in P$ such that $x \leq a$ for all $x \in P$ then a is called greatest element, if it exists, will be comparable with all elements of the *poset*. It is generally denoted by *u or l.*

Also an element $b \in P$ will be called least or zero element of P if $b \leq x, \forall x \in P$. It is denoted by 0. Least element (if it exists) will be unique.

Let $X = \{1,2,3\}$, then $P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$ form a peset under usual '$\leq$' with ϕ as least element and $\{1,2,3\}$ as greatest element. An element a in a *poset* P is called maximal element of P if $a < x$ for no $x \in P$. In the *poset* $\{1,2,4,6\}$ under divisibility 4 and 6 are both maximal elements. Greatest element is the unique maximal element in figure 1.1. An element b in a *poset* P is called a minimal element of P if $x < b$ for no x in P. 2 and 3 are both minimal elements in figeure 1.1.

Theorem 1.1.2 : If S is a nonempty finite subset of a *poset* P then S has *maximal* and *minimal* elements.

Proof : Let $x_1 . x_2$ ...., x_n be all the distinct elements of S in any random order. If x_1 is *maximal* element, we are done. If x_1 is not *maximal* then there exists some $x_i \in S$ such that $x_1 < x_i$. If x_i is *maximal*. We are done. If not, there exists some $x_j \in S$ such that $x_i < x_j$. Continuing like this, we will reach a stage where some element will be *maximal*. Similarly, we can show that S has *minimal* elements. ■

Theorem 1.1.3: The cardinal product of two *posets* is a *poset*.

Proof : Let P_1 and P_2 be two *posets* then we show that $P_1 \times P_2 = \{(x, y) / x \in P_1, y \in P_2\}$ forms a *poset* under the relation defined by. $(x_1, y_1) \le P_1 \times P_2 (x_2, y_2) \Leftrightarrow x_1 \le P_1 x_2$ in P_1, $y_1 \le P_2 y_2$ in P_2

i) Reflexivity : $(x, y) \le P_1 \times P_2 (x,y)$ $\forall$ $(x, y) \in P_1 \times P_2$ as $x \le P_1$ in P_1 and $y \le P_2 y$ in P_2 $\forall x \in P_1, y \in P_2$

(ii) Anti – symmetry : Let $(x_1, y_1) \le P_1 \times P_2 (x_2, y_2)$ and $(x_2,y_2) \le P_1 \times P_2 (x_1, y_1)$. Then $x_1 \le P_1 x_2, y_1 \le P_2 y_2$ *and* $x_2 \le P_1 x_1, y_2 \le P_2 y_2$, implies that $x_1 = x_2$, $y_1 = y_2$ implies that $(x_1, y_1) = (x_2, y_2)$.

(iii) Transitive: Let $(x_1, y_1) \le P_1 \times P_2 (x_2, y_2)$ and $(x_2, y_2) \le P_1 \times P_2 (x_3, y_3)$. Then $x_1 \le P_1 x_2, y_1 \le P_2 y_2$ and $x_2 \le P_1 x_3$, $y_2 \le P_2 y_3$, implies that $x_1 \le P_1 x_3$, $y_1 \le P_1 y_3$ *implies* $(x_1, y_1) \le P_1 \times P_2 (x_3, y_3)$.

Hence the product of two *posets* is a *poset*. ■

Definition(Suprimum and Infimum): Let S be a non empty subset of a *poset* P. An element $a \in P$ is called an upper bound of S if $x \le a \forall x \in S$. Further if a is an upper bound of S such that, $a \le b$ for all upper bounds b of S then a is called least upper bound or *supremum* of S. We write

Sup S for *supremum of S.* Then a is called least upper bound or *supremum of S.* An element $a \in P$ will be called a *lower bound of S* if S if $a \leq x \forall x \in S$ and a will be called the *greatest lower bound* or *Infimum of S* if $b \leq a$ for all lower bounds b of S.

Example : Let $< Z, \leq >$ be the *poset* of integers under usual '$\leq$'

Let $S = \{\ldots\ldots\ldots\ldots -3, -2, -1, 0, -2, 3\}$ then $3 = Sup\ S$.

Definition(Lattice): *Lattices* are defined in two ways; (i) *set theoretically* and (ii) *Algebraically*

Set theoretically (define a lattice): A *poset* $< L; \leq >$ is said to form a *lattice* if for every $a, b \in L$, $Sup\{a,b\}$ and $Inf\{a,b\}$ exist in L. So we can write $Sup\{a,b\} = a \vee b$ and $Inf\{a,b\} = a \vee b$

Example: 1.1.4: Let X be a non empty set, then the *poset* $< P(X); \subseteq >$ of all subsets of X under set inclusion '$\subseteq$' is *lattice*.

Here, for $A, B \in P(X), A \wedge B = A \cap B$ and $A \vee B = A \cup B$. As a particular case when $X = \{1, 2, 3\}$ then

$P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

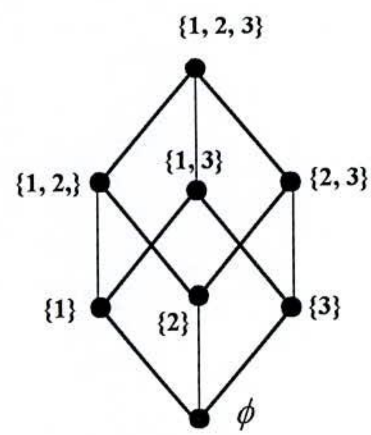


Figure 1.3

Now we give an example of a *poset* which is not a *lattice*. ■

Example: 1.1.5: The set {2,3,4,12} under divisibility is a *poset* but is not a *lattice*. Since $2 \wedge 3 = 6$ does not exists.

The algebraic definition of a lattice: A nonempty set L together with two binary operations $\wedge$ and $\vee$ is said to form a *lattice* if $\forall\ a, b, c \in L$ the following conditions hold;

i) Idempotency : $a \wedge a = a, a \vee a = a$

ii) Commutativity : $a \wedge b = b \wedge a,\ a \vee b = b \vee a$

iii) Associativity : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.
$a \vee (b \vee c) = (a \vee b) \vee c$

iv) Absorption: $a \wedge (a \vee b) = a,\ a \vee (a \wedge b) = a$.

Example:1.1.6: The set $L = \{0, a, b, 1\}$ forms a *lattice.*

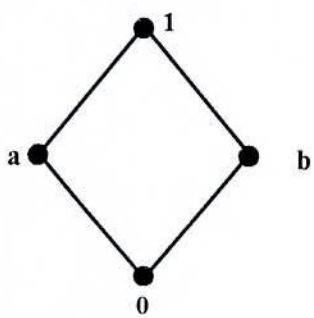


Figure 1.1.4

The *meet* table and the *join* table of $L = \{0, a, b, 1\}$ are as follows:

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Table - 1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Table - 2

Theorem: 1.1.7: (a) Let the *poset* $L=<L;\leq>$ be a *lattice*.

Set $Sup\{a,b\}=a\vee b$ and $Inf\{a,b\}=a\wedge b$, then the algebra $L^{a}=<L;\wedge,\vee>$ is a *lattice*.

(b) Let the algebra $L=<L;\leq>$ be a *lattice*. Set $a\leq b$ if and only if $a\wedge b=a$, then $L^{p}=<L;\leq>$ is a *poset* and the *poset* L^{p} is a *lattice*.

Proof: a) We have L is non empty and $\wedge$ and $\vee$ are two binary operations in L.

i) $a\wedge a=Inf\{a,a\}=a$, $a\vee a=Sup\{a,a\}=a$

$\therefore$ $\wedge$ and $\vee$ satisfy idempotent law.

ii) $a\wedge b=Inf\{a,b\}=Inf\{b,a\}=b\wedge a$

$a\vee b=Sup\{a,b\}=Sup\{b,a\}=b\vee a$

$\therefore$ $\wedge$ and $\vee$ satisfy commutative law.

iii) $a\wedge(b\wedge c)=a\wedge Inf\{b,c\}=Inf\{a,b,c\}$

$=Inf\{a,b\}\wedge c=(a\wedge b)\wedge c$

$a\vee(b\vee c)=a\vee Sup\{b,c\}=Sup\{a,b,c\}$

$=Sup\{a,b\}\vee c=(a\vee b)\vee c$

$\therefore$ $\wedge$ and $\vee$ satisfy associative law.

iv) $a\wedge(a\vee b)=a\wedge Sup\{a,b\}=Inf\{a,Sup\{a,b\}\}=a$

$a\vee(a\wedge b)=a\vee \inf\{a,b\}=\sup\{a,\inf\{a,b\}\}=a$

$\therefore$ $\wedge$ and $\vee$ satisfy absorption law.

So $L^{a}=<L;\wedge,\vee>$ is a *lattice*. ∎

b) Given that the *algebra* $L=<L;\leq>$ be a *lattice* set $a\leq b$ if and only if $a\wedge b=a$; then $L^{p}=<L;\leq>$ is a *lattice*.

i) $a=a\wedge b$ set $a\leq b$ if and only if $a=a\wedge b$. Since $\wedge$ is idempotent.

$\therefore$ $a\wedge a=a$, Implies that $a\leq a,\ a\in L$ $\therefore$ $\leq$ is reflexive.

ii) Since $\wedge$ is commutative then $a \wedge b = b \wedge a$ implies that $a < b$ and $b \leq a$.

implies that $a = b$ where $a, b \in L$.

$\therefore \leq$ is anti -symmetric.

iii) Let $a \leq b$ and $a \leq b$ then $a = a \wedge b$ and $b = b \wedge c$

$a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$,

So $a \leq c$ where $a, b, c \in L$

$\therefore \leq$ is transitive.

Hence $L = < L; \leq >$ *is* a *poset*.

Let $a, b, c \in L$ then $a \wedge b \in L$

Now $(a \wedge b) \wedge a = a \wedge (b \wedge a) = a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b$

and $(a \wedge b) \wedge b = a \wedge (b \wedge b) = a \wedge b$

So, $a \wedge b \leq a, b$

i.e. $(a \wedge b)$ is the another lower bound of *a* and *b*.

Let c be the another lower bound of *a* and *b*. $\therefore c \leq a, c \leq b$

Then $c \wedge a = c$ and $c \wedge b = c$. i.e., $c \leq a \wedge b$

$\therefore (a \wedge b)$ is greatest lower bound of *{a, b}*

$\therefore (a \wedge b) = Inf\{a, b\}$

By absorption law,

$a \wedge (a \wedge b) = a$ and $b \wedge (a \wedge b) = b$

i.e., *a* and *b* is lower bound of $a \vee b$.

Therefore $b \leq a \vee b$.

Then $a \vee b$ is an upper bound of *a* and *b*

Let c be the another upper bound of *a* and *b*, then $a \leq c$, $b \leq c$.

So, $a \vee c = (a \wedge c) \vee c = c$, $b \vee c = (b \wedge c) \vee c = c$

Thus $(a \vee b) \wedge c = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge (a \vee b \vee c)$

$$=(a\vee b)\wedge((a\vee b)\vee c)$$
$$=(a\vee b)\text{[by absorption law]}$$

i.e. $(a\vee b)\leq c$

and so $a\vee b=Sup\{a,b\}$

Hence $L^p=<L;\leq>$ is a *lattice.* ▦

Theorem 1.1.8 : The cardinal product of two *lattices* is a *lattice.*

Proof: Let L_1 and L_2 be two *lattices* then we have already proved that [Th-1.1.3] $L_1\times L_2=\{x,y:x\in L_1,y\in L_2\}$ is a *poset* under the relation $\leq$ define by. $(x_1,y_1)\leq L_1\times L_2(x_2,y_2)\Leftrightarrow x_1L_1x_2$ in $L_1, y_1\leq L_2y_2$ in L_2.

We shall show that $L_1\times L_2$ forms a *lattice*.

Let $(x_1,y_1),(x_2,y_2)\in L_1\times L_2$ be any elements. Then $x_1,x_2\in L_1$ and $y_1,y_2\in L_2$. Since L_1 and L_2 are *lattices* ,then $\{x_1,x_2\}$ and $\{y_1,y_2\}$ have sup and inf in L_1 and L_2 respectively.

Let $x_1\wedge x_2=\inf\{x_1,x_2\}$ and $y_1\wedge y_2=\inf\{y_1,y_2\}$

Then $x_1\wedge x_2\leq L_1x_1$, $x_1\wedge x_2\leq L_1x_2$, $y_1\wedge y_2\leq L_2y_1$, $y_1\wedge y_2\leq L_2y_2$

Impliesthat($x_1\wedge x_2, y_1\wedge y_2$)$\leq L_1\times L_2$ (x_1,y_1),($x_1\wedge x_2, y_1\wedge y_2$)$\leq L_1\times L_2$ (x_2,y_2). Implies that ($x_1\wedge x_2, y_1\wedge y_2$) is a lower bound of $\{(x_1,y_1),(x_2,y_2)\}$. Suppose (p,q) is any lower bound of $\{(x_1,y_1),(x_2,y_2)\}$.

then (p, q) $\leq L_1\times L_2$ (x_1,y_1) and (p, q) $\leq L_1\times L_2$ (x_2,y_2)

Implies that $p\leq L_1x_1, q\leq L_2y_1, p\leq L_1x_2, q\leq L_2y_2$

Implies that $p\leq L_1x_1, p\leq L_1x_2$, and $q\leq L_2y_1$,$q\leq L_2y_2$

Implies that p is a lower bound of $\{x_1,x_2\}$ in L.

q is a lower bound of $\{y_1,y_2\}$ in L

Implies that $p\leq L_1\,x_1\wedge x_2=\inf\{x_1,x_2\}$, q$\leq$ $L_1\,y_1\wedge y_2=\inf\{y_1,y_2\}$

Implies that $(p,q) \leq_{L_1 \times L_2} \{x_1 \wedge x_2, y_1 \wedge y_2\}$

impliesthat $(x_1 \wedge x_2, y_1 \wedge y_2)$ is greatest lower bound

of $\{(x_1, y_1), (x_2, y_2)\}$.

Similarly,we can say that $(x_1 \wedge x_2, y_1 \wedge y_2)$ is least upper bound of $\{(x_1, y_1), (x_2, y_2)\}$. Hence $L_1 \times L_2$ is a *lattice*. ■

Definition(Complete lattice): A *lattice* L is called a *complement lattice* if every nonempty subset of L has its *Sup* and *Inf* exists in L.

Example: $I(L)$ the *lattice* of all *ideals* of a *lattice* L is *complete* if $0 \in I$.

Definition(Meet semi lattice): A *poset* $<P;\leq>$ is called a *meet semi lattice* if for all $a,b \in P$, $Inf\{a,b\}$ exists. Equivalently, a nonempty set L together with a binary operation $\wedge$ is called a *meet semi lattice* if $\forall a,b,c \in L$,

(i) $a \wedge a = a$ (ii) $a \wedge b = b \wedge a$, (iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Definition(Sublattice): A nonempty subset S of a *lattice* L is called a *sublatice* of L if $a,b \in S$ implies that $a \wedge b, a \vee b \in S$. If L is any *lattice* and $a \in L$ be any element then $\{a\}$is a *sublattice* of L.

Theorem 1.1.9 : Union of two *sublattices* may not be a *sublattice*.

Proof: Consider the *lattice* $L = \{1,2,3,4,6,12\}$ of factors of 12 under divisibility.

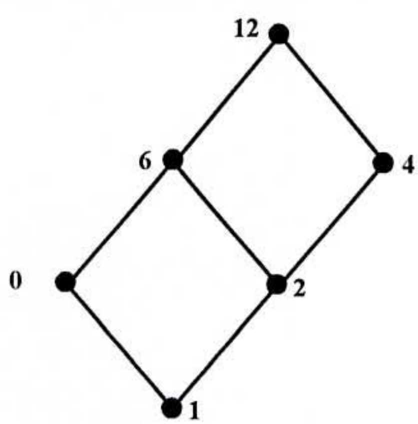


Figure1.4

Then $S = \{1,2\}$ and $T = \{2,3\}$ are *sublattices* of L.

But $S \cup T = \{1,2,3\}$ is not *sublattice* as $2,3 \in S \cup T$

but $2 \vee 3 = 6 \notin S \cup T$. ■

Theorem 1.1.10: A *lattice* L is a *chain* if and only if every non empty subset of it is a *sublattice*.

Proof: Let S be a non empty subset of a *chain* L then $a,b \in S$

implies that $a,b \in L$,

implies that a,b comparable, let $a \leq b$

then $a \wedge b = a \in S$, $a \vee b = b \in S$, therefore S is a *sublattice*.

Conversely, Let L be a *lattice* such that every nonempty subset of L is a *sublattice*. We show that L is a *chain*. Let $a,b \in L$ be any elements, than {a, b} being a non empty subset of L will be a *sublattice* of L. Thus by defination of *sublattice* $a \wedge b = \{a,b\}$ implies that $a \wedge b = a$ or $a \wedge b = b$ implies that $a \leq b$ or $a \leq b$ i.e, a, b are comparable, Hence L is a *chain*. ■

Definition(Convex sub lattice): A sudset K of a *lattice* L is called a *convex* if $a,b \in K$; $c \in L$ and $a \leq c \leq b$ implies that $c \in K$. Any interval $[a,b]$ in a lattice is a *convex sublattice*.

Now we give an example which is not *convex sublattice*.

In the *lattice* $\{1,2,3,4,6,12\}$ under divisibility $\{1,6\}$ is a *sublattice* which is *non-convex* as $2,3 \in [1,6]$, but $2,3 \notin \{1,6\}$.

Thus $[1,6] \not\subset \{1,6\}$.

Definition(Bounded lattice): A *lattice* is called finite if it contains a finite nuber of elements. A *lattice* with a largest and smallest elements is called a bounded *lattice*. Smallest element is denoted by *zero* and the largest element is denoted by *one*.

Let L_1 and L_2 be *lattices*. A mapping $\varphi : L_1 \to L_2$ is called a *meet homomorphism* if $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$. It is called a join *homomorphism* if $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$. If φ is both *meet* as well as *join homomorphism,* it is called a *homomorphism.*

Example: Let L_1 and L_2 be the *lattices* of figure 1.6(a) and 1.6(b) respectively.

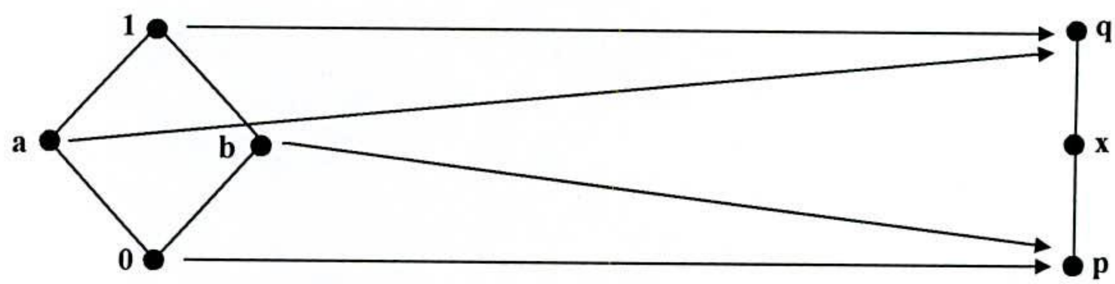


Figure 1.6 (a)

Define $\varphi : L_1 \to L_2$ such that $\varphi(0) = p, \varphi(a) = q, \varphi(b) = p, \varphi(u) = q$.

Then φ is a *homomorphism* for

$\varphi(a \wedge b) = \varphi(0) = p, \varphi(a) \wedge \varphi(b) = q \wedge p = p$

implies that $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$,

$$\varphi(0 \vee a) = \varphi(a) = q,$$

$$\varphi(0) \vee \varphi(a) = p \vee q = p$$

implies that $\varphi(0 \vee a) = \varphi(0) \vee \varphi(a)$

Similarly for all other elements.

A map $\varphi : P_1 \to P_2$ is called *isotone* if $x \leq P_1 y$ implies that $f(x) \leq P_2 f(y)$.

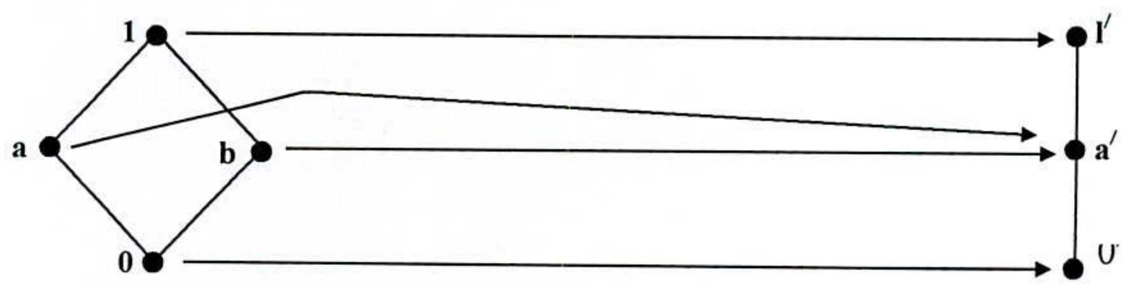


Figure 1.6(b)

Theorem 1.1.11: The *algebra* $<L;\wedge,\vee>$ is a *lattice* if and only if $<L;\wedge>$ and $<L;\vee>$ *semi-lattices* and $a = a \wedge b$ is equivalent to $b = a \vee b$.

Proof : Let $\wedge$ and $\vee$ are two binary relations on L. Since $<L;\vee>$ is a *lattice* then $\wedge$ and $\vee$ satisfy the following conditions : For all $a,b,c \in L$, $a \wedge a = a, a \vee a = a$; $a \wedge b = b \wedge a$ and $<L;\vee>$ are I. Let $a = a \wedge b$ then $a \vee b = (a \wedge b) \vee b = b$,

Conversely, let $<L;\wedge>$ and $<L;\vee>$ are *semi-lattices* then the above three conditions hold. So we need only to show the absorption identities hold in L. $a \wedge (a \vee b) = a \wedge b = a$ and $a \vee (a \wedge b) = a \vee a = a$, so $<L;\wedge,\vee>$ is a *lattice*. ■

2. Ideals of a lattice.

Definition(Ideal): A sub *lattice* I of a *lattice* L is called an *ideal* of L if, $i \in I$ and $a \in L$ implies that $a \wedge i \in I$

Equivalently,

A non empty subset I of a *lattice* L is an *ideal* if

(i) $a, b \in I,\ a \vee b \in I$

(ii) $a \in I$ and $l \in L$ implies that $a \wedge l \in I$

Let $L = \{1,2,3,5,6,10,15,30\}$ be a *lattice* of factors of 30 under divisibility.

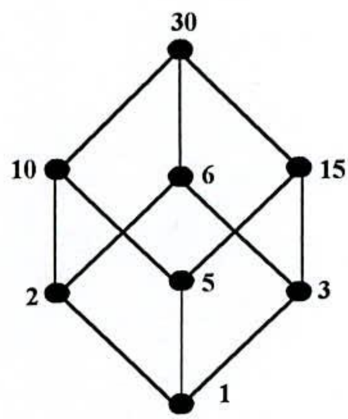


Figure 1.7

Then $\{1\},\{1,2\},\{1,3\},\{1,5\},\{1,2,5,10\},\{1,3,5,15\},\{1,2,3,6\},\{1,2,3,5,6,10,15\}$ are all the ideals of L.

Theorem: 1.2.1: Intersection of two *ideals* is an *ideal.*

Proof: Let I_1 and I_2 are two *ideals* of a *lattice* L. Since I_1, I_2 are non empty, there exists some $a \in I_1,\ b \in I_2$. Now $a \in I_1,\ b \in I_2 \subseteq L$ implies that $a \wedge b \in I_1$. Similarly $a \wedge b \in I_2$. Thus $I_1 \cap I_2 \neq \phi$.

Let $x, y \in I_1 \cap I_2$ be any elements,

implies that $x, y \in I_1$ and $x, y \in I_2$

implies that $x \vee y \in I_1$ and $x \vee y \in I_2$ as I_1, I_2, are *ideals*, So, $x \vee y \in I_1 \cap I_2$. Again if $x \in I_1 \cap I_2$ and $l \in L$ be any elements then $x \in I_1, x \in I_2, l \in L$ implies that $x \wedge l \in I_1$ and $x \wedge l \in I_2$

implies that $x \wedge l \in I_1 \cap I_2$.

Hence $I_1 \cap I_2$ is an ideal. ■

Theorem 1.2.2: Union of two ideals is an ideal if and only if one of them is contained in the other.

Proof: Let I_1, I_2 be two *ideals* of a *lattice* L such that either

$I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. We have to show that $I_1 \cup I_2$ is an *ideal.*

Since $I_1 \neq \phi, I_2 \neq \phi$ then $I_1 \cup I_2 \neq \phi$ (as I_1, I_2 are two ideals).

Let $I_1 \subseteq I_2$ then $I_1 \cup I_2 = I_2$. If $I_2 \subseteq I_1$ then $I_1 \cup I_2 = I_1$.

In this case $I_1 \cup I_2$ is an *Ideal.*

Conversely, let I_1 and I_2 be two *ideals* of L and $I_1 \not\subset I_2$ and $I_2 \not\subset I_1$, such that $I_1 \cup I_2$ is an *ideal*. As $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$ there exists $x \in I_1, x \in I_2$ and $y \in I_1, y \in I_2$. Now $x, y \in I_1 \cup I_2$ implies that $x \vee y \in I_1 \cup I_2$ implies that $x \vee y \in I_1$ or $x \vee y \in I_2$ if $x \vee y \in I_1$ then $x \leq x \vee y$, $y \leq x \vee y$ implies that $x, y \in I_1$

which is contradiction.

If $x \vee y \in I_2$ then $x \leq x \vee y$, $y \leq x \vee y$ implies that $x, y \in I_2$,

which is contradiction.

Hence $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. ■

Theorem 1.2.3: A nonempty subset I of a *lattice* L is an *ideal* if and only if

(i) $a, b \in I$ implies that $a \vee b \in I$

(ii) $a \in I, x \leq a$ implies that $x \in I$.

Proof : Let I be an ideal of a *lattice* L. By definition of *ideal* given condition $a \wedge l \in I$. Hence I is an *ideal.*

(i) is satisfied. Let $a \in I, x \leq a$ then $x = a \wedge x \in I$.

Conversely, we need show that $a \in I, l \in L$ implies that $a \wedge l \in I$.

since $a \wedge l \leq a$ and $a \in I$. By given condition $a \wedge l \in I$.

Hence I is an *ideal*. ■

Theorem 1.2.4: The set of all *ideals I(L)* of a *lattice* L forms a *Lattice* under '$\subseteq$' relation.

Proof: Let *I(L),* be the set of all *ideals* of L. We shall show that $< I(L); \subseteq >$ is a *lattice*. Now as $L \in I(L)$ then $I(L) \neq \phi$.

First we show $< I(L); \subseteq >$ is a *poset*.

Reflexivity : $I_1 \subseteq I, \ \forall \ I \in I(L)$

Anti-symmetry: Let $I_1, I_2 \in I(L)$ such that $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$

Implies that $I_1 = I_2$.

Transitivity: Let $I_1, I_2, I_3 \in I(L)$ and $I_1 \subseteq I_2 \subseteq I_3$ implics that $I_1 \subseteq I_3$.

Hence $< I(L); \subseteq >$ is a *poset*.

Again let $I_1, I_2 \in I(L)$ then $I_1 \wedge I_2 = I_1 \cap I_2 \in I(L)$.

Therefore $Inf\{I_1, I_2\} = I_1 \wedge I_2 \in I(L)$.

Now we claim that $I_1 \vee I_2 = \{x \in L / x \leq i_1 \vee i_2\}$ for some $i_1 \in I_1, i_2 \in I_2$

To prove this, let $x, y \in R.H.S$ then $x \leq i_1 \wedge i_2$ for some $i_1 \in I_1$, $i_2 \in I_2$ and $y \leq j_1 \vee j_2$ for some $j_1 \in I_1$, $j_2 \in I_2$

So $x \vee y \leq (i_1 \vee i_2) \vee (j_1 \vee j_2) = (i_1 \vee j_1) \vee (i_2 \vee j_2)$

(where $i_1 \vee j_1 \in I_1, i_2 \vee j_2 \in I_2$,)

Which implies $x \vee y \in$ R.H.S. If $x \in$R.H.S and $t \in L$ with $t \leq x$ then $x \leq i_1 \vee i_2$ for some $i_1 \in I_1$, $i_2 \in I_2$. So $t \leq i_1 \vee i_2$ implies $t \in$R.H.S. Therefore R.H.S is an *ideal*. Obviously this contains both I_1 and I_2. Suppose K is an *ideal* containing both I_1 and I_2, Let $x \in$R.H.S then $x \leq i_1 \vee i_2$ for some $i_1 \in I_1$, $i_2 \in I_2$, Since K is an ideal containing I_1 and

I_2. So $i_1 \vee i_2 \in K$ and $x \in K$ i.e., R.H.S $\leq$ K i.e., R.H.S is the smallest *ideals*.Therefore R.H.S = $I_1 \vee I_2$ and so $I(L)$ is a *lattice*. i.e., *Sup* $\{I_1, I_2\}$ = $I_1 \vee I_2$. Hence $< I(L);\subseteq >$ is a *lattice*. ∎

Definition (dual ideal): A nonempty subset D of a *lattice* L is called *dual ideal* of L if

(i) $a,b \in D$ implies that $a \wedge b \in D$

(ii) $d \in D, a \in L$ implies that. $d \vee a \in D$.

Let I = *{1, 2, 5, 10}* be the *lattice* under divisibility. Then *{10}*, *{5,10}*, *{2,10}* are all dual ideals of lattice L.

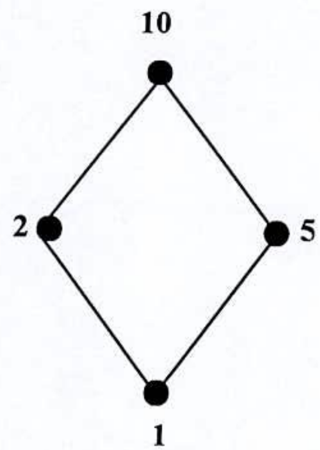


Figure 1.8

An *ideal* I of L is proper if $I \neq L$

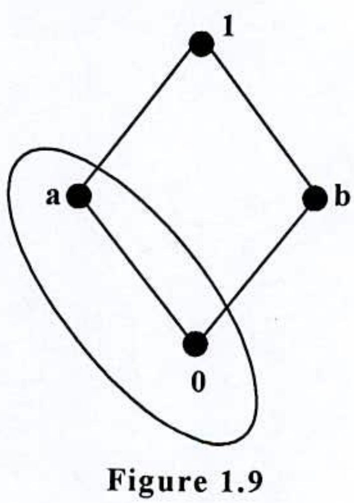


Figure 1.9

A proper *ideal* P of L is called a *prime ideal* if for any x, $y \in L$ and $x \wedge y \in P$ implies either $x \in P$ or $y \in P$. Let $L=\{ 1,2,3,4,6,12 \}$ factors of 12 under divisibility forms a *lattice* then $\{ 1,2,4 \}$ be a *prime ideal* of L.

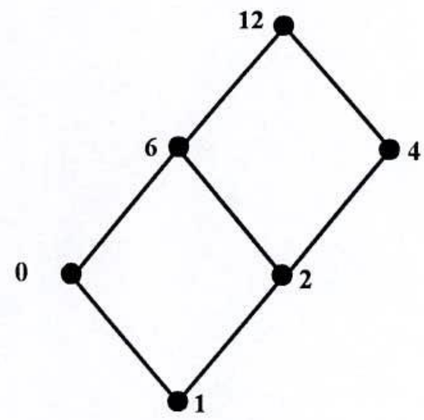


Figure 1.9

Theorem 1.2.5: Every *ideal* of a *lattice* L is *prime* if L is chain.

Proof: Let $a, b \in L \therefore a \wedge b \in L$. Consider $(a \wedge b)$ by hypothesis $I = (a \wedge b)$ is *prime* implies that either $a = a \wedge b$ or $b = a \wedge b$ implies that either $a \leq b$ or $b \leq a$ .Hence L is *chain*.

Conversely, Let L be a *chain* and I be an *ideal* of L.Suppose $a \wedge b \in P$, since L is *chain*, either $a \leq b$ or $b \leq a$ implies that $a \in I$ or $b \in I$, therefore I is *prime*. ■

CHAPTER TWO

CONGRUENCES OF A LATTICE

1. Congruence and Distributive lattices

Introduction: Congruence of lattices, Distributive lattices, Modular lattices and Boolean algebras has been studied by several authors including Katrinak [10], H. Lakser [13], A. S. A. Noor & M. A. Latif [23], W. H. Cornish [4], A. Davey [6], G. Gratzer [7] and Vijay K. Khanna [18]. In this chapter, we discuss congruence of lattices, distributive lattices, modular lattices, complemented lattices and Boolean algebras which are basic concept of this thesis.

Definition (Congruence): An equivalence relation Θ (that is, a *reflexive symmetric*, and *transitive* binary relation) on a *lattice* L is called a *congruence* relation of L if and only if $a_0 \equiv b_0(\Theta)$ and $a_1 \equiv b_1(\Theta)$ imply that $a_0 \wedge a_1 = b_0 \wedge b_1(\Theta)$ and $a_0 \vee a_1 \equiv b_0 \vee b_1(\Theta)$

Lemma.2.1.1: Let Θ be a *congruence* relation of L. Then for every $a \in L$, $[a]\Theta$ is a *convex sub lattice.*

Proof: Let $x,\ y \in [a]\Theta$; then $x \equiv a(\Theta)$ and $y \equiv a(\Theta)$.

Therefore $x \wedge y \equiv a \wedge a = a\ (\Theta)$ and $x \vee y \equiv a \vee a = a(\Theta)$, proving that $[a]\Theta$ is a *sub lattice*. If $x \leq t \leq y$ and $x, y \in [a]\Theta$ then $x \equiv a(\Theta)$ and $y \equiv a\ (\Theta)$. Therefore, $t = t \wedge y = t \wedge a(\Theta)$

and $t = t \vee x \equiv (t \wedge a) \vee x \equiv (t \wedge a) \vee a = a(\Theta)$,

Hence $[a]\Theta$ is *convex*. ■

Sometimes a long computation is required to prove that a given binary relation is a *congruence* relation. Such computations are often facilitated by the following lemma (G. Gratzer and E. T. Schmidt [1958e] and F. Maeda [1958]):

Lemma.2.1.2: *A reflexive binary relation* Θ *on a lattice L is a congruence relation if and only if the following three properties are satisfied; forall* $x, y, z, t \in L$;

(i) $x \equiv y(\Theta)$ *iff* $x \wedge y \equiv x \vee y\ (\Theta)$

(ii) $x \leq y \leq z$, $x \equiv y$ *and* $y \equiv z(\Theta)$ *imply that* $x \equiv z(\Theta)$.

(iii) $x \leq y$ *and* $x \equiv y(\Theta)$ *imply that* $x \wedge t \equiv y \wedge t(\Theta)$ *and* $x \vee t \equiv y \vee t(\Theta)$.

Proof: The "only if" part being trivial, assume now that a symmetric and reflexive binary relation Θ satisfies conditions (i) - (iii).Let $b, c \in [a, d]$ and $a \equiv d(\Theta)$, we claim that $b \equiv c(\Theta)$. Indeed $a \equiv d(\Theta)$ and $a \leq d$ by (iii) imply that $b \wedge c = a \vee (b \wedge c) \equiv d \vee (b \wedge c) = d(\Theta)$. Now $b \wedge c \leq d$ and (iii) imply that $b \wedge c = (b \wedge c) \wedge (b \vee c) \equiv d \wedge (b \vee c) = b \vee c(\Theta)$;

Thus by (i), $b \equiv c(\Theta)$.

To prove that Θ is transitive, let $x \equiv y(\Theta)$ and $y \equiv z(\Theta)$.

Then by (i), $x \wedge y \equiv x \vee y\ (\Theta)$ and

by (iii), $y \vee z = (y \vee z) \vee (y \wedge x) \equiv (y \vee z) \vee (y \vee x) = x \vee y \vee z\ (\Theta)$,

and similarly, $x \wedge y \wedge z \equiv y \wedge z(\Theta)$.

Therefore $x \wedge y \wedge z \equiv y \wedge z \equiv y \vee z \equiv x \vee y \vee z\ (\Theta)$

and $x \wedge y \wedge z \leq y \wedge z \leq y \vee z \leq x \vee y \vee z$.Thus applying (ii) twice, we get $x \wedge y \wedge z \equiv x \vee y \vee z(\Theta)$. Now we apply the statement of the previous paragraph with $a = x \wedge y \wedge z, b = x, c = z, d = x \vee y \vee z$ to conclude that $x \equiv z(\Theta)$.

Let $x \equiv y(\Theta)$; we claim that $x \vee t \equiv y \vee t(\Theta)$.

Indeed, $x \wedge y \equiv x \vee y(\Theta)$ by (i); thus by (iii), $(x \wedge y) \vee t \equiv x \vee y \vee t(\Theta)$

Since $x \vee t, y \vee t \in [(x \wedge y) \vee t, x \vee y \vee t]$, we conclude that $x \vee t \equiv y \vee t\,(\Theta)$.

To prove the substitution Property for $\vee$, let $x_0 \equiv y_0(\Theta)$ and $x_1 \equiv y_1(\Theta)$.

Then $x_0 \vee x_1 \equiv x_0 \vee y_1 \equiv y_0 \vee y_1\,(\Theta)$,

Implying that $x_0 \vee x_1 \equiv y_0 \vee y_1(\Theta)$, since Θ is transitive .

The substitution property for $\wedge$ is similarly proved. ■

Lemma 2.1.3: C(L) is a *lattice* . For $\Theta, \Phi \in C(L)$, $\Theta \wedge \Phi = \Theta \cap \Phi$. The join $\Theta \vee \Phi$ can be described as follows:

$x \equiv y(\Theta \vee \Phi)$ if and only if there is a sequence $z_0 = x \wedge y$, $z_1, \ldots\ldots\ldots, z_{n-1} = x \vee y$ of elements of L such that $z_0 \leq z_1 \leq \ldots\ldots\ldots \leq z_{n-1}$ and for each i, $0 \leq i \leq n-1, z_i \equiv z_{i+1}(\Theta)$ or $z_i \equiv z_{i+1}(\Phi)$.

Proof: $\Theta \wedge \Phi = \Theta \cap \Phi$ is obvious. To prove the statement for the join ,let Ψ be the binary relation described in this theorem . Then $\Theta \subseteq \Psi$ and $\Phi \subseteq \Psi$ are obvious. If Γ is a congruence relation $\Theta \subseteq \Gamma$, $\Phi \subseteq \Gamma$ and $x \equiv y(\psi)$ and $x \equiv y(\psi)$,then for each i, either $Z_i \equiv Z_{i+1}(\Theta), z_i \equiv z_{i+1}(\Gamma)$ .By the transitivity of $\Gamma, x \wedge y \equiv x \vee y(\Gamma)$; thus $x \equiv y(\Gamma)$. Therefore, $\psi \subseteq \Gamma$. this shows that if Ψ is a *congruence* relation , then $\Psi = \Theta \vee \Phi$. Ψ is obviously reflexive and satisfies Lemma 2.1.2. If $x \leq y \leq z, x \equiv y(\Psi)$ and $y \equiv z(\Psi)$ then $x \equiv z(\Psi)$ is established by putting together the sequences showing $x \equiv y(\Psi)$ and $y \equiv z(\Psi)$; this verifies Lemma 2.1.2(ii). To show lemma 2.1.2(iii), Let $x \equiv y(\Psi), x \leq y$ with $z_0, \ldots\ldots\ldots\ldots z_{i-1}$ establishing this, and $t \in L$ .Then $x \wedge t \equiv y \wedge t(\Psi)$ and $x \vee t \equiv y \vee t(\Psi)$ can be shown with the sequences $z_i \wedge t, 0 \le i < n, z_i \vee t, 0 \le i < n$, respectively. Thus the hypotheses of Lemma 2.1.2 hold for Ψ and we conclude that Ψ is a *congruence* relation. *Homomorphism* and *congruence* relations express two sides of the same phenomenon. To establish this fact we first define *quotient lattices* (also called *factor lattices*). Let L be a *lattice* and let Ψ be a *congruence* relation on L. Let L/Θ denote the set of blocks of the Partition of L induced by Θ, that is $L/\Theta=\{[a]\Theta : a \in L\}$.

set $$[a]\Theta \wedge [b]\Theta = [a \wedge b]\Theta$$

and $$[a]\Theta \vee [b]\Theta = [a \vee b]\Theta .$$

This defines $\wedge$ and $\vee$ on L/Θ. Indeed, if $[a]\Theta = [a_1]\Theta$ and $[b]\Theta = [b_1]\Theta$, then $a \equiv a_1(\Theta)$ and $b \equiv b_1(\Theta)$;

therefore , $a \wedge b \equiv a_1 \wedge b_1(\Theta)$, that is $[a \wedge b](\Theta) = [a_1 \wedge b_1]\Theta$. Thus $\wedge$ and (dually) $\vee$ are well defined on L/Θ .The *lattice* axioms are easily verified. The lattice L/Θ is the *quotient lattice* of L modulo Θ.

Example: the *lattice L* and a *congruence sub lattice S* of *L* that cannot be represented as [a] Θ for any *congruence* relation Θ of L.

Consider the *lattice*

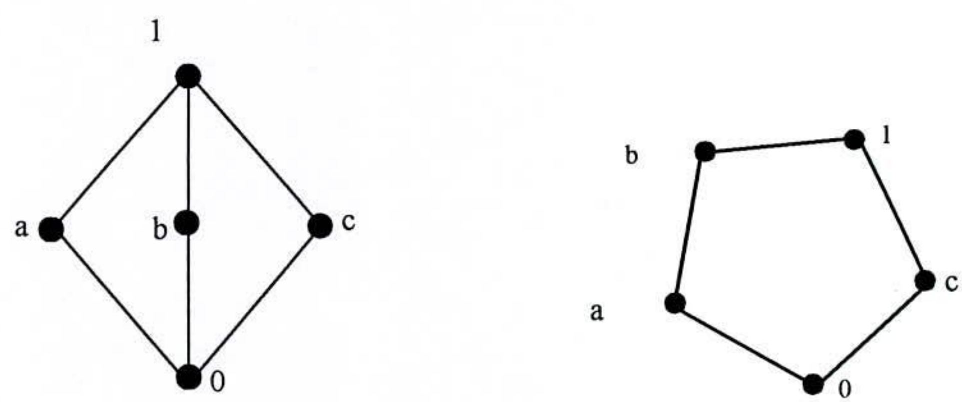


Figure 2.1

Consider the *convex sub lattice* $\{0, a\}$.

Now if $0 \equiv [a]\Theta$ for some *congruence* Θ

then $c \vee o \equiv c \vee a$ *or,* $c \vee [a]\Theta$

and $c \wedge b = c \wedge b\Theta$ or $o \equiv b\Theta$. This implies $b \in [a]\Theta$, i.e. *Convex sub lattice*. $\{o, a\}$ is not a *congruence* class for any *Congruence*. ■

Theorem 2.1.4: Construct a *lattice* that has exactly three *congruence* relations.

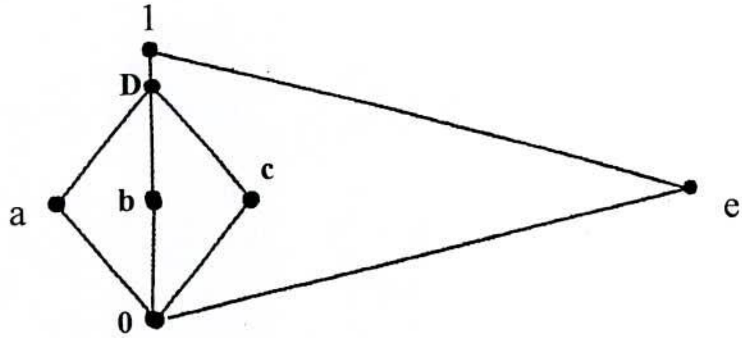


Figure-2.2

Observe that only *congruence* of above lattice are φ, 1 and Θ where $\Theta = \{o, a, b, c, 1\}, \{e, 1\}$, so above *lattice* has exactly three *congruence*.

Theorem 2.1.5: (THE HOMOMORPHISM THEOREM)

Every *homomorphic* image of a *lattice* L is isomorphic to a suitable *quotient lattice* of L. In fact, if $\varphi: L \rightarrow L_1$ is a homomorphism of L onto L_1 and if Θ is the *congruence* relation of L defined by $x \equiv y(\Theta)$ if and only if $x\varphi = y\varphi$, then $L/\Theta \cong L_1$; an isomorphism figure 1.14 is given by $\Psi: [x]\Theta \rightarrow x\varphi$, $x \in L$.

Proof: Since φ is a *homomorphism* and (Θ) is obviously a *congruence* to prove that Ψ is an isomorphism we need to check

i) Θ is well defined: Let $[x]\Theta = [y](\Theta)$. Then $x \equiv y(\Theta)$; thus $x\varphi = y\varphi$ $\Rightarrow ([x]\Theta)\Psi = ([y]\Theta)\Psi$

i,e,. Ψ is well defined.

(ii) To show that Ψ is one-one Ψ $([x](19)) = \Psi$ $(y), \Theta) \Rightarrow \varphi$ $(x) = \varphi$ (y) then $x \equiv y(\Theta)$ and so $[x](\Theta) \equiv [y](\Theta)$. i.e., Ψ is one-one.

(iii) To show that ψ is onto: Let $x \in L_1$. Since φ is onto, There is any $\in L$ with φ $(y) = x$. Thus $([y]\Theta)\, y\psi = x$. i.e., ψ is onto.

(iv) To show that ψ is a homomorphism Let $[x]\Theta$, $[y]$ $\Theta \in$ L/Θ, therefore ψ $([x]\Theta \wedge [y]\Theta)$ $=$ $\psi([x \wedge y]\Theta)$ $=$ $\varphi(x \wedge y)$ $=\varphi(x)$ $\wedge(\varphi(y)$ $=$ ψ $(|\ x|\ \Theta) \wedge \psi(|\ y|\ \Theta)$. And $\psi([x]\Theta \vee [y]\Theta)$ $= \psi([x \vee y]\Theta\} = \varphi(x \vee y) = \varphi(x) \vee (\varphi(y)) = \psi([x]\Theta) \vee \psi(|\ y|\ \Theta)$

i.e., ψ is homomorphism then the theorem is proved. ▦

Theorem: 2.1.6: L/Θ is a *lattice* under the operations $\wedge$ and $\vee$ defined by $[a]\Theta \wedge [b]\Theta) = [a \wedge b]\Theta$ and $[a]\Theta \vee [b]\Theta = [a \vee b]\Theta$.

Proof: Let L be a *lattice* and Θ be a *congruence* relation on L defined by $a_1 \equiv b_1(\Theta)$ and $a_2 \equiv b_2(\Theta)$ where $a_1 \wedge a_2 \equiv b_1 \wedge b_2(\Theta)$ and $a_1 \vee a_2 \equiv b_1 \vee b_2(\Theta)$. We also define $[a]$ $(\Theta) = \{x \in L\,/\,x \equiv a\,(\Theta)\}$.

Then L/Θ $= \{[a]$ $\Theta\,|\ a \in L\}$.

Now define $\wedge$ and $\vee$ on L by $[a]\Theta \wedge [b]\Theta = [a \wedge b]\Theta$ and $[a]\Theta \vee [b]\Theta =$ $[a \vee b]\Theta$.

Idempotency: $[a]\Theta \wedge [a]\Theta = [a \wedge a]\Theta = [a]\Theta$ and $[a]\Theta \vee [a]\Theta = [a \vee a]$ $\Theta = [a]\Theta$.

Commutativity: $[a]\Theta \wedge [\mathrm{b}]\Theta = [\mathrm{a} \wedge \mathrm{b}]\Theta = [\mathrm{b} \wedge \mathrm{a}]\Theta = [\mathrm{b}]\Theta \wedge [\mathrm{a}]\Theta$.

$[a]\Theta \vee [b]\Theta = [a \vee b]\Theta = [b \vee a]\Theta = [b]\Theta \vee [a]\Theta$.

Associativity: $[a]\Theta \wedge ([b]\Theta \wedge [c]\Theta) = [a]\Theta \wedge ([b \wedge c]\Theta)$.

$= [a \wedge (b \wedge c)]\Theta = [(a \wedge b) \wedge c]\Theta$

$= ([a \wedge b]\Theta) \wedge [c]\Theta = ([a]\Theta \wedge [b]\Theta) \wedge [c]\Theta$.

Similarly, $[a]\Theta \vee ([b]\Theta \vee [\mathrm{c}]\Theta) = ([a]\Theta \vee [b]\Theta) \vee [c]\Theta$.

Absorption: $[a]\Theta \wedge ([a]\Theta \vee [\mathrm{b}]\Theta) = [a]\Theta \wedge ([a \vee b]\Theta)$.

$= [a \wedge (a \vee b)]\Theta = [a]\Theta$

$$[a]\Theta \vee ([a]\Theta \wedge [b]\Theta) = [a]\Theta \vee ([a \wedge b]\Theta).$$

$$= [a \vee (a \wedge b)]\Theta = [a]\Theta.$$

Hence L/Θ is a lattice. ■

Definition (Modular Lattice): A *lattice L* is called *modular lattice* if all $a, b, c \in L$ with $a \geq b$ then $a \wedge (b \vee c) = b \vee (a \wedge c)$.

Definition (Distributive Lattice): A *lattice L* is called *distributive lattice* if all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Lemma.2. 1.7: The following inequalities hold in any *lattice*

i) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$

ii) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$

iii) $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$

iv) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee (x \wedge z))$

Proof: (i) In any *lattice* $x \wedge y \leq x$, $x \wedge y \leq y$, $y \leq y \vee z$

implies that $x \wedge y \leq x$, $x \wedge y \leq y \vee z$

implies that $x \wedge y$ is a lower of $\{ x, y \vee z \}$:

$\therefore$ $x \wedge y \leq x \wedge (y \vee z)$ ....(i).

Again in any *lattice* $x \wedge z \leq x$, $x \wedge z \leq z$, $z \leq y \wedge z$

implies that $x \wedge z \leq x$, $x \wedge z \leq y \vee z$

implies that $x \wedge z$ is a lower hound of $\{ x, y \vee z \}$

$\therefore x \wedge z \leq x \wedge (y \vee z)$ ....(ii) .

From (i) and (ii) we can say that $x \wedge (y \wedge z)$ is upper bound of $\{x \wedge y, x \wedge z \}$. Therefore $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$.

(ii) In any *lattice*, $x \leq x \vee y$, $y \leq x \vee y$, $y \wedge z \leq y$

implies that $x \vee y \geq x$, $x \vee y \geq y \geq y \wedge z$

implies that $x \vee y \geq x$, $x \vee y \geq y \wedge z$.

Implies that $x \vee y$ is upper bound of $\{x, y \wedge z\}$.

$\therefore x \vee y \geq x \vee (y \wedge z)$.

Implies that $x \vee (y \wedge z) \leq x \vee y$............ (iii)

Again, $x \leq x \vee z,\ z \leq x \vee z,\ y \wedge z \leq z$

implies that $x \vee z \geq x,\ x \vee z \leq z,\ z \geq y \vee z$

implies that $x \vee z \geq x,\ x \vee z \geq y \wedge z$

implies that $x \vee z$ is upper bound of $\{x, y \wedge z\}$..........(iv).

Form (iii) and (iv) we get $x \vee (y \wedge z)$ is a lower bound of $\{x \vee y, x \vee z\}$.

There fore $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.

(iii) Any lattice, $x \wedge y \leq x,\ x \leq x \vee y$

Implies that $x \wedge y \leq x \vee y$........... (v)

Again $x \wedge y \leq y,\ y \leq y \vee z$

Implies that $x \wedge y \leq y \vee z$............. (vi).

Also $x \wedge y \leq x,\ x \leq z \vee x$

Implies that $x \wedge y \leq z \vee x$............. (vii).

Form (v), (vi), (vii) we can say that

$x \wedge y$ is lower bound of $\{x \vee y, y \vee z, z \vee x\}$,

$\therefore\ x \wedge y \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$...........(A).

Again $y \wedge z \leq y,\ y \leq x \vee y$

implies that $y \wedge z \leq x \vee y$...........(viii).

Also $y \wedge z \leq z,\ z \leq y \vee z$

Implies that $y \wedge z \leq y \vee z$.......... (ix)

and $y \wedge z \leq z$, $z \leq z \vee x$.

$\therefore\ y \wedge z \leq z \vee x$..............(x).

From (viii), (ix) and (x) we can say that

$y \wedge z$ is lower hound of $\{x \vee y, y \vee z, z \vee x\}$.

$\therefore y \wedge z \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$............(B).

Similarly, $z \wedge x \leq (x \vee y) \wedge (y \vee Z) \wedge (z \vee x)$............(C).

From (A), (B) and (C) we can say that

$(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ is upper bound of $\{x \wedge y, y \wedge z, z \wedge x\}$.

$\therefore\ (x \vee y) \wedge (y \vee z) \wedge (z \vee x) \leq (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

iv) Since $x \wedge y \leq x \wedge z \leq x$, So we get $(x \wedge y) \vee (x \wedge z) \leq x$................(xi),

And $x \wedge y \leq y \leq y \vee (x \wedge z)$ and $x \wedge z \leq y \vee (x \wedge z)$

$\therefore (x \wedge y) \vee (x \wedge z) \leq y \vee (x \wedge z)$ .... (xii)

From (xi) and (xii) we get $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee (x \wedge z)$. ■

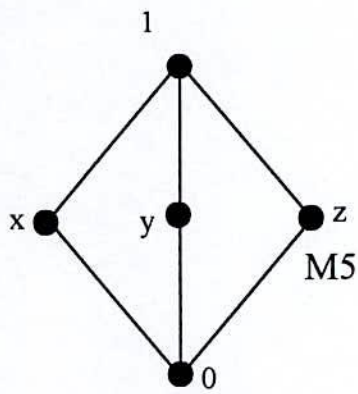
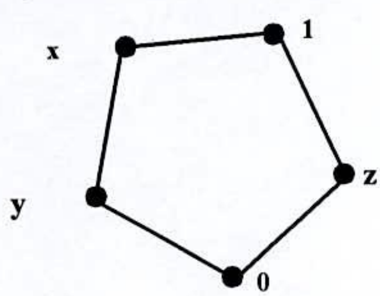


Figure 2.3

Example: The *pentagonal lattice* is not *modular*.



R_5

Figure-2.4

Here, $x \wedge (y \vee z) = x \wedge 1 = x$

And $y \vee (x \wedge z) = y \vee 0 = y$

Since $x \wedge (y \vee z) \neq y \vee (x \wedge z)$

Hence the *pentagonal lattice is not modular.* ■

Theorem.2.1.8: Two *lattices* L_1 and L_2 are *modular* if $L_1 \times L_2$ is *Modular*

Proof: Let L_1 and L_2 be *modular*. Let (x_1,y_1), (x_2,y_2), $(x_3, y_3) \in L_1 \times L_2$ be three elements with $(x_1, y_1) \geq (x_3,y_3)$.

Then $x_1, x_2, x_3 \in L_1$, $x_1 \geq x_3$, $y_1, y_2, y_3 \in L_2$, $y_1 \geq y_3$

and since L_1 and L_2 are *Modular*.

We get $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee x_3$, $y_1 \wedge (y_2 \vee y_3) = (y_1 \wedge y_2) \vee y_3$.

Thus $(x_1, y_1) \wedge [(x_2,y_2) \vee (x_3,y_3)]$

$$= (x_1, y_1) \wedge [x_2 \vee x_3, y_2 \vee y_3]$$

$$= (x_1 \wedge (x_2 \vee x_3)\ y_1 \wedge (y_2 \vee y_3))$$

$$= ((x_1 \wedge x_2) \vee x_3, (y_1 \wedge y_2) \vee y_3)$$

$$= ((x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_3, y_3))$$

$$= [(x_1, y_1) \wedge (x_2, y_2)] \vee (x_3, y_3)$$

Hence $L_1 \times L_2$ is *modular*.

Conversely, Let $L_1 \times L_2$ be *modular*. Let x_1, x_2, $x_3 \in L_1$, $x \geq x_3$ and $y_1, y_2, y_3 \in L_2, y_1 \geq y_3$ then $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in L_1 \times L_2$ and $(x_1,y_1) \geq (x_3,y_3)$. Since $L_1 \times L_2$ is *modular*.

We find $(x_1,y_1) \wedge [(x_2, y_2) \vee (x_3, y_3)] = [(x_1, y_1) \wedge (x_2, y_2)] \vee (x_3, y_3)]$

Or, $(x_1, y_1) \wedge [(x_2 \vee x_3), (y_2, \vee y_3)] = [(x_1 \wedge x_2),(y_1 \wedge y2) \vee (x_3,y_3)]$

Or, $(x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) = ((x_1 \wedge x_2) \vee x_3,(y_1 \wedge y_2) \vee y_3))$

Or, $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee x_3\ y_1 \wedge (y_2 \vee y_3) = (y_1 \wedge y_2) \vee y_3$

$\therefore L_1$ and L_2 are *modular*. ■

Theorem.2.1.9: If a, b are any elements of a *modular lattice* then $[a \wedge b, a] \cong [b, a \vee b]$

Proof: We know an interval in a *lattice* is a *sub lattice*. We establish the isomorphism define a map ψ: $[a \wedge b,\ a] \rightarrow [b,\ a \vee b]$ such that $\psi(x) = x \vee b,\ x \in [a \wedge b, a]$. Then ψ is well defined as $x \in [a \wedge b, a]$

implies that $a \wedge b \leq x \leq x \leq a$

implies that $(a \wedge b) \vee b \leq x \vee b \leq a \vee b$

implies that $b \leq x \vee b \leq a \vee b$

implies that $x \vee b \in [b,\ a \vee b]$. also $x_1 = x_2$.

implies that $x_1 \vee b = x_2 \vee b$

implies that $\psi(x_1) = \psi(x_2)$,

ψ is one-one as let ψ $(x_1) = \psi(x_2)$ then $x_1 \vee b = x_1 \vee b$

implies that $a \wedge (x_1 \vee b) = a \wedge (x_2 \vee b)$

implies that $x_1 \vee (a \wedge b) = x_2 \vee (a \wedge b)$

implies that $x_1 = x_2$,

ψ is onto as let $y \in [b,\ a \vee b]$ be any element.

We show that $a \wedge y$ is the required pre-image.

$y \in [b, a \vee b]$ implies that $b \leq y \leq a \vee b$

implies that $a \wedge b \leq a \wedge y \leq a \wedge (a \vee b)$

implies that $a \wedge b \leq a \wedge y \leq a$

implies that $a \wedge y \in [a \wedge b,\ a]$.

Also, $\psi(a \wedge b) = (a \wedge y) \vee b$, so we need show $y = (a \wedge y) \vee b$

Now, $y \leq a \vee b$ implies that $y \wedge (a \vee b) = y$

Implies that $y = y \wedge (b \vee a) = b \vee (y \wedge a)$.

Hence ψ is onto.

Again, $x_1 \leq x_2$, implies that $x_1 \vee b \leq x_2 \vee b$

Implies that ψ $(x_1) \leq \psi(x_2)$

Now, $x_1 \vee b \leq x_2 \vee b$ Implies that $a \wedge (x_1 \vee b) \leq a \wedge (x_2 \vee b)$

Implies that $x_1 \vee (a \wedge b) \leq x_2 \vee (a \wedge b)$

Implies that $x_1 \leq x_2$.

Thus $x_1 \leq x_2$

Implies that $\psi(x_1) \leq \psi(x_2)$.

Hence ψ is an *isomorphism.* ■

Theorem.2.1.10: A *lattice L* is *modular* if it does not contain a *Sub lattice isomorphic* to *pentagonal lattice.*

Proof: Suppose a *lattice L* is *modular*, then its every *sub lattice* is also *modular*; Since N ={0, a, b, c, 1}

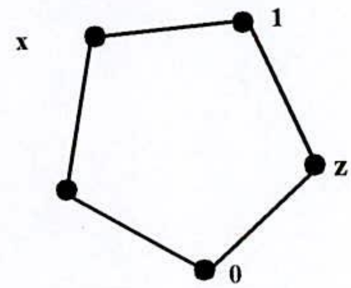


Figure 2.5

Where $b \leq a$, $a \wedge b = a \wedge c = b \wedge c = 0$ and $a \vee b = a \vee c = b \vee c = 1$ is not *Modular* So, L does not contain any *sub lattice isomorphic* to N

To prove the converse, let L is not *modular*, then there exists elements x,y,z∈L with $z \leq x$ such that $x \wedge (y \vee z) \neq (x \wedge y) \vee z$. But $x \wedge (y \vee z) > (x \wedge y) \vee z$. Then the elements $x \wedge y$, y, $(x \wedge y) \vee z$, $x \wedge (y \vee z)$, $y \vee z$ form a *lattice*

Diagram as follows:

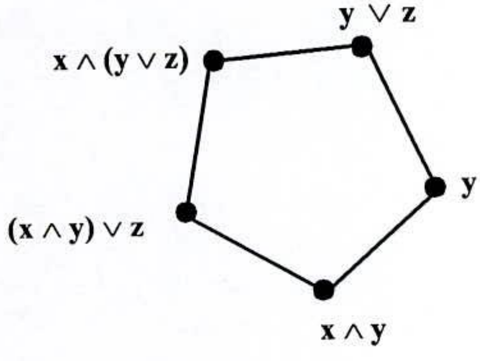


Figure-2.6

Observe that $(x \wedge (y \vee z)) \wedge y = x \wedge [(y \vee z) \wedge y] = x \wedge y$

And so, $y \wedge ((x \wedge y) \vee z) = x \wedge y$

Again, $y \vee ((x \wedge y) \vee z) = [y \vee (y \wedge x)] \vee z = y \vee z$

And so, $y \vee (x \wedge (y \vee z)) = y \vee z$. If $y = x \wedge y$ then we have $y \leq x$

And so, $y \vee z. = (x \wedge y) \vee z$,

Also, $y \leq x$ and $z \leq x$ implies that $y \vee z \leq x$ and so $x \wedge (y \vee z) = y \vee z$,

So we have $x \wedge (y \vee z) = (x \wedge y) \vee z$ which gives a contradiction. Since L is not *modular*. So $y \neq x \wedge y$. Similarly, we can show that

$(x \wedge y) \vee z \neq x \wedge y$, $y \neq y \vee z$, $x \wedge (y \vee z) \neq y \vee z$

Hence the five elements are distinct and they form a *sub lattice* of L. which is isomorphic to N_5. Hence *L* is *modular*.

A *lattice* $< L; \wedge, \vee >$ is called *distributive lattice* if for all x, y, $z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, dually, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ of course every *distributive lattice* is *modular*. ■

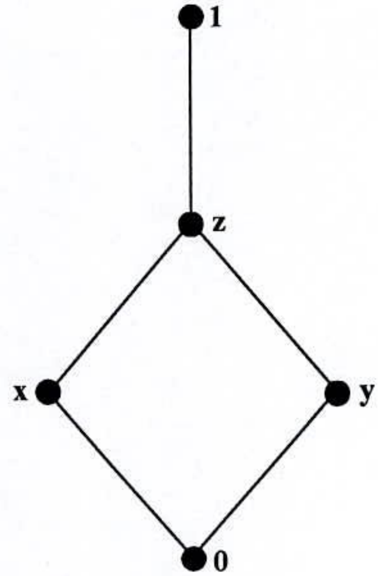


Figure -2.7

Theorem: 2.1.11: Two *lattices* L_1 *and* L_2 *are distributive* if $L_1 \times L_2$ is *distributive*.

Proof: Let L_1, and L_2 are *distributive*, let (x_1,y_1), (x_2,y_2),(x_3,y_3) be any three elements of $L_1 \times L_2$ then $x_1, x_2, x_3 \in L_1$, $y_1, y_2, y_3, \in L_2$.

Now, $(x_1, y_1)\ [(x_2, y_2) \vee (x_3, y_3)] = (x_1,y_1) \wedge (x_2 \vee x_3, y_2 \vee y_3)$

$$= (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3))$$

$$= ((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2) \vee (y_1 \wedge y_3))$$

$$= [(x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3)]$$

$$= [(x_1, y_1) \wedge (x_2, y_2)] \vee [(x_1, y_1) \wedge (x_3, y_3)]$$

Shows $L_1 \times L_2$ *is distributive.*

Conversely, *Let* $L_1, \times L_2$ be *distributive.*

let $x_1, x_2, x_3. \in L_1$ and $y_1, y_2\ y_3 \in L_2$ be any elements, then

$(x_1, y_1), (x_2, y_2), (x_3,y_3) \in L_1, \times L_2$ and as $L_1 \times L_2$ is *distributive.*

$(x_1,y_2) \wedge [(x_2, y_2) \vee (x_3, y_3)]$

$=[(x_1, y_1) \wedge (x_2,y_2) \vee [(x_1,y_1) \wedge (x_3,y_3)]$

i.e., $(x_1, y_1) \wedge (x_2 \vee x_3, y_2 \vee y_3) = (x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1)$

or, $((x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3))$

$=((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2) \vee (y_1 \wedge y_3))$

Which gives, $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$

$y_1 \wedge (y_2 \vee y_3) = (y_1 \wedge y_2) \vee (y_1 \wedge y_3)$

implies that L_1 and L_2 are *distributive* . ■

Theorem: 2.1.12: A *distributive lattice* is always *modular* but Converse is not true.

Proof: Suppose L is *distributive*, let $a,b,c \in L$ with $c \leq a$,

then $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = (a \wedge b) \vee c$, Thus L is *modular*.

Conversely, consider the *lattice*

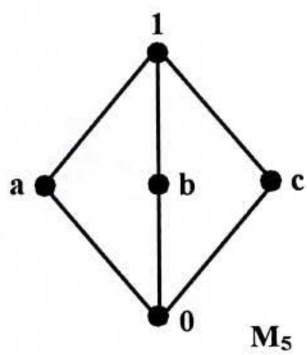


Figure -2.8

It is says to check that M_5 is *modular*: $a \wedge (b \vee c) = a \wedge 1 = a$,

$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$ i.e., $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$.

Therefore L is not *distributive*. ■

Theorem 2.1.13: Let L be a *distributive lattice*, I be an *ideal*. Let D be a *dual ideal* of L and let $I \cap D = \Phi$ Then there exists a *prime ideal* P of L such that $P \supseteq I$.

Proof: Let X be the set of all ideals of L containing I that are disjoint form D. Clearly X is non empty as $I \in X$.

Let C be a chain in X and Let $M = U\ \{X|\ X \in C\}$. If $a\ , b \in M$ then $a \wedge X,\ b \wedge Y$, for some $X, Y \in C$. Since C is chain either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$ then $a,\ b \in Y$. Since Y is an ideal $a \vee b \in Y \subseteq M$.

Also if $a \in M$ and $b \leq a$, then $a \in X$ for some $X \in C$.

Since X is an ideal, so $b \in X \subseteq M$. Therefore M is an ideal contain I.

Obviously $M \cap D = \Phi$. Hence $M \in C$,

so by zorn's Lemma, X has a maximal element, say P,

We claim that p is a prime ideal.

If P is not prime, then there exists $a, b \in L$ with $a, b \in P$ such that $a \wedge b \in P$.

By the maximality of P $((a] \vee P) \cap D \neq \varphi$, $((b] \vee P) \cap D \neq \varphi$

Let $p \vee a \in D$ and $q \vee b \in D$ for some $p, q \in P$

Then $x = (p \vee q) \wedge (a \vee b) = (p \wedge q) \vee (a \wedge q) \vee (p \wedge b)(a \wedge b) \in P$

Which implies that $x \in P \cap D$ which gives a contradiction.

Therefore φ must be a *prime ideal.* ■

Theorem 2.1.14: *Dual of a distributive lattice is distributive.*

Proof: Let $< L;\ \wedge, \vee >$ be *distributive* and $< L;\ \wedge, \vee >$ be its *dual.*

Now for any $a,\ b,\ c \in L = L$, $a \wedge^{d} \wedge (b \vee^{d} c) = a(b \wedge c) = (a \vee b)\ \ (a \vee c) = (a \wedge^{d} b) \vee^{d} (a \wedge^{d} c)$ as L is *distributive* .

This implies that L is also *distributive*. ■

2. Complemented and Boolean lattices.

Definition (Complemented Lattice): In a bounded *lattice L*, *a* is a complement of b if $a \wedge b = 0$ and $a \vee b = 1$. A *complemented lattice* is a bounded *lattice* in which every element has a complement.

Now, let [a, b] be an interval in a *lattice L*. Let $x \in [a,b]$ be any element. If there exists $y \in L$ such that $x \wedge y = a, x \vee y = b$. We say y is a complement of x relative to $[a, b]$ or y is relative complement of x in $[a, b]$.In every element x of an interval $[a, b]$ has at least one complement relative to [a, b], the interval $[a, b]$ is said to complement. Further, if every interval in a lattice is complement, the *lattice* is said to relative complemented.

Theorem 2.2.1: Two *lattices* L_1 and L_2 are relatively complemented if and only if $L_1 \times L_2$ is relatively complemented.

Proof: Let L_1 and L_2 be relatively complemented. Let $[(x_1, y_1)(x_2, y_2)]$ be any interval of $L_1 \times L_2$ and suppose (*a, b*) is any element of this interval.Then $(x_1, y_1) \leq (a,b) \leq (x_2, y_2)$ *where* $x_1, y_1, a \in L_1$ *and* $y_1 y_2, b \in L_2$.

implies that $x_1 \leq a \leq x_2$, $y_1 \leq b \leq y_2$.

implies that $a \in [x_1, x_2]$a an interval in L_1 and $b \in [y_1, y_2]$ be an interval in L_2. Since L_1, L_2 are relatively complemented, a, b have complements relative to $[x_1, x_2]$ and $[y_1, y_2]$ respectively.

Let a' and b' be these complements,

Then $a \wedge a' = x_1, a \vee a' = x_2, b \wedge b' = y_2$.

Now, $(a,b) \wedge (a',b') = (a \vee a', b \wedge b') = (x_1, x_2)$

$(a,b) \wedge (a',b') = (a \vee a', b \wedge b') = (y_1, y_2)$

i.e, (a',b') is complement of (a, b) relative to $[(x_1,y_1),\ (x_2,y_2)]$. Thus any interval in $L_1 \times L_2$ is complemented. Hence $L_1 \times L_2$ is relatively complemented.

Conversely, Let $L_1 \times L_2$ be relatively complemented, L let $[x_1,x_2]$ and $[y_1,y_2]$ be relatives in L_1 and L_2. Let $a \in [x_1,x_2]$ and $b \in [y_1,y_2]$ be any elements. Then $x_1 \le a \le x_2, y_1 \le b \in y_2$

implies that $(x_1,y_1) \le (a,b) \le (x_2,y_2)$

implies that $(a,b) \in [(x_1,y_1)\ (x_2,y_2)]$ an interval in $L_1 \times L_2$

. implies that $(a,\ b)$ has a complement, say (a',b') relative to this interval.

Thus $\quad (a,b) \wedge (a',b') = (x_1,y_1)$

$$(a,b) \vee (a',b') = (x_2,y_2)$$

implies that $(a \vee a', b \wedge b') = (x_1,y_1)$

$(a \vee a', b \wedge b') = (x_2,y_2)$ implies that $a \wedge a' = x_1, a \vee a' = x_2$

$$b \wedge b' = b_1, b \vee b' = y_2$$

implies that aa', is complement of a relative to $[x_2,y_2]$, b' is complement of b relative to $[x_2,y_2]$.

Hence L_1 and L_2 are relative complemented. ∎

Theorem 2.2.2: A complemented *modular lattice* is relatively complemented.

Proof: Let L be a complemented *modular lattice.* Let $[a,\ b]$ be any interval in L and $x \in [a,b]$ be any element, Since L is complemented, x has a complement, say x'. Then $y = a \vee (b \wedge x')$

$x \wedge x' = 0.\ x' = 1,\ a \le x \le b$.

Take $\ y = a \vee (b \wedge x')$

Then $x \wedge y = x[a \vee (b \wedge x')]$

$= a \vee (x \wedge (b \wedge x'))$ [as $x \geq a, L$ is *modular*]

$= a \vee (b \wedge x, b \wedge x')$

$= a \vee (b \wedge 0)$

$= a \vee 0$

$= a$

$x \vee y = x \vee [a \vee (b \wedge x')] = (x \vee a) \vee (b \wedge x') = x \vee (b \wedge x') = b \wedge (x \vee x') = b \wedge 1 = b$.

Hence $y = a \vee (b \wedge x')$ is relative complement of x in $[a, b]$. ■

Theorem 2.2.3: Let L be a *distributive lattice* and let $a \in L$ then the map $\varphi: x \to < x \wedge a, x \vee a >$, $x \in L$ is an embedding of L into $(a] \times [a)$: it is an isomorphism if a has a complement.

Proof: $\varphi: L \to (a] \times [a)$ is defined by $\varphi(x) = < x \wedge a, x \vee a >$

for any $x, y \in L$

$\varphi(x \wedge y) = < (x \wedge y) \wedge a, (x \wedge y) \wedge a >$

$= < (x \wedge a) \vee (y \wedge a), (x \vee a) \wedge (y \vee a) >$

$= < x \wedge a, x \vee a > \wedge < y \wedge a, y \vee a >$

$= \varphi(x) \wedge \varphi(y)$

i.e. φ is a homomorphism.

Let $\varphi(x) = \varphi(y)$, then $< x \wedge a, x \vee a > = < y \wedge a, y \vee a >$

implies that $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$

Now, $x = x \wedge (x \vee a) = x \wedge (y \vee a) = (x \wedge y) \vee (x \wedge a)$

$= (x \wedge y) \vee (y \wedge a) = y \wedge (x \vee a) = y \wedge (y \vee a) = y$

i.e. φ is one- one.

Now suppose a has a complement a'. To show on tones.

Let $< r, s > \in (a] \times [a)$,

Then $[(a' \wedge s) \vee r] \wedge a = (a' \wedge s \wedge a) \vee (r \wedge a) = 0 \vee (r \vee a) = 0 \vee (r \wedge a)$

$= r \wedge a = r$

and $[(a \wedge s) \vee r] \vee a \ \ (a \vee r \vee a) \wedge (s \vee r \vee a) = 1 \wedge (s \vee r \vee a) = s$

i.e. $< r, s >= [(a' \wedge s) \vee r] \wedge a, [(a' \wedge s) \vee a] \vee a = \varphi(a' \wedge s) \vee r$

So φ is onto and hence $L \cong (a] \times [a)$. ■

Definition (Boolean Lattice): A complemented *distributive lattice* is called a *Boolean lattice.*

Since complements are unique in a *Boolean lattice* we can regard a *Boolean lattice* as an algebra with two binary operations $\wedge$ and $\vee$ and one unary operation $'$. *Boolean lattices* so considered are called *Boolean algebras.* In other words, by a *Boolean algebra,* we mean a system $< L, \wedge, \vee, ', 0, 1 >$ where L is a non empty set with the binary operations $\wedge$ *and* $\vee$ and a unary operation $'$, and nullary operations 0, 1 is called a Boolean algebra if it satisfy the following condition:

i) $a \wedge a = a, a \vee a = a, \ \forall a \in L$

ii) $a \wedge b = b \wedge a, a \vee b = b \vee a, \ \forall a, b \in L$

iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c, a \vee (b \vee c) = (a \vee b) \vee c, \forall a, b, c \in L$

iv) $a \wedge (a \vee b) = a, a \vee (a \wedge b) = a, \ \forall a, b \in L$

v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \ \forall a, b, c \in L$

vi) There exists $0 \in L, 1 \in L$ such that $a \vee 0 = a, a \wedge 1 = a \forall a \in L$

vii) Each $a \in L, a' \in L$ such that $a \wedge a' = 0, a \vee a' = 1$

viii) $0' = 1$

ix) $1' = 0$

x) $(a \wedge b)' = a' \vee b'$

xi) $(a \vee b)' = a' \wedge b'$

Theorem 2.2.4: The infinite distributive laws hold in a complete *Boolean algebra.*

Proof: We have for *distributive lattice* $y \wedge (\vee x_i) = \vee(y \wedge x_i)$, even when there are infinitely many terms in the unions. These unions certainly exist since the *lattice* is complete.

Let $z = \vee(y \wedge x_i)$ then $y \wedge x_i \leq z$

and $x_i \leq y' \vee x_i = y' \vee (y \wedge x_i) = y' \vee z$ for each i.

Hence $\vee x_i \leq y' \vee z$ and so $y \wedge (\vee x_i) \leq y \wedge (y' \vee z) = y \wedge z \leq z$.

That is to say $y \wedge (\vee x_i) = \vee(y \wedge x_i)$.

We there fore have by anti- symmetric property the distributive law $y \wedge (\vee x_i) = \vee(y \wedge x_i)$. Its dual may be obtained in the same way. ■

An element a of a *lattice* is called join irreducible if $a = b \vee c$ implies either $a = b$ or $a = c$.

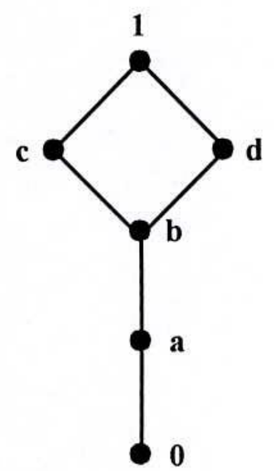


Figure 2.9

Here l is not join- irreducible but *a, b, c, d* all are join- irreducible.

Now zero join- irreducible element x which cover 0.

i.e. *x* ,*0* are called atoms.

[a,b means $b \leq a$ and if $b \leq c \leq a$ then either $b = c or a = c$]

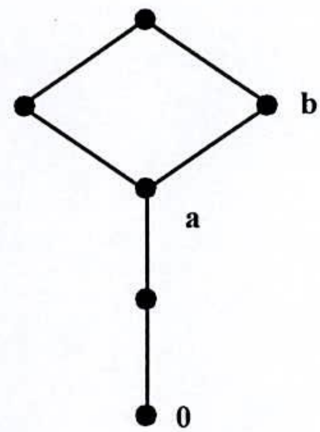


Figure-2.1o

Theorem.2.2.5: In a *Boolean lattice* $x \neq 0$ be join- irreducible if and only if x is an atom.

Proof: Let L be a *Boolean lattice* and let $x \neq 0$ be join- irreducible. We have to show that x is an atom.

Let $t \in [0,x]$ then there exists t' such that $t \wedge t' = 0, t \wedge t' = x$. Since x is join- irreducible, then either $t = x$ or $t' = x$. If $t \wedge x$ then $t' = x$ $\therefore t = t \wedge x = t \wedge t' = 0$ implies that x is an atom.

Conversely, Let x is an atom. We have to prove that x is join- irreducible. Let $a \vee b = x$, then $0 \leq a \leq x$, $0 \leq x$ implies that $0 = a$ *or* $a = x$; $0 = b$ or $b = x$ implies that x is join- irreducible. ■

CHAPTER THREE

PSEUDOCOMPLEMENTED LATTICE.

Introduction: In lattice theory there are difference classes of *lattice* knows as variety, Of course the most powerful variety. Throughout this chapter we will be concerned with another large variety known as the class of *distributive pseudocomplemented lattice*. *Pseudocomplemented lattice* have been studied by several authors [9], [10], [13], [14], [15], [16]. There are two concepts that we should be able to distinguish a *lattice* $<L;\wedge,\vee>$ in which every element has a *pseudocomplement* and an algebra, $<L;\wedge,\vee,*,0,1>$. Where $<L;\wedge,\vee,0,1>$ is a bounded *lattice* and where, for every $a \in L$, the element a^* is a *pseudocomplement* of a. We shall call the former a *pseudocomplemented lattice* and the later a *lattice* with pseudocomplementation (as an operation). In this chapter we have also studied *algebraic lattice.*

Construction of pseudocomplemented lattices.

Let L be a bounded *distributive lattice,* let $a \in L$, an element $a^* \in L$ is called a *pseudocomplement* of a in L if the following conditions hold:

$(i) a \wedge a^* = 0$, (ii) $\forall x \in L, a \wedge x = 0$ implies that $x \leq a^*$

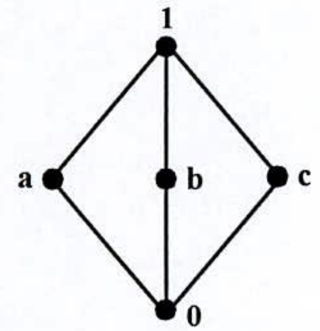


Figure 3.1

a has no pseudocomplement.

A bounded *lattice L* is called a *pseudocomplemented lattice* if its every element has a *pseudocomplement*.

Example :

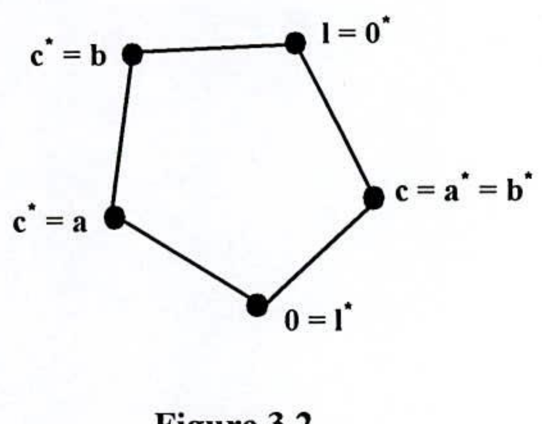


Figure 3.2

The *lattice* $L = \{0, a, b, c, 1\}$ show by the figure 3.2 is *pseudocomplemented.*

An algebra, $<L;\wedge,\vee,*,0,1>$ where $\wedge$ *and* $\vee$ are binary operation, * is a unary operation and 0, 1 are nullary operations is called a *lattice* with *pseudocomplementation* if.

i) $<L,\wedge,\vee,0,1>$ is bounded *lattice*

ii) $*$ is a unary operation i.e. $\forall a \in L$ there exists $a*$ such that $a \wedge a* = 0$ and $a \wedge x = 0$ implies that $x \wedge a* = x, \forall x \in L.$

A bounded *distributive lattice* L is called a *pseudocomplemented distributive lattice* if its every element has a *pseudocomplement*.

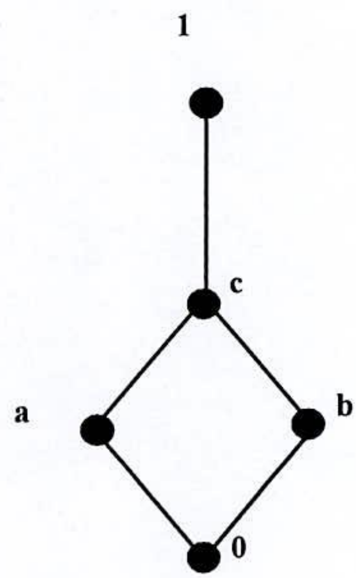


Figure – 3.3

1. Pseudocomplemented distributive lattice.

To see the difference in view point, consider the finite *distributive lattice* of figure (3.3). As a *distributive lattice* it has twenty-five *sublattice* and eight *congruences*; as a lattice with pseudocomplementation it has three subalgebras and five *congruencies*.

L as lattice:

Sub lattice: $\{0\},\{a\},\{b\},\{c\},\{1\},\{0,a\},\{0,b\},\{0,c\},\{0,1\},\{0,a,b,c\},L,$

$\{a,c\},\{a,c,1\},\{b,c\},$
$\{a,1\},\{b,1\},\{b,c,1\},\{c,1\},\{0,a,1\},\{0,b,1\},\{0,c,1\},\{0,a,c\},$

$\{0,b,c\},\{0,a,c,1\}\ \{0,b,c,1\}=25:$

L as a *lattice* with *pseudocomplementation* $\{0,1\}$, L, $\{0,c,1\}$

Congruence:

As a *lattice*:

$\omega=\{0\},\{a\},\{b\},\{c\},\{1\}$

$\tau=\{0,a,b,c,1\}$

$0=\{0,a\},\{b,c\},\{1\}$

$\varphi=\{0,a\},\{b,c,1\}$

$\psi=\{0,b\},\{a,c\},\{1\}$

$\iota=\{0,b\},\{a,c,1\}$

$\zeta=\{0,a,b,c\},\ \{1\}$

$\eta=\{c,1\},\{a\},\{b\},\{0\}$

Congruence as a *lattice* with *pseudocomplementation* $\omega,\tau,\varphi,\iota,\eta$

Theorem 3.1.1: Let L be a *pseudocomplemented distributive lattice.* $S(L)=\{a*/a\in L\}$ and $D(L)=\{a/a*=0\}$. Then for $a,b,\in L$:

$(i) a\wedge a*=0$

$(ii) a\leq b$ implies that $a*\geq b*$

$(iii) a \le a**$

$(iv) a* = a***$

$(v)(a \vee b)* = a* \wedge b*$

$(vi)(a \wedge b)** = a** \wedge b**$

(vii) $a \wedge b = 0$ iff $a** \wedge b** = 0$

$(viii)$ $a \wedge (a \wedge b)* = a \wedge b*$

(ix) $0* = 1$ and $1* = 0$

(x) $a \in S(L)$ iff $a = a**$

(xi) $a, b \in S(L)$ implies that $a \wedge b \in S(L)$

(xii) Sup $_{S(L)}$ $\{a, b\} = (a \vee b)** = (a* \wedge b*)*$

$(xiii)$ $0,1 \in S(L), 1 \in D(L)$ and $S(L) \cap D(L) = \{1\}$

(xiv) $a, b \in D(L)$ implies that $a \wedge b \in D(L)$

(xv) $a \in D(L)$ and $a \le b$ implies that $b \in D(L)$

(xvi) $a \vee a* \in D(L)$

$(xvii)$ $x \to x**$ is a meet- homomorphism of L onto $S(L)$

Proof: (i) By the definition of *pseudocomplement*, $a \wedge a* = 0. \forall a \in L$.

(ii) For $b \wedge b* = 0$ and $a \le b \Rightarrow a \wedge b* = 0$ which implies $a* \ge b*$

(iii) By the definition of *pseudocomplement* $a \wedge a* = a* \wedge a = 0$

Similarly, $a* \wedge (a*)* = 0 \Rightarrow a* \wedge a** = 0$ and $a* \wedge a = 0 \Rightarrow a* \le a**$,

$\Rightarrow a \le a**$. Hence $a \le a**$.

(iv) From (iii) we have $a \le a**$

implies that $a* \ge a**$ .... (A) [by (ii)]

Again $a* \wedge a** = 0$, i.e. $a** \wedge a* = 0$.

Similarly $a** \wedge (a**)* = 0$, implies that $a** \wedge a*** = 0$,

and $a** \wedge a* = 0$ implies that $a* \le a***$ .... (B).

From (A) and (B)

We have $a* = a***$ Hence $a* = a***$

(v) We have $(a \vee b) \wedge (a^* \wedge b^*) = (a \wedge a^* \wedge b^*) \vee (b \wedge a^* \wedge b^*)$

$= (0 \wedge b^*) \vee (a^* \wedge 0)$ [by (i)]

$= 0 \vee 0$

$= 0$

Let $(a \vee b) \wedge x = 0$

implies that $(a \wedge x) \vee (b \wedge x) = 0$

implies that $a \wedge x = 0$ and $b \wedge x = 0$

implies that $x \leq a^*$ and $x \leq b^*$

Implies that $x \leq a^* \wedge b^*$

There fore $a^* \wedge b^*$ is the *pseudocomplement* of $a \vee b$.

Hence $(a \vee b)^* = a^* \wedge b^*$.

(vi) Let $a, b \in L$ implies that $a^*, b^* \in L$ implies that $a^{**}, b^{**}, \in S(L)$.

implies that $a^{**} \wedge b^{**} \in S(L)$. But $a^{**} \wedge b^{**}$ is the smallest element of $S(L)$ containing $a \wedge b$. So $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

(vii) If $a \wedge b = 0$ by (vi) then $a^{**} \wedge b^{**} = (a \wedge b)^{**} = 0^{**} = 0$.

So $a^{**} \wedge b^{**} = 0$.

Conversely, if $a^{**} \wedge b^{**} = 0$ by (iii) $a \leq a^{**}, b \leq b^{**} \forall a, b, \in L$,

then $a \wedge b \leq a^{**} \wedge b^{**} = 0$

$\therefore a \wedge b = 0$, Hence $a \wedge b = 0$ if and only if $a^{**} \wedge b^{**} = 0$.

(viii) Since $a \wedge b \leq b$ so $(a \wedge b)^* \leq b^*$ and

so $a \wedge (a \wedge b)^* \geq a \wedge b^*$ .... *(A)*.

Again $(a \wedge b) \wedge (a \wedge b)^* = 0$ implies that $(a \wedge (a \wedge b)^*) \wedge b = 0$,

there fore $a \wedge (a \wedge b)^* \leq b^*$

implies that $a \wedge a \wedge (a \wedge b)^* \leq a \wedge b^*$ .... *(B)*.

Form *(A)* and *(B)* $a \wedge (a \wedge b)^* = a \wedge b^*$.

Hence $a \wedge (a \wedge b)^* = a \wedge b^*$.

(ix) We have $0 \wedge x = 0 \, \forall x \in L$ and $0 \wedge 1 = 0$.

But $x \leq 1 \forall x \in L$. Hence $0^* = 1$.

Again $0^* = 1$ implies that $0^{**} = 1^*$

implies that $0 = 1^* \therefore 1^* = 0$.

(x) If $a \in S(L)$ then, $a = b^*$ for some $b \in L$.

but a* = a*** , ∀ a ∈ L.

Now $a^{**} = b^{***} = b^* = a$

Hence $a^{**} = a$

Conversely if $a = a^{**}$ then $a = b^*$, thus $a \in S(L)$.

Hence $a \in S(L)$ if and only if $a = a^{**}$.

(xi) Let $a, b \in S(L)$ then $a = a^{**}, b = b^{**}$, Since $a \wedge b \leq a$

implies that $(a \wedge b)^{**} \leq a^{**} = a$,

$\therefore a \geq (a \wedge b)^{**}$,

Again since $a \wedge b \leq b$ implies that $(a \wedge b)^{**} \leq b^{**} = b$

$\therefore (a \wedge b)^{**} \leq b$ implies that $b \geq (a \wedge b)^{**}$

implies that $a \wedge b \geq (a \wedge b)^{**}$..................(*A*).

But $(a \wedge b) \leq (a \wedge b)^{**}$..........................(B).

From *(A)* and $(B)\, a \wedge b = (a \wedge b)^{**}$ implies that $a \wedge b \in S(L)$.

If $x \in S(L)$ such that $x \leq a$ and $x \leq b$ then $x \leq a \wedge b$.

i.e $a \wedge b$ is a greatest lower bound of *S(L)*.

Therefore $a \wedge b =$ Inf $_{S(L)}\{a, b\} \in S(L)$.

(xii) For $a, b \in S(L)$. since $a^* \geq a^* \wedge b^*$

implies that $a^{**} \leq (a^* \wedge b^*)^*$ [by (ii)]

implies that $a \leq (a^* \wedge b)^*$ [by (i)]

Again $b^* \geq a^* \wedge b^*$ implies that $b^{**} \leq (a^* \wedge b^*)^*$ [by (ii)]

Implies that $b \leq (a^* \wedge b^*)^*$ [by (i)]

$(a^* \wedge b^*)^*$ is a upper bound of $\{a, b\}$ in S(L).

Let $x \in S(L)$ such that $a \leq x, b \leq x$ then $a^* \geq x^*, b^* \geq x^*$ [by (ii)].

$\therefore a^* \wedge b^* \geq x^*$ implies that $(a^* \wedge b^*)^* \leq x^{**} = x$

implies that $(a^* \wedge b^*)^* \leq x$

$\therefore (a^* \wedge b^*)^*$ is a least upper bound of $\{a, b\}$ in $S(L)$

Sup $_{S(L)}$ $\{a, b\} = (a^* \wedge b^*)^*$

Again $(a \wedge b)^{**} = ((a \wedge b)^*)^* = (a^* \wedge b^*)^*$

Hence Sup $_{S(L)}$ $\{a, b\} = (a \vee b)^{**} = (a^* \wedge b^*)^*$

(xiii) From (ix) we have $0^* = 1, 1^* = 0$ then $0, 1 \in S(L)$ and $1 \in D(L)$.

Let $x \in S(L) \cap D(L)$ then $x \in S(L)$ and $x \in D(L)$

such that $x = x^{**}$, $x^* = 0$ then $x = (x^*)^* = 0^* = 1$.

Hence $S(L) \cap D(L) = \{1\}$.

(xiv) Let $a, b \in D(L)$ then $a^* = 0, b^* = 0$ implies that $a^{**} = b^{**} = 0^* = 1$

Now, $(a \wedge b)^{**} = a^{**} \wedge b^{**} = 1 \wedge 1 = 1$ [by (iv)]

$(a \wedge b)^* = (a \wedge b)^{***} = 1^* = 0$ implies that $a \wedge b \in D(L)$.

(xv) If $a \in D(L)$ then $a^* = 0$ and $a \leq b$ implies that $a^* \geq b^*$

implies that $b^* \leq a^* = 0$

implies that $b^* = 0$. Hence $b \in D(L)$.

(xvi) From (v) we have $(a \vee a^*)^* = a^* \wedge a^{**} = a^* \wedge (a^*)^* = 0$.

Hence $a \vee a^* \in D(L)$.

(xvii) Let $\varphi : L \to S(L)$ defined by $\varphi(x) = x^{**}$. Then $\varphi(x \wedge y)$

$$= (x \wedge y)^{**} == x^{**} \wedge y^{**}$$

$$= \varphi(x) \wedge \varphi(y).$$

$\therefore \varphi$ is meet homomophism. ■

An identity $x \wedge \vee(x_i \mid i \in I) = \vee(x \wedge x_i \mid i \in I)$ is called the join Infinite Distributive Identity.

Lemma 3.1.2: Let B be a complete *Boolean lattice*. Then B satisfies the *Join Infinite Distributive Identity (JID)*

Proof: $x \wedge x_i \leq x$ and $x \wedge x_i \leq \vee(x_i | i \in I)$;

therefore $x \wedge \vee(x_i | i \in I)$ is an upper bound for $\{x \wedge x_i | i \in I\}$. Now let u be any upper bound, that is, $x \wedge x_i \leq u$ for all $i \in I$.

Then $x_i = x_i \wedge (x \vee x') = (x_i \wedge x) \vee (x_i \wedge x') \leq u \vee x'$.

Thus $x \wedge \vee(x_i | i \in I) \leq x \wedge (u \vee x') = (x \wedge u) \vee (x \wedge x') = x \wedge u \leq u$.

Showing that $x \wedge \vee(x_i | i \in I)$ is the least upper bound for $\{x \wedge x_i' | i \in I\}$. ■

Theorem 3.1.3: Any complete *lattice* that satisfies the *Join Infinity Distributive Identity (JID)* is a *pseudocomplemented distributive lattice.*

Proof: Let L be a complete *lattice*. For $a \in L$. set

$a^* = \vee(x / x \in L,\ a \wedge x = 0)$.

Then by $(JID), a \wedge a^* = a \wedge \vee(x / a \wedge x = 0) = \vee(a \wedge x /\ a \wedge x = 0) = \vee(0) = 0$.

Suppose $a \wedge x = 0$, then $x \leq a^*$ by the definition of a^*; Thus a* is the *pseudocompoement* of a and so L is *pseudocompoemented.*

Recall that a *distributive lattice* L is a complete *distributive* if $\wedge H$ and $\vee H$ exists in l for any subset H of L.

The following figure 3.4 is an example of a complete *distributive lattice* which is not *pseudocompoemented.*

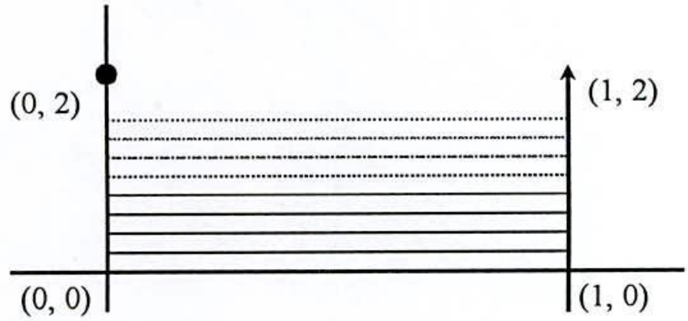


Figure 3.4

Here $L = \{(o, y) | 0 \le y < 2\} \cup \{(1, y) | 0 \le y \le 2\}$, so *(0,0)* is the smallest and *(l,2)* is the largest element. Observe that *(0,2)* $\notin$ L. This is a complete distributive lattice, where ≤' is the usual '≤' relation. But this is not pseudocomplemented as *(1, 0)* has no pseudocompoement. ■

2. Algebraic lattices.

Definition (Algebraic lattice) : A set $(L;\wedge,\vee)$ with two binary operation $\wedge$ and $\vee$ is called an *algebraic lattice* if it satisfy the following properties :

(i) for all $a\in L, a\wedge a=a, a\vee a=a$

(ii) for all $a,b\in L, a\wedge b=b\wedge a,\ a\vee b=b\vee a.$

(iii) for all $a,b,c\in L,\ a\wedge(b\wedge c)=(a\wedge b)\wedge c.$

$$a\vee(b\vee c)=(a\vee b)\vee c.$$

(iv) for all $a,b\in L,\ a\wedge(a\vee b)=a.$

$$a\vee(a\wedge b)=a.$$

A complete *lattice* is called *algebraic* if every element is the join of compact elements

Example: Let L be a with 0 then $I(L)$, the set of all *ideals* of L under '$\subseteq$' is an *algebraic lattice*.

In the literature, *algebraic lattices* are also called compactly generated *lattices*. Just as for *lattices*, a nonvoid subset l of a *join - semi lattice S* is an *ideal* if, for $a,b\in S$, we have $a\vee b\in L$ if and only if a, $a,b\in L$. Again, *I(S)* is the *poset* of all *ideals* of *S* partially ordered under set inclusion. If *S* has a zero, then *I(S)* is a *lattice*.

Using *I(S),* We give a useful characterization of *algebraic lattices*. ■

Theorem 3.2.1: A *lattice L* is *algebraic* if and only if it is isomorphic to the *lattice* of all *ideals* of a *join semi- lattice* with *0*.

Proof: Let *S* be a *join semi-lattice* with *0*. We have to prove that *I(S)* is *algebraic*. Since $0\in S, I(S)$ is a *complete lattice*, We claim that $\forall a\in S$ $(a]$ is a compact in *I(S)*.

Let $X \subseteq I(S)$ and $(a] \subseteq \vee(I | I \in X)$.

Now $\vee (I | I \in X) = \{X | x \leq t_1 \vee .... \vee t_n, t_i \in I_i, I_i \in X\}$

There fore, $a \leq t_1 \vee .... \vee t_n, t_i \in I_i, I_i \in X$

Thus with $X_1 = \{I_1 .... I_n\}$

$$(a] \leq \vee(I_i \in X_1 \subseteq X).$$

Therefore *(a]* is compact in *I(S)*.

Now, for any $I \in I(S), I = \vee((a] \ / \ a \in L)$. Hence *I(S)* is *algebraic* and so any *lattice* L is isomorphic to *I(S)* is also *algebraic*.

Conversely, let L be an *algebraic lattice* and let S be the set of all compact element of L. Obviously $0 \in S$.

Moreover, clearly join of two compact elements is again a compact element. So S is a *join semi-lattice* with *0*. Now consider the map $\varphi : L \to I(L)$ is defined by $\varphi(a) = \{x \in S | x \leq a\}$.

Obviously, φ maps L into $I(S)$. By the definition of an *algebraic lattice* $a = \vee\varphi(a)$, and so φ is one- one. To prove that φ is onto. Let $I \in I(S)$, $, a = \vee I$ then $\varphi(a) \supseteq I$. Now, let $x \in \varphi(a)$, then $x \in S, x \leq a$.

$\vee I_1$, By compactness of *x*, there exists a finite subset $I_1 \subseteq I$ such that $x \leq \vee I_1$. This implies $x \in I$ and so $I \in \varphi(a)$. There fore φ is onto.

Also $\varphi(a \wedge b) = \{x \in S | x \leq a \wedge b\} = \{x \in S | x \leq b\}$

$$= \varphi(a) \wedge \varphi(b)$$

Also $\varphi(a \vee b) = \{x \in S | x \in\leq a \vee b\} = \{x \in S | x \leq a\} \vee \{x \in S | x \leq b\}$

$$= \varphi(a) \vee \varphi(b)$$

i.e. φ is a *homomorphism*

Therefore it is an *isomorphism*. ■

Corollary 3.2.2: Let L be an arbitrary *lattice C(L)* is an *algebraic lattice*.

Proof: We already know that *C(L)* is a *complete distributive lattice*.

Suppose $\Theta \in C(L)$. Observe that $\Theta = \vee(\Theta(a,b) | a \equiv b\Theta, a,b \in L)$. Since every *principal congruence* is *compact*, So *C(L)* is *algebraic*. ■

Corollary 3.2.3 : Every *distributive algebraic lattice spseudocomplement*.

Proof: Let L be a *distributive algebraic lattice*. Then $L \cong I(S)$, for some *distributive join semi lattice* S with 0, $I(L)$ is *complete*.

Let $I, I_K \in I(S)$, we have to show that $I \wedge (\vee I_K) = \vee(I \wedge I_K)$

Of course, $\vee(I \wedge I_K) \subseteq I \wedge (\vee I_K)$..............................(1).

Let $x \in I \wedge (\vee I_K)$ then, $x \in I$ and $x \in \vee I_K$

implies that $x \leq i_{K1} \vee .... i_{Kn}$, for some $i_{K1} \in I_{K1}, i_{K2} \in I_{K2} .... i_{Kn} \in I_{Kn}$

implies that $x \in I_{K1} \vee .... \vee I_{Kn}$

implies that $x \in I \wedge (I_{K1} \vee .... \vee I_{Kn})$

$$(I \wedge I_{K1}) \vee .... \vee (I \wedge I_{Kn}) \leq (I \wedge I_K).$$

implies that $(I \wedge \vee I_K) \subseteq \vee(I \wedge I_K)$.. (ii)

From (i) and (ii)

$\vee(I \wedge I_K) = I \wedge (\vee I_K)$

implies that *I*(*S*) holds JID

implies that *I*(*S*) is *pscudocomplemdnted*.

implies that L is *pscudocomplemented*. ■

Theorem 3.2.4: Let L be a *pseudocomplemented meet semi-lattice*. $S(L) = \{a^* | a \in L\}$.Then the partial ordering of L partially orders $S(L)$ and makes S(L) into a *Boolean lattice*.

For $a,b \in S(L)$ we have $a \wedge b \in S(L)$ and the join in $S(L)$ is described by $a \vee b = (a^* \wedge b^*)^*$.

Proof: The following results have already been proved in theorem 3.1.1.

(*i*) $a \le a^{**}$

(*ii*) $a \le b$ implies that $a^* \ge b^*$

(*iii*) $a^* = a^{***}$

(*vi*) $a \in S(L)$ iff $a^* = a^{**}$

(*v*) $a, b \in S(L)$ implies that $a \wedge b \in S(L)$

(*vi*) For $a, b \in S(L)$, Sup $s_{(L)}\{a,b\} = (a^* \wedge b^*)^*$

For $a, b \in S(L)$ define $a \vee b = (a^* \wedge b^*)^*$

then by (*v*) and (*vi*) we get $< S(L); \wedge, \vee >$ is a bounded *lattice*.

Since, for $a \in S(L), a \wedge a^* = 0$ and $a \vee a^* = (a^* \wedge a^{**})^* = 0^* = 1$,

implies that *S*(*L*) is *Complemented lattice*.

Now we need only to show that *S*(*L*) is *distributive*.

For $x, y, z, \in S(L), x \wedge z \le . x \vee (y \wedge z)$ and $y \wedge z \le x \vee (y \wedge z)$;

there fore $x \wedge z \wedge (x \vee (y \wedge z))^* = 0$

implies that $x \wedge (z \wedge (x \vee (y \wedge z))^*) = 0$

implies that $z \wedge (x \vee (y \wedge z))^* \le x^*$

Again $y \wedge z \wedge (x \vee (y \wedge z))^* = 0$

Or $y \wedge (z \wedge (x \vee (y \wedge z)^*) = 0$

$\therefore$ $z \wedge (x \vee (y \wedge z))^* \le y^*$

We can write $z \wedge (x \vee (y \wedge z))^* \le x^* \wedge y^*$

Consequently, $z \wedge (x \vee (y \wedge z))^* \wedge (x^* \wedge y^*)^* = 0$,

which implies that $z \wedge (x^* \wedge y^*)^* \le (x \vee (y \wedge z))^{**}$.

Now the left- hand side is $z \wedge (x \vee y)$ [by for a, b $\in$ S(L).

Sup $s_{(L)}$ $\{a, b\} = (a^* \wedge b^*)^*$]

and the right hand side is $x \vee (y \wedge z)$ [by $a \in S(L)$ iff $a = a^{**}$].

Thus we $z \wedge (x \vee y) \le x \vee (y \wedge z)$ which is distributivity. ■

Theorem 3.2.5: Let L be a *pseudocomplemented lattice*.

Then $a^{**} \vee b^{**} = (a \vee b)^{**}$ for all $a, b \in L$.

Proof: We know that if L is a *pseudocomplemented meet semi-lattice.*

then $a \vee b = (a^* \vee b^*)^*$ where $a, b \in S(L)$.

Now for $a, b \in L$, $a^{**}, b^{**} \in S(L)$

So $a^{**} \vee b^{**} = (a^{***} \wedge b^{***})^*$

$$= (a^* \wedge b^*)^*$$

$$= (a \vee b)^{**}$$

implies that $a^{**} \vee b^{**} = (a \vee b)^{**}$. ■

Theorem 3.2.6: Let L be a *pseudocomplemented meet semi-lattice* and let $a, b \in L$ then $(a \wedge b)^* = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$

Proof: Since L is a *pseudocomplemented meet semi-lattice.*

Then $a \leq a^{**}$ implies that $a \wedge b \leq a^{**} \wedge b$

implies that $(a \wedge b)^* \geq (a^{**} \wedge b)^*$ .... (i)

Again $b \leq b^{**}$ implies that $a^{**} \wedge b \leq a^{**} \wedge b^{**}$

implies that $a^{**} \wedge b \leq (a \wedge b)^{**}$

implies that $(a^{**} \wedge b)^* \geq (a \wedge b)^{****}$

implies that $(a^{**} \wedge b)^* \geq (a \wedge b)^*$ .... (ii)

Form (i) and (ii) we have $(a \wedge b)^* = (a^{**} \wedge b)^*$ .... (iii)

Again, $b \leq b^{**}$ implies that $a^{**} \wedge b \leq a^{**} \wedge b^{**}$

Implies that $(a^{**} \wedge b)^* \geq (a^{**} \wedge b^{**})^*$ .... (iv)

Again, $a^{**} \leq a^{****}$ implies that $a^{**} \wedge b^{**} \leq a^{****} \wedge b^{**}$

$$= (a^{**} \wedge b)^{**}$$

implies that $(a^{**} \wedge b^{**})^* \geq (a^{**} \wedge b)^{***}$ implies that

$(a^{**} \wedge b^{**})^* \geq (a^{**} \wedge b)^*$ .... (v).

From (iv) and (v)

$(a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$ .... *(v)*

From (iii) and (vi)

$$(a \wedge b)^* = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*. \quad \blacksquare$$

Theorem 3.2.7: Let L be a *pseudocomplemented distributive lattice*. Then for each $a \in L, (a]$ is a *pseudocomplement distributive lattice* in fact the *pseudocomplement* of $x \in (a]$ in $(a]$ is $x^* \wedge a$.

Proof: Let $x \in (a]$ then $x \wedge (x^* \wedge a) = (x^* \wedge a) = (x \wedge x^*) \wedge a = 0 \wedge a = 0$. Further if $x \wedge t = 0$ then $t \leq x^*$ implies that $t \wedge a \leq x^* \wedge a$ implies that $t \leq x^* \wedge a$ implies that $x^* \wedge a$ is the *pseudocomplement* of x, implies that $(a]$ is a *pseudocomplemented distributive lattice*. $\blacksquare$

Theorem 3.2.8: Let $\wedge$ be a binary operation on L, let $*$ be a unary operation on L (that is, for every $a \in L, a^* \in L$) and let 0 be a nulary operation (that is $0 \in L$). Let us assume that the following hold for all $a, b, c \in L : a \wedge b = b \wedge a$.

$(a \wedge b) \wedge c = a \wedge (b \wedge c), a \wedge a = a, 0 \wedge a = 0, a \wedge (a \wedge b)^* = a \wedge b^*,$

$a \wedge 0^* = a, (0^*)^* = 0$. Show that $< L; \wedge >$ is a *meet semi-lattice* with 0 as zero, and for all, $a \in L, a^*$ is the *pseudocomplement* of a *(R. Balbes and A. Horn [**1970a**])*

Proof: Let $a \in L, a^* \in L$ then

i) $a \wedge a = a$ [by given condition]

ii) $a \wedge a = b \wedge a$ [by given condition]

iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ [by given condition]

Define '$\leq$' on L by $a \leq b \Leftrightarrow a = a \wedge b$.

$\therefore < L; \wedge >$ is a *meet semi-lattice.*

Now $0 \wedge a = 0 \;\; \forall a \in L$ implies that $0 \leq a$

So, 0 is the zero element of L.

Second part: $0 = a \wedge 0 = a \wedge 0^{**} = a \wedge (a \wedge 0^*)^* = a \wedge a^*$ and $a \wedge x = 0$.

Then $x \wedge a^* = x \wedge (x \wedge a)^* = x \wedge 0^* = x = x \wedge a^* = x$ implies that $x \leq a^*$

Hence $a*$ is the pseudocomplement of a. ■

Theorem 3.2.9: For as *pseudocomplemented distributive lattice L*. Define the relation R by: $x \equiv y(R)$ if and only if $x* = y*$. Then R is a *congruence* on L and $L|R \cong S(L)$.

Proof: Given that $x \equiv y(R) \Leftrightarrow x* = y*$, then $x* = x*$ implies that $x = x(R)$ implies that R is reflexive.Also if $x \equiv y(R)$, then $x* = y*$ implies that $y* = x*$ implies that $y \equiv x(R)$ implies that R is symmetric. Let $x \equiv y(R)$ and $y \equiv z(R)$, then $x* = y*$ and $y* = z*$ implies that $x* = z*$ implies that $x \equiv z(R)$ implies that R is transitive.implies that R is an equivalence relation.

Now, suppose $x \equiv y(R)$ and $t \in L$ then $x* = y*$ implies that $x** = y**$.

Now, $(x \wedge t)** = x** \wedge t** = y** \wedge t** = (y \wedge t)**$

implies that $(x \wedge t)** = (y \wedge t)**$

implies that $(x \wedge t)* = (y \wedge t)*$

implies that $x \wedge t \equiv y \wedge t(R)$

and $(x \vee t)* = x* \wedge t* = y* \wedge t* = (y \vee t)*$

implies that $x \vee t \equiv y \vee t(R)$.

So R is a *congruence* relation on L.

Define $\varphi : L/R \to S(L)$ by $\varphi((a]R) = a**$,

then $\varphi([a] \wedge [b]) = \varphi([a \wedge b])** = (a \wedge b)** = a** \wedge b**$

$$= \varphi([a]) \wedge \varphi([b])$$

And $\varphi([a] \vee [b]) = \varphi([a \vee b]) = (a \vee b)** = (a* \wedge b*)*$

$$= (a*** \wedge b***)*$$

$$= a** \vee b**$$

$$= \varphi([a]) \vee \varphi([b])$$

$\therefore \varphi$ is a homomorphism.

To show that φ is one- one. Let $a** = b**$

implies that $a* = b*$

implies that $a \equiv b(R)$ implies that $[a] = [b]$,

$\therefore$ φ is one- one.

Let $a \in S(L)$ then $a = a**$ implies that $a = \varphi[a]$

implies that φ is onto.

Hence $\varphi : L / R \to S(L)$ is an isomorphism.

Therefore $L / R \cong S(L)$. ■

CHAPTER FOUR

STONE LATTICES

Introduction: Stone lattices have been studied by several authors including Cornish [5], G. Gratzer & E.T. Schmidt [9], Katrinak [11], T.P.Speed [25], J.Verlet [26]. In this chapter, we discuss the *Stone lattices, Stone algebras* and some basic concepts to *Stone lattices*. In section 1 of this chapter, we give some basic properties of *Stone algebra* which will be needed in the next part.

In section 2 of this chapter, we have given characterization of *minimal prime ideals of a pseudocomplemented distributive lattice*. Then we have shown that every *pseudocomplemented lattice is generalized Stone* if and only if every two *minimal prime ideals are co-maximal*.

Definition (Stone lattice): A *distributive pseudocomplemented lattice* L is called a *Stone lattice* if for each $a \in L$, $a^* \vee a^{**} = 1$.

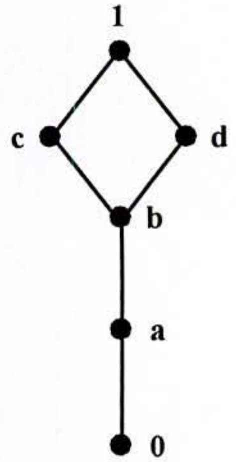


Figure 4.1

Definition (Stone algebra): A *pseudocomplemented distributive lattice* L is called a *stone algebra* if and only if it satisfies the condition $a^* \vee a^{**} = 1$ which is called *stone identity*, for each $a \in L$.

Definition (Generalized stone lattice): A *lattice* L with 0 is called *generalized stone lattice* if $(x]^* \vee (x]^{**} = L$ for each $x \in L$.

1. Properties of Stone Lattices.

Theorem 4.1.1: For a *distributive lattice* L with *pseudocomplementation*,
the following conditions are equivalent.

i) L is a *Stone algebra*

ii) $(a \wedge b)^* = a^* \vee b^*$ for all $a, b \in L$

iii) $a, b \in S(L)$ implies that $a \vee b \in S(L)$.

iv) $S(L)$ is a *sub algebra* of L.

Proof: (i) implies (ii), Let L be a *Stone algebra*, we shall show that $a^* \vee b^*$ is the *pseudocomplement* of $a \wedge b$, Indeed.

$(a \wedge b) \wedge (a^* \vee b^*) = (a \wedge b \wedge a^*) \vee (a \wedge b \wedge b^*)$

$= (0 \wedge b) \vee (a \wedge 0)$

$= 0 \vee 0$

$= 0$

If $(a \wedge b) \wedge x = 0$ then $(b \wedge x) \wedge a = 0$.

and so $b \wedge x \leq a^*$, Meeting both sides by a^{**}

Yields, $b \wedge x \wedge a^{**} \leq a^* \wedge a^{**} = 0$;

that is, $b \wedge (x \wedge a^{**}) = 0$, implying that $a^{**} \wedge x \leq b^*$

We have, $a^* \vee a^{**} = 1$, by *Stone's identity*.

$\therefore \; x = x \wedge 1 = x \wedge (a^* \vee a^{**}) = (x \wedge a^*) \vee (x \wedge a^{**}) \leq a^* \vee b^*$.

implies that $a^* \vee b^*$ is the *pseudocomplement* of $a \wedge b$

implies that $(a \wedge b)^* = a^* \vee b^*$.

(ii) implies (iii).

Let $a, b \in S(L)$, then $a = a^{**}$, $b = b^{**}$

$\therefore a \vee b = a^{**} \vee b^{**} = (a^* \wedge b^*)^* = (a \vee b)^{**}$

implies that $a \vee b \in S(L)$

(iii) implies (iv), For $a,b \in S(L), a \vee b \in S(L)$

Also $a = a^{**}, b = b^{**}$

Now, $a \vee b = a^{**} \vee b^{**} = (a^* \wedge b^*)^* = (a \vee b)^{**} = a \vee b$

i.e. $S(L)$ is a *sub algebra* of L.

(iv) implies (i) Let $S(L)$ is a *sub algebra* of L.

Then $a^* \vee a^{**} = (a \wedge a^*)^* = 0^* = 1$.

Hence L is a *Stone algebra.* ■

Theorem 4.1.2: If L is a *complete Stone lattice*, then so is *I(L).*

Proof: Let $I^* = (a]$, where $a = \wedge(x^* \mid x \in I)$ and let $x \in I \cap I^*$, then $x \in I$ and $x \in I^* = (a]$ implies that $x \in I$ and $x \in (a]$ implies that $x \in I$ and $x \le y^* \forall y \in I$ implies that $x \le x^*$ implies that $x = x \wedge x^* = 0$, implies that $I \wedge I^* = (0]$,

Let $I \wedge J$, choose any $j \in J$, then $i \wedge j = 0$ $\forall i \in I$ implies that $j \le i^*, i \in I$ implies that $j \le \wedge(I^* \mid i \in I)$ implies that $j \le a$ implies that $j \in I^*$ implies that $J \subseteq I^*$ implies that I^* is a *pseudocomplemented.* Since $0 \in L$, so *I(L)* is *complete*. Finally, we have to show that $I^* \vee I^{**} = L$.

Now $I^* \vee I^{**} = (a] \vee (a]^* = (a]^{**} \vee (a]^*$

$$= (a^{**}] \vee (a^*]$$

$$= (a^{**} \vee a^*]$$

$$= L$$

Hence *I(L)* is a *Stone.*

Thus *I(L)* is a *complete Stone lattice.* ■

Theorem 4.1.3: A *distributive pseudocomplemented lattice* is a *Stone lattice* if and only if $(a \vee b)^{**} = a^{**} \vee b^{**}$ for *a, b* $\in L$.

Proof: Let L be a *Stone lattice*. Then we have $(a \wedge b)^* = a^* \vee b^*$ for *a, b* $\in L$. Now $(a \vee b)^{**} = (a \vee b^*)^* = (a^* \wedge b^*)^* = a^{**} \vee b^{**}$

Conversely, let $(a \vee b)^{**} = a^{**} \vee b^{**}$ for $a, b \in L$.

Since L is a *pseudocomplemented lattice*. Then for $a \in L, a \wedge a^* = 0$

implies that $(a \wedge a^*)^{**} = 0^{**}$

implies that $a^{**} \wedge a^{***} = 0$

implies that $a^{**} \wedge a^* = 0$

Now, $(a \vee a^*)^* = a^* \wedge a^{**} = 0$

implies that $(a \vee a^*)^{**} = 0^*$

implies that $a^{**} \vee a^{***} = 1$

implies that $a^{**} \vee a^* = 1$

Hence L is a *Stone lattice*. ■

2. Minimal prime ideals.

A *prime ideal* P of a lattice L is called *minimal* if there does not exists a *prime ideal* Q such that $Q \subset P$.

The following lemma is a fundamental result in *lattice theory*;

e.f. [7], lemma 4pp. 169]. Though our proof is similar to their proof, we include the proof for the convenience of the reader.

Theorem 4.2.1: Let L be a *lattice* with 0. Then every *prime ideal* contains a *minimal prime ideal*.

Proof: Let P be a *prime ideal* of L and Let R denote the set of all *prime ideals* Q contained in P. Then R is non-void, since $0 \in Q$ and Q is an *ideal*; infact, Q is *prime*. Indeed, if $a \wedge b \in Q$ for some $a, b \in L$, then $a, b \in X$ for all $X \in C$; since X is *prime*, either $a \in X$ *or* $b \in X$. Thus either $Q = \cap\ (X : a \in X)$ *or* $Q = \cap\ (X : b \in X)$ proving that a or b $\in Q$. Therefore, We can apply to R the *dual* form of Zorn's lemma to conclude the existence of a *minimal* member of R. ■

Lemma 4.2.2: Let L be a *pseudocomplemented distributive lattice* and let P be a *prime ideal* of L. Then the following four conditions are equivalent.

i) P is *minimal*.

ii) $x \in P$ implies that $x* \notin P$.

iii) $x \in P$ implies that $x** \in P$.

iv) $P \cap D(L) = \phi$.

Proof: (i) implies (ii).

Let P be *minimal* and (ii) fail, that is $a* \in P$ for some $a \in P$. Let $D = (L - P) \vee [a)$, We claim that $0 \notin D$. Indeed, if $0 \in D$, then

$q \wedge a = 0$ for some $q \in L - P$, which implies that $q \leq a \in P$, a contradiction. Thus (by theorem 1.4.8) there exists a *prime ideal* Q disjoint to D. Then $Q \subseteq P$ since $Q \cap (L - P) = \phi$, and $Q \neq P$. since : a $\notin$ Q, contradicting the minimally of P.

(ii) implies (iii)

Indeed, $x^* \wedge x^{**} = 0 \in P$ for any x $\in L$ thus if $x \in P$, then by (ii) $x^* \in$P, implying that $x^{**} \in$P.

(iii) implies (iv)

If $a \in P \cap D(L)$ for some $a \in L$, then $a^{**} = 1 \notin P$, a contradiction to (iii), thus $P \cap D(L) = \phi$.

(iv) implies (i)

If P is not *minimal*, then $Q \subset P$ for some *prime ideal* Q of L .

Let $x \in P - Q$. Then $x \wedge x^* = 0 \in Q$ and $x \notin Q$: then $x^* \in Q \subset P$, which implies that $x \vee x^* \in P$. By theorem 3.1.1. (xvi), $x \vee x^* \in D(L)$; thus we obtain $x \vee x^* \in P \cap D(L)$, contradicting (iv).

Hence P is *minimal*. ■

Theorem 4.2.3: In a *Stone algebra* every *prime ideal* contains exactly one *minimal prime ideal*.

Proof: Let L be a *stone algebra* and let P be a *prime ideal* of L. We need prove that P contains exactly one *minimal prime ideal*. Suppose P contains two distinct *minimal prime ideals* Q_1 and Q_2.

Choose $x \in Q_1 - Q_2$ ($Q_1 \not\subset Q_2$, since Q_2 is *minimal* and $Q_2 = Q_1$, hence $Q_1 - Q_2 \neq \phi$);

Since $x \wedge x^* = 0 \in Q_2$, $x \notin Q_2$ and Q_2 is prime, so $x^* \in Q_2$, L- Q_1 is *maximal dual prime ideal*, hence it is a *maximal dual ideal* of L.

Thus $(L - Q_1) \vee [x) = L$ and so, $x \wedge a = 0$ for some $a \in L - Q_1$. Therefore, $x^* \geq a \in$L $-$ Q$_1$ implies that $x^* \in Q_1$. Hence $x^* \in Q_2 - Q_1$. Similarly, $x^* \in Q_1$, so x^* and x^{**} both contained in P.

implies that $1 = x* \vee x** \in P$, which is a contradiction that P is a *prime ideal* of L. Thus in a *Stone algebra* every *prime ideal* contains exactly one *minimal prime ideal.* ■

Theorem 4.2.4: A *prime ideal* P of a *Stone algebra* L is *minimal* if and only if $P = (P \cap S(L))_L$.

Proof: Suppose P is *minimal*, Let $x \in (P \cap S(L)]_L$. Then $x \leq r$ for some $r \in P \cap S(L)$ implies that $r \in P$ and $r \in S(L)$ implies that $x \in P$ implies that $r \in P$ and $r \in S(L)$ implies that $r \in P$ implies that $x \in P$. implies that $(P \cap S(L)]_L \subseteq P$ .... (i)

Again let $x \in P$, since P, is *minimal* so, $x** \in P$, Then $x \in P \cap S(L)$, as $x \leq x**$. So $x \in (P \cap S(L)]_L$.

implies that $P \subseteq (P \cap S(L)]_L$ .... (ii)

Form (i) and (ii) $P = (P \cap S(L)]_L$

Conversely, let $P = (P \cap S(L)]_L$ and let $x \in P$ then $x \leq r$ for some $r \in P \cap S(L)$, implies that $x** \leq r** = r$ implies that $x** \in P$.

Hence P is *minimal.* ■

Theorem 4.2.5: *A distributive lattice* with *pseudocomplementation* is a *Stone algebra* if and only if every *prime ideal* contains exactly one *minimal prime ideal* (G. Gratzer and E. T Schmidt [1957b])

Proof: Let L be *distributive lattice* with *pseudocomplementation*. If L is a *Stone algebra*, then by theorem 4.2.3 every *prime ideal* contains exactly one *minimal prime ideal*.

Conversely, let L is not a *Stone lattice* and let $a \in L$ such than $a* \vee a** \neq 1$. Then there exist a *prime ideal* R such that, $a* \vee a** \in R$. We claim that $(L - R) \vee [a*) \neq L$. If $(L - R) \vee [a*) \neq L$ then there exist an $x \in L - R$ such that $x \wedge a* = 0$. Then $a** \geq x \in L - R$ implies $a** \in L - R$. Which is a contradiction. So $(L - R) \vee [a*) \neq L$. Let F be a *minimal dual prime ideal* containing $(L - R) \vee [a*)$ and let G be a *minimal dual prime ideal* containing $(L-R)\vee[a*)$. We set $P = L - F$ and $Q = L - G$. Then P and Q are *minimal prime ideals* such that $P,\ Q \subseteq R$. Moreover $P \neq Q$, because $a* \in F = L - P$ and hence $a* \notin P$; thus $a** \in P$ but $a** \notin Q$. ■

Theorem 4.2.6: Let L be a *distributive* with 0 and 1 . For an *ideal* I of L. We set $I^* = \{x \mid x \wedge i = 0 \text{ for all } i \in I\}$. Let P be a *prime ideal* of L. Then P is *minimal prime ideal* if and only if $x \in P$
implies that $(x]^* \subseteq P$ (T. P. Speed).

Proof: By the definition of $I*, (x]* = \{y \mid y \wedge x = 0\}$ as $x* \wedge x = 0$ implies that $x* \in (x]*$ implies that $(x*] \subseteq (x]*$, again let $z \in (x]*$, then $z \wedge x = 0$ implies that $z \leq x*$ implies that $z \in (x*]$ implies that $(x]* \subseteq (x*]$ implies that $(x]* = (x*]$. Now suppose P be a *minimal prime ideal* and $x \in P$, then by the theorem $x* \notin P$, implies that $(x*] \not\subset P$ implies that $(x*] \subseteq P$.

Conversely, if for $x \in P, (x]* \not\subset P$ and if possible. Let P is not *minimal* then there exist a *prime ideal* Q such that $Q \subset P$. Let $x \in P = Q$.
Now $x* \wedge x = 0 \in Q$ implies that $x* \in Q$ implies that $x \in P$ implies that $(x*] \subseteq P$ implies that $(x]* \subseteq P$, which is a contradiction.
Hence the proof. ■

Theorem 4.2.7: Every *Boolean lattice* is a *Stone lattice* but the conversely is not necessary true.

Proof: Let L be a *Boolean lattice*. Then for each $a \in L$, it's complement a' is also the *pseudocomplement* of a.
Moreover, $a* \vee a** = a' \vee a'' = a' \vee a = 1$. Hence L is also *Stone*.
Observe that 3- elements chain is a *Stone lattice*.

For $a^* \vee a^{**} = 0 \vee 0^* = 0 \vee 1 = 1$. But it is not *Boolean*, as a has no *complement*.

Figure – 4.2

In theorem 4.2.3, we have proved that in a *Stone lattice* every *prime ideal* contains a unique *minimal prime ideal*. In the following lattice, observe that $(c]$ is a *prime ideal* and it contains two *minimal prime ideals* $(a]$ and $(b]$.

Hence it is not a *Stone lattice*.

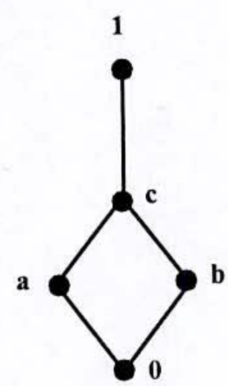


Figure – 4.3

Also by 4.1.1. we know that in a *Stone lattice* L, $a \wedge b \in S(L)$ for all $a, b \in L$. In above *lattice* observe that $a \vee b = c \notin S(L)$.

Hence L is not *Stone*.

Definition(Skeleton of a lattice): Let L be a *Stone lattice*, then $S(L) = \{a* : a \in L\}$ is called *skeleton* of L. The elements of $S(L)$ are called *skeletal*. L is *dense* if $S(L) = \{0,1\}$,

$< S(L);\wedge,\vee,*,0,1 >$ is a *Boolean algebra*.

Corollary 4.2.8: A *finite distributive lattice* is a *Stone lattice* if and only if it is the direct product of finite *distributive dense lattices* that is *finite distributive lattices* with only one *atom*.

Proof: By theorem 4.1.1 a *Stone lattice* L has a complemented element $a \notin \{0,1\}$ iff $S(L) \neq \{0,1\}$; thus the decomposition of theorem 2.1.14 can be repeated until each factor L_i satisfies $S(L) = \{0,1\}$. In a direct product, $*$ is formed component wise: Therefore all the L_i are *Stone lattices*; For a finite lattice K with $S(K) = \{0,1\}$ the condition that K has one *atom* is equivalent to K being a *Stone lattice*. ■

Theorem 4.2.9: *A distributive pseudocomplemented lattice* is a *Stone lattice* L if and only if for any two *minimal prime ideals* P and Q,

$P \vee Q = L$

Proof : Suppose L is a *Stone lattice* and P, Q are two minimal *prime ideals*. If $P \vee Q \neq L$ then by theorem 2.1.17 there exists a *prime ideal* R containing $P \vee Q$. This means that R contains two *minimal prime ideals*, which is a contradiction to theorem 4.2.5. as L is a *Stone*, there fore $P \vee Q = L$.

Conversely, suppose the given condition holds and R is a *prime ideal* of L. Then R can not contain two *minimal prime ideals* P and Q, as other wise $R \supset P \vee Q = L$, Therefore again by theorem 4.2.5. L is *Stone*. ■

Definition (Dense set): $D(L) = \{a \in L : a* = 0\}, D(L)$ is called the *dense set*. $D(L)$ is a *filter* or *Dual ideal*, $1 \in D(L)$.

We can easily cheek that *D(L)* is a *dual ideal* of L and that $I \subset D(L)$; thus $D(L)$ is a *distributive lattice* with 1. Since $a \vee a \subset D(L)$ for every a ∈ L, we can interpret the identity $a \vee a ** \wedge (a \vee a*)$.

To mean that every $a \in L$ can be represented in the form $a = b \wedge c$.

Where $b \in S(L)$, $c \in D(L)$. Such an interpretation correctly suggests that if we know *S(L)* and *D(L)* and the relation ships between element of *S(L)* and *D(L)*,

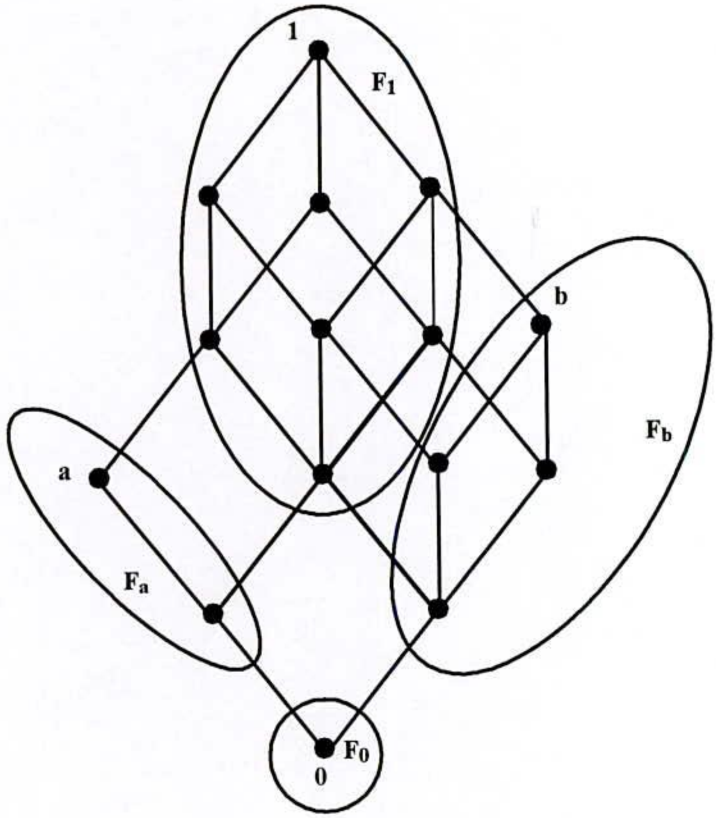


Figure : 4.4

Then we can describe *L*. The relation ship is expressed by the *homomorphism* $\varphi(L) : S(L) \rightarrow \wp(D(L))$ defined by

$\varphi(L) : a \rightarrow \{x \mid x \in D(L); x \geq a*\}$

Now we prove a theorem which givens an *ideal* of construction of *Stone algebra's*.

Theorem 4.2.10: (C. C. Chen and G. Gratzer [1969b]) Let L be a Stone algebra. Then *S(L)* is a *Boolean algebra D(L)* is a *distributive lattice* with L and $\varphi(L)$ is a {0, 1} *homomorphism* of *S(L)* into $\wp D(L))$. The triple $< S(L).\ D(L).\wp(L) >$ characterizes L up to *isomorphism*.

Proof : The first statement is easily verified. For $a \in S(L)$,

set $F_a = \{x : x** = a\}$.

The sets $\{F_a \mid a \in S(L)\}$ form a partition of *L;* for simple example figure 4.4. Obviously, $F_0 = \{0\}$ and $F_1 = D(L)$; The map $x \to x \vee a*$ sends F_a into $F_1 = D(L)$; infact the map is an *isomorphism* between F_a and $a\varphi(L) \subseteq D(L)$. Thus $x \in F_a$ is completely determined by a and $x \vee a* \in a\varphi(L)$- that is by a pair $< a, z >$ where $a \in S(L), z \in a\varphi(L)$- and every such pair determines one and only one element of L. To complete our proof we have to show how the partial ordering on L can be determined by such pairs.

Let $x \in F_a$ and $y \in F_b$. Then $x \leq y$ implies that $x** \leq y**$, that is $a \leq b$. Since $x \leq y$ if and only if, $a \vee x \leq a \vee y$ and $x \vee a* \leq y \vee a*$ and since the first of these two conditions is trivial, we obtain: $x \leq y$ iff $a \leq b$ and $x \vee a* \leq y \vee a*$. Identifying x with $< x \vee a*, a >$ and y with $< y \vee b*, b >$, we see that the preceding conditions are stated in terms of the components of the ordered pairs, except that $y \vee a*$ will have to be expressed by the triple. Because $\varphi(L)$ is a *{0,1} homomorphism* and $a**$ is the *complement* of $a*$, we conclude that $a**\varphi(L)$ and $a*\varphi(L)$ are complementary dual ideals of *D(L)*. Therefore, by theorem 2.2.3. for any $z \in D(L), [z)$ is the direct product of $[z \vee a*)$ and $[z \vee a**)$. Thus every z can be written in a unique fashion in the form $z = z(a*) \wedge z(a**)$, where $z(a*) \in a\varphi(L)$ and $z(a**) \in a*\varphi(L)$. Let $y\rho_\alpha$ denoted the element $(y\varphi(L))(a*)$ and observe that ρ_a is expressed interims of the triple. Finally, $y \vee a* = y \vee b* \vee a* = (y\varphi(L)) \vee a* = y\rho_a$. Thus for $u \in a\varphi(L)$ and $v \in b\varphi(L)$, we have $< u,a > \leq < v,b >$ if and only if $a \leq b$ and $u \leq vp_a$. ■

CHAPTER FIVE

MODULAR AND DISTRIBUTIVE LATTICE WITH n-IDEAL.

Introduction: An idea of *standard n-ideals* of a *lattice* was introduced by A.S.A.Noor and M.A. Latif in [20]. Then they studied those *n-ideals* extensively and included several properties in [19] and [21]. Moreover, in [22] Latif has *generalized isomorphism* theorems for *standard ideals* in terms of *n-ideals*. In this section we give a nice *idea* of *distributive* and *modular lattice* with *n-ideals*.

An *n-ideal* S of a *lattice* L is called a *standard n-ideal* if it is a standard element of the *lattice* I_n (L). That is, S is called standard if for all

$I, J \in I_n(L),\ \ I_n \wedge (s \vee J) = (I \cap s) \vee (I \cap J)$.

Distributive elements and ideals were studied extensively by Gratzer and Schmidt in [9]. On the other hand [24] have studied the *distributive* elements and *ideals* in *Join semi lattices* which are directed below:

An element d of a *lattice* L is called *distributive* if for all $x, y \in L, d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$. An *ideal* I is called *distributive* if it is a *distributive* element of the *ideal Lattice* $I(L)$.

In [24] Talukder and Noor have given an idea of a modular element and a *modular ideal* of a *Lattice*. According to them, an element n of a *lattice* L is called *modular* if for all $x, y \in L$ with $y \leq x$, $x \wedge (n \vee y) = (x \wedge n) \vee y$.

An ideal of L is called *modular* if it is a *modular* element of $I(L)$.

An element $s \in L$ is *standard* if for all

$x, y \in L,\ x \wedge (s \vee y) = (x \wedge s) \vee (x \wedge y)$

An element $n \in L$ is called *neutral* if it is *standard* and for all $x, y \in L, (a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a)$ That is, n is *dual distributive*.

In section 1, we have introduced some idea of *distributive lattice* with n-ideals. We have given several characterizations of *distributive lattice with n-ideals*. For a *distributive lattice of n-ideal I* of a *lattice L* we have also given some definition of $\Theta(I)$. The *congruence* generated by *I*. We have also explained *neutral* element n of a *lattice L*, *Principal n-ideal* $\langle a \rangle_n$ or $P_n(L)$ in *distributive Lattice.*

1. n-Ideal of a lattice.

A non-empty subset I of a *lattice* L is said be an *ideal* of L if

(i) $a, b \in I \Rightarrow a \vee b \in I$

(ii) $a \in I,\ l \in L \Rightarrow a \wedge l \in I$.

If L is bounded then {o} is always an *ideal* of L and is called the *zero ideal*. The *n-ideal* of a *lattice* have been studies extensively by A.S.A Noor and M.A. Latif in [19], [20], [21], [22] and [23]. For a fixed element n of a *lattice L*, a *convex sub lattice* containing n is called an *n-ideal*. If L has "o", then replacing n by "o" an *n-ideal* becomes a filter by replacing n by 1. Thus the *idea of n-ideals* is a kind of generalization of both *ideals* and *filters of lattices*. So any result involving *n-ideals* of a *lattice L* will give a generalization of both *ideals* and *filters of lattices*. So any result involving *n-ideals* of a *lattice L* with give a generalizations of the results on *ideals* if $0 \in L$ and filters if $1 \in L$.

The set of all *n-ideals* of a *lattice* L is denoted by $I_n(L)$. Which is an *algebraic lattice* under set inclusion. Moreover, {n} and L are respectively the smallest and the largest elements of $I_n(L)$, while the set theoretic intersection is the *infimum*. For any two *n-ideals* H and K, of a *lattice* L, it is easy to say that $H \cap K = \{x : x = m(h, n, k)$ for some $h \in H, k \in K\}$

Where $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and

$H \vee K = \{x : h_1 \wedge k_1 \leq x \leq h_2 \vee k_2$, for some $h_1, h_2 \in H$. and $k_1, k_2 \in K$.

The *n-ideal* generated by $p_1, p_2, \ldots\ldots\ldots\ldots, p_m$ is denoted by $\langle p_1, p_2 \ldots\ldots\ldots\ldots, p_m \rangle_n$,

clearly, $\langle p_1, p_2 \ldots\ldots\ldots\ldots, p_m \rangle_n = \langle p_1 \rangle_n \vee \langle p_2 \rangle_n \vee \ldots\ldots\ldots\ldots \langle p_m \rangle_n$.

The *n-ideal* generated by a finite number of elements is called a finitely generated *n-ideal*. The set of all *finitely generated n-ideal* is denoted by $F_n(L)$, is a *lattice*. The *n-ideal* generated by a single element is called a *principal n-ideal*. The set of all *principal n-ideals* of a *lattice* L is denoted by $P_n(L)$. We have $\langle a\rangle_n = \{x \in L : a \wedge n \le x \le a \vee n\}$.

Standard element of a *Lattice*: An element s of a *lattice* L is called *standard* if $x \wedge (s \vee y) = (x \wedge s) \vee (x \wedge y)\, for\, all\, x, y \in L.$

Theorem 5.1.1: If $I_n(L)$ be an *n-ideal* of a *lattice* L is *distributive* if and only if $(I \vee \langle a\rangle_n) \cap (I \vee \langle b\rangle_n) = I \vee (\langle a\rangle_n \cap \langle b\rangle_n).\ for\ a, b \in L.$

Proof: Let J and K be two *ideals* of a *lattice* L and I is *distributive lattice*. Again let $x \in (I \vee J) \cap (I \vee K)$.

Then $x \in I \vee J\ and\ x \in I \vee K$.

Then $i_1 \wedge j_1 \le x \le i_2 \vee j_2\ and\ i_3 \wedge k_3 \le x \le i_4 \vee k_{4.}$

for some $i_1, i_2, i_3, i_4 \in I,\ j_1, j_2 \in J\ and\ k_3, k_4, \in K.$

Now, $n \le x \vee n \le i_2 \vee j_2 \vee n$ implies that $x \vee n \in I \vee \langle j_2 \vee n\rangle_n$

Similarly, $n \le x \vee n \le i_4 \vee k_4 \vee n$ implies that

Thus, $x \vee n \in (I \vee \langle J_2 \vee n\rangle_n) \subseteq (I \vee (J \cap K))..$

If I is *distributive*, then the condition clearly holds from the definition. To prove the converse, suppose given equation holds for all $a, b \in L$, let J and K be any two *n-ideals* of L.

Obviously, $I \vee (J \cap K) \subseteq (I \vee J) \cap (I \vee K)$. ■

Theorem.5.1.2: An element a of a *lattice* L is *distributive* if and only if the relation θ_a defined by $x \equiv y\theta_a$ if and only if $x \vee a = y \vee a$ is a *congruence*.

Theorem5.1.3: If I be *n-ideal* of a *lattice* L, is *distributive* if and only if the relation $\Theta(I)$ defined by $y \equiv x\, \Theta(I)\ \forall x, y \in L$ if and any if

$x \vee i_1 = y \vee i_1$ and $x \wedge i_2 = y \wedge i_2$ for some $i_1, i_2 \in I$ in the *congruence* generated by I.

Proof: At first we shall show that

$y \equiv x\ \Theta(I)$ *if and only if* $\langle y\rangle_n = \langle x\rangle_n \Theta_1$ in $I_n(L)$. Let $y \equiv x\Theta(I)$,

Then $y \vee i_1 = x \vee i_1$ *and* $y \wedge i_2 = x \wedge i_2$. *for some* $i_1, i_2 \in I$.

Now $y \wedge i_2 = x \wedge i_2 \le x \le x \vee i_1 = y \vee i_1$ implies that $x \in \langle y\rangle_n \vee I$.

Therefore, $\langle y\rangle_n \vee I = \langle x\rangle_n \vee I$.

Which implies that $\langle y\rangle_n \equiv \langle x\rangle_n \Theta(I)$ *in* $I_n(L)$.

Conversely, $\langle y\rangle_n = \langle x\rangle_n \Theta_1$ in $I_n(L)$

then $\langle y\rangle_n \vee I = \langle x\rangle_n \vee I$.

Again, $y \in \langle x\rangle_n \vee I$, *and os* $x \wedge n \wedge i_1 \le y \le x \vee n \vee i_2$.

Similarly y, $x \wedge n \wedge i_3 \le x \le y \vee n \vee i_4$.

This $y \le x \vee n \vee i_2 \le y \vee n \vee i_2 \vee i_4$

Which implies $y \vee n \vee i_2 \vee i_4 = x \vee n \vee i_2 \vee i_4$.

Similarly $y \wedge n \wedge i_1 \wedge i_3 = x \wedge n \wedge i_1 \wedge i_3$.

That is $y \vee i = x \vee i$ *and* $y \wedge j = x \wedge j$

Where $i = n \vee i_2 \vee i_4$ *and* $j = n \wedge i_1 \wedge i_3$.

Therefore $y \equiv x\Theta(I)$.

Above proof shows that $\Theta(I)$ is a *congruence* in L if and only if Θ_1 is a *congruence* in $I_n(L)$. But by lemma 5.1.2 Θ_1 is a *congruency* if and only if I is *distributive* in $I_n(L)$ and completes the proof. ■

Theorem: 5.1.4: If n be a *neutral element of a lattice* L and $P_1 \wedge n, \ldots\ldots\ldots, P_m \vee n$ are *distributive* in L. Then finitely generated *n-ideals* $\langle P_1, P_2, \ldots\ldots\ldots, P_m\rangle_n$ is distributive.

Proof: Suppose $P_1 \wedge n, \ldots\ldots\ldots, P_m \wedge n$ are *dual distributive* and $P_1 \vee n, \ldots\ldots\ldots\ldots, P_m \vee n$ are *distributive* in a *lattice* L. let $J, K \in I_n(L)$.

Suppose $x \in (\langle P_1, \ldots\ldots, P_m \rangle_n \vee J) \cap (\langle P_1 \ldots\ldots\ldots, P_m \rangle_n \vee K)$.

Then by using *distributivity* of $P_1 \vee n, \ldots\ldots\ldots\ldots P_m \vee n$.

We have, $x \leq (P_1 \vee \ldots\ldots\ldots \vee P_m \vee n \vee j) \wedge (P_1 \vee \ldots\ldots\ldots \vee P_m \vee n \vee K)$

$=(p_1 \vee n) \vee [(p_2 \ldots\ldots\ldots\ldots \vee p_m \vee n \vee j) \wedge (p_2 \vee \ldots\ldots\ldots \vee p_m \vee n \vee k)]$

for some $j \in J$, $k \in K$.

$=(p_1 \vee n) \vee (p_2 \vee n) \vee \ldots\ldots\ldots\ldots\ldots\ldots \vee (p_m \vee n) \vee (j \wedge k)$.

$= (p_1 \vee p_2 \vee \ldots\ldots\ldots\ldots \vee p_m \vee n) \vee [(j \vee n) \wedge (k \vee n)]$

But, $(j \vee n) \wedge (k \vee n) = m(j \vee n, n, k \vee n) \in J \cap K$.

Dually using the dual *distributivity* of $p_1 \wedge n, \ldots\ldots\ldots . p_m \wedge n$,

It is easy to see that,

$p_1 \wedge p_2 \wedge \ldots\ldots\ldots\ldots \wedge p_m \wedge n \wedge ((J_1 \wedge n) \vee (K_1 \wedge n)) \leq x$

for some $j_1 \in J$, $k \in K$.

Moreover, $(j_1 \wedge n) \vee (k_1 \wedge n) = m(j_1 \wedge n, n, k_1 \wedge n) \in J \cap K$.

Thus by convexity, Since the reverse in inclusion is $x \in \langle p_1, p_2, \ldots\ldots\ldots\ldots, p_m \rangle_n \vee (J \cap K)$.

so $\langle p_1, p_2, \ldots\ldots\ldots p_m \rangle_n$ is *distributive*.

It should be mentioned that the converse of above result is not necessarily true. For example consider the following *lattice*.

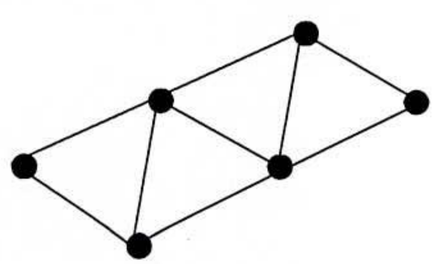


Figure: 5.1

Here $\langle a,f\rangle_n = L$ which is of course *distributive* in $I_n(L)$.

But neither $a \vee n$ nor $f \vee n$ is *distributive* in L.

But the converse holds for *principal n-ideals*. ■

Definition (neutral element of a lattice): An element $n \in L$ is called *neutral* if it is *standard* and for all $x, y, \in L$. $n \wedge (x \vee y) = (n \wedge y)$. By [15], we know that $n \in L$ is *neutral* if and only if for all $x, y \in L$.

$m(x,n,y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n) = (x \vee y) \wedge (x \vee n) \wedge (y \vee n)$.

Ofcourse 0 and 1 of a *lattice* are always *neutral*, of course every element of a *distributive lattice* is *distributive, standard* and *neutral*.

Theorem : 5.1.5: Suppose n be a *neutral* element of $I_n(L)$. Then $a \wedge n$ is *dual distributive and* $a \vee n$ is *distributive* if and only if $\langle a\rangle_n$ is *distributive*.

Proof: Suppose $\langle a\rangle_n$ is *distributive* and $b, c \in L$.

Then $\langle a\rangle_n \vee (\langle b\rangle_n \cap \langle c\rangle_n) = (\langle a\rangle_n \vee \langle b\rangle_n) \cap (\langle a\rangle_n \vee \langle c\rangle_n)$.

Thus, $[a \wedge n, a \vee n] \vee ([b \wedge n, b \vee n] \cap [a \wedge c \wedge n, a \vee c \vee n])$

$= [a \wedge b \wedge n, a \vee b \vee n] \cap [a \wedge c \wedge n, a \vee c \vee n]$

This implies,

$a \wedge n \wedge ((b \wedge n) \vee (c \wedge n)) = (a \wedge b \wedge n) \vee (a \wedge c \wedge n)$

and $a \vee n \vee ((b \vee n) \wedge (c \vee n)) = (a \vee b \vee n) \wedge (a \vee c \vee n)$

That is $(a \wedge n) \wedge (b \vee c) = (a \wedge b \wedge c) \vee (a \wedge c \wedge n)$

and $(a \vee n) \vee (b \wedge c) = (a \vee b \vee n) \wedge (a \vee c \vee n)$,

as n is *neutral* Therefore , $a \wedge n$ is *dual distributive* and $a \vee n$ is *distributive* in a lattice L.

To prove the converse, suppose $a \wedge n$ is *dual distributive* and $a \vee n$ is *distributive*. Then by theorem 5.1.4 $\langle a\rangle_n$ is *distributive*.

Theorem: 5.1.6: Let I be a *distributive n-ideal* of a *lattice L*. Then $I_n(L)$ is isomorphic with the *lattice* of all *n-ideals* of L containing I, that is, with [I, L] in $I_n(L)$.

Proof: Let φ be the *homomorphism* x$\rightarrow[x]\Theta(I)$ onto$\frac{L}{\Theta(I)}$.

Then it is easily to see that the map $\psi : K \rightarrow K\varphi^{-1}$ maps $I_n(\frac{L}{\Theta(I)})$ into [I,L]. To show that Ψ is onto, it is sufficient to see that [J] $\Theta(I) = J$ for all $j \supseteq I$. Indeed, if $j \in J$ *and* $a \in L$ with $j \equiv a\Theta(I)$, then $J \vee i = a \vee i$ *and* $j \wedge i_1$ for some $i, i_1 \in I$. Thus $j \wedge i_1 \leq a \leq j \vee i$. Since $j \wedge i_1, j \vee i \in j$, so by convexity $a \in J$. Moreover, Ψ is obviously an *isotone* and *one-one*. Therefore, it is an *isomorphism*. ■

1. n-Ideal of a lattice.

A non-empty subset I of a *lattice* L is said be an *ideal* of L if

(i) $a,b \in I \Rightarrow a \vee b \in I$

(ii) $a \in I,\ l \in L \Rightarrow a \wedge l \in I.$

If L is bounded then {o} is always an *ideal* of L and is called the *zero ideal*. The *n-ideal* of a *lattice* have been studies extensively by A.S.A Noor and M.A. Latif in [19], [20], [21], [22] and [23]. For a fixed element n of a *lattice L*, a *convex sub lattice* containing n is called an *n-ideal*. If L has "o", then replacing n by "o" an *n-ideal* becomes a filter by replacing n by 1. Thus the *idea of n-ideals* is a kind of generalization of both *ideals* and *filters of lattices*. So any result involving *n-ideals* of a *lattice L* will give a generalization of both *ideals* and *filters of lattices*. So any result involving *n-ideals* of a *lattice L* with give a generalizations of the results on *ideals* if $0 \in L$ and filters if $1 \in L$.

The set of all *n-ideals* of a *lattice* L is denoted by $I_n(L)$. Which is an *algebraic lattice* under set inclusion. Moreover, {n} and L are respectively the smallest and the largest elements of $I_n(L)$, while the set theoretic intersection is the *infimum*. For any two *n-ideals* H and K, of a *lattice* L, it is easy to say that $H \cap K = \{x : x = m(h,n,k)$ for some $h \in H, k \in K\}$

Where $m(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and

$H \vee K = \{x : h_1 \wedge k_1 \le x \le h_2 \vee k_2$, for some $h_1, h_2 \in H$. and $k_1, k_2 \in K$.

The *n-ideal* generated by $p_1, p_2, \ldots\ldots, p_m$ is denoted by $\langle p_1, p_2 \ldots\ldots, p_m \rangle_n$,

clearly, $\langle p_1, p_2 \ldots\ldots, p_m \rangle_n = \langle p_1 \rangle_n \vee \langle p_2 \rangle_n \vee \ldots\ldots \langle p_m \rangle_n$.

2. Modular n-ideals of a lattice

Introduction: An *n-ideal M* of a *lattice L* is called a *modular n-ideal* if it is a *modular* element of the *lattice* $I_n(L)$. In other words is called *Modular* if for all $H,K \in I_n(L)$ with $K \subseteq I$,

$H \cap (M \vee K) = (H \cap M) \vee K$.

We know from [24] that a *lattice L* is *modular* if and only if its every element is *modular.* Also from [20]. We know that for a *neutral* element n of a *lattice L, L* is *modular* if and only if $I_n(L)$ is so.

Thus for a *neutral* element *n,* the *lattice L* is *modular* if and only if it every *n-ideal* is *modular*. Following result gives a characterization of *modular n-ideals* of a *lattice.*

Theorem :5.2.1: An *n-ideal M* of a *lattice L* is *modular* if and only if for any $J, K \in P_n(L)$ with $K \subseteq J, (J \cap M) \vee K = J \cap (M \vee K)$.

Proof: Suppose M is *modular lattice* of $I_n(L)$. The above relation obviously holds from the definition. Conversely, Suppose $(J \cap K) \vee K = J \cap (M \vee K)$ for all $J,K \in P_n(L)$ with $K \subseteq J$. Let $S.T \in I_n(L)$ with $T \subseteq S$.

We have to show that, $(S \cap M) \vee T = S \cap (M \vee T)$.

Clearly, $(S \cap M) \vee T \subseteq S \cap (M \vee T)$.

To prove the reverse inclusion let $x \in S \cap (M \vee T)$.

Then $x \in S$ and $x \in (M \vee T)$.

Then, $m \wedge t \leq x \leq m_1 \vee t_1$. for some $m_1 m_1 \in M, t, t_1 \in T$.

Thus, $x \vee n \leq x \leq m_1 \vee t_1 \vee n$.

Which implies $x \vee n \in \langle m_1 \vee n \rangle_n \vee \langle t_1 \vee n \rangle_n \subseteq M \vee \langle t_1 \vee n \rangle_n$

Moreover, $x \vee n \in \langle x \vee t_1 \vee n \rangle_n$ and $\langle x \vee t_1 \vee n \rangle_n \supseteq \langle t_1 \vee n \rangle_n$.

Hence by the given Condition, $x \vee n \in \langle x \vee t_1 \vee n \rangle_n \cap (M \vee \langle t_1 \vee n \rangle_n)$

$= ((\langle x \vee t_1 \vee n \rangle_n \cap M) \vee \langle t_1 \vee n \rangle_n \subseteq (S \cap M) \vee T.$

By a dual proof of above we can easily see that $x \wedge n \in (S \cap M) \vee T.$

Thus by Convexity $x \in (S \cap M) \vee T.$ ■

Theorem.5.2.2: Suppose n is a *neutral* element of a *lattice* L. Then $M \in I_n(L)$ is *modular* if and only if for and only if for any $x \in M \vee \langle y \rangle_n$ with $\langle Y \rangle_n \subseteq \langle x \rangle_n$, $x = (x \wedge m_1) \vee (x \wedge y) = (x \vee m_2) \wedge (x \vee y)$ for some $m_1, m_2 \in M$.

Proof: Suppose M is *modular* and $x \in M \vee \langle y \rangle_n$.

Then $x \in \langle x \rangle_n \cap (M \vee \langle y \rangle_n = ((\langle x \rangle_n \cap M) \vee \langle y \rangle_n.$

This impels $p \wedge y \wedge n \leq x \leq q \vee y \vee n.$

for some $p, q \in \langle x \rangle_n \cap M.$

By Proposition 1.1.1, $q \in \langle x \rangle_n \cap M.$

Implies that $q = (x \vee q) \vee (x \wedge n) \vee (q \wedge n) = (x \wedge (q \vee n)) \vee (q \wedge n).$

Thus, $x \vee n \leq (x \wedge (q \vee n)) \vee y \vee n \leq x \vee n,$

which implies $x \vee n = (x \wedge (q \vee n)) \vee y \vee n =$

$(x \wedge (q \vee n)) \vee y \wedge (x \vee n)) \vee n.$

$= (x \wedge (q \vee n)) \vee (x \wedge y) \vee n$, an n is *neutral*. Hence by the neutrality of n again, $x = x \wedge (x \vee n) = x \wedge [x \wedge (q \vee n)) \vee (x \wedge y) \vee n]$

$= (x \wedge [(x \wedge (q \vee n)) \vee (x \wedge y)]) \vee (x \wedge n)$

$= (x \wedge (q \vee n)) \vee (x \wedge y) \vee (x \wedge n).$

$= (x \wedge (q \vee n)) \vee (x \wedge y),$

Which is the first relation where $m_1 = q \vee n \in M.$

A dual Proof of above establishes the second relation.

Conversely, let $\langle y\rangle_n \subseteq \langle x\rangle_n$, By theorem 5.2.1, we need to show that

$\langle x\rangle_n \cap (M \vee \langle y\rangle_n) = .\langle x\rangle_n \cap (M \vee \langle y\rangle_n) =$

Clearly R.H.S $\subseteq$L.H.S.

To prove the reverse inclusion let $t \in \langle x\rangle_n \cap (M \vee \langle y\rangle_n$.

Then $t \in \langle x\rangle_n$ *and* $t \in M \vee \langle y\rangle_n$.

Then $m \wedge y \wedge n \leq t \leq m_1 \vee y \vee n$. for some $m, m_1 \in M$.

Thus, $t \vee y \vee n \leq m_1 \vee y \vee n,$ *and so* $t \vee y \vee n \in M \vee \langle y \vee n\rangle_n$

and $\langle y \vee n\rangle_n \subseteq \langle t \vee y \vee n\rangle_n$.

So by the given condition $t \vee y \vee n = ((t \vee y \vee n) \wedge m') \vee (y \vee n)$ for some $m' \in M$. Since $t, y \in \langle x\rangle_n$,

So $t \vee y \vee n \in \langle x\rangle_n$.

Moreover, by the neutrality of n,

$((t \vee y \vee n) \wedge m') \vee (y \vee n)$

$= ((t \vee y \vee n) \wedge (m' \vee n)) \vee y.$

$= m(t \vee y \vee n, n, m') \vee y \in (\langle x\rangle_n \cap M) \vee \langle y\rangle_n$.

Therefore, $t \vee y \vee n \in (\langle x\rangle_n \cap M) \vee \langle y\rangle_n$.

By the dual proof we can show that $t \wedge y \wedge n \in (\langle x\rangle_n \cap M) \vee \langle y\rangle_n$.

Thus, by the convexity, $t \in (\langle x\rangle_n \cap M) \vee \langle y\rangle_n$.

Therefore, $\langle x\rangle_n \cap (M \vee \langle y\rangle_n) = (\langle x\rangle_n \cap M) \vee \langle y\rangle_n$.

and so by Theorem 5.2.1, *M* is *Modular*. ■

Theorem.5.2.3: Let *M* is a *modular n-ideal* and *I* be any n-ideal of *L* and I be only n-ideal of L and n be a neutral element of a lattice L. Then $I_n(L)$ is principal if $M \vee I = \langle a\rangle_n$ *and* $M \cap I = \langle b\rangle_n$.

Theorem.5.2.4: Let I and J be *ideals* of a join *Semi-lattice* then $I \vee J = \{t / t \le i \vee j, i \in I, j \in J\}$.

Proof: Suppose a *modular lattice L* is *distributive*. Then clearly, R.H.S $\le I \vee J$. Now let, $t \in I \vee J$.

Then we have $t \le i \vee j$ for some $i \in I \; and \; j \in J$.

$\therefore t = t \wedge (i \vee j)$.

$= (t \wedge i) \vee (t \wedge j)$

$= i' \vee j' \; where \; i' = t \wedge i \in I$ and $j' = t \wedge j \in J$.

Hence $t \in R.H.S.$

$\therefore I \vee J \le R.H.S.$

Therefore, $I \vee J = \{i \vee j / i \in I, j \in J\}$

Conversely, Suppose L is not *distributive*.

Therefore it contains elements a,b,c is M_5 or N_5.

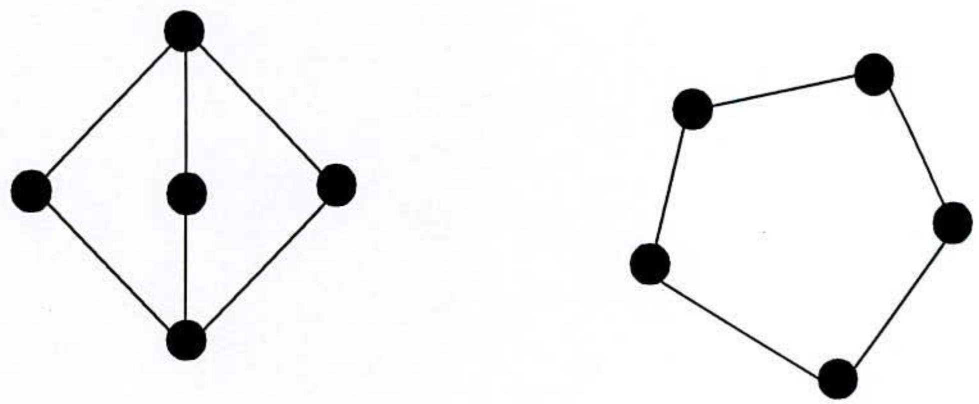


Figure-5.2

Let $I = (b] \; and \; J = (c]$ since $a \le b \vee c$, Then we have $a \in I \vee J$.

However a has no representation as in given theorem.For if $a = i \vee j, i \in I, J \in J$

Then $j \leq a$. also $j \leq c$

Therefore $j \leq a \wedge c < b$. Thus $j \in I$

Which gives a contradiction.

Hence L is distributive. ■

REFERENCES:

1. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloq. Publ. 25. 3rd Edition. (1967).
2. W. H. Cornish n-Normal Lattices,Proc.Amer. Math. Sco. 1(45) (1974), 48-54.
3. W. H. Cornish, Pseudocomplemented modular semilattices. J. Austral. Math. Soc. 18 (1974), 239 – 251.
4. W. H. Cornish, The multiplier extension of a distributive lattice, J. Algebra 32 (1974), 339 – 355.
5. W. H. Cornish , Normal Lattices J. Austral. Math. Soc. 14 (1972), 200 – 215.
6. A. Davey. Some annihilator conditions on distributive lattices. Alg. Univ. 3(4) (1974). 316- 322.
7. G. Gratzer, lattice theory, First concepts and distributive lattices, Freeman, San Francisco, 1971.
8. G. Gratzer, General lattice theory, Birkhauser, Verlag. Basel and Stuttgart, 1978.
9. G. Gratzer & E. T. Schmidt, On a problem of M. H. Stone. Acta. Math. Acad. Sci. Hung. 8(1957), 455 – 460.
10. Katrinak, Congruence lattices of distributive P- algebras, Algebra Universalis, 7 (1977), 265 – 271.
11. Katrinak, Remarks on Stone lattices. Math. Fyz. casopis 16 (1966). 128 – 142.
12. T. Katrinak & P. medarly, constructions of p-algebras . Algebra Universalis. 17(1983), 288-316.
13. H. Lakser, principal congruence's of pseudocomplemented distributive lattices, proc. Amer. Math. Soc.37(1973) , 32 – 36.
14. H. Lakser, The structure of pseudocomplemented distributive

lattices I. Subdirec decomposition. Trans. Math. Amer. Soc. 156 (1971). 335 – 342.

15. K. B. Lee, Equations classes of a distributive seudocomplemented lattices, Canad. J. math. 22 (1970) 881 – 891.

16. P. Medarly, A characterization of modular pseudocomplemented semilattices, Colloq. Math. Soc. J. Bolyai. Szeged. 14 (1975), 231 – 248.

17. D. E. Rutherford, Introduction to lattice theory, Oliver and Boyd. 1965.

18. Vijay K. Khanna, Lattices and Boolean algebras (First concepts).

19. A. S. A. Noor & M. A. Ail- Minimal prime n-ideal of a lattice, The Journal of Science Math. Bangle University Review, India 9 (1), (1998), 32 – 36.

20. A. S. A. Noor & M. A. Latif- Standard n-ideal of a lattice, SEA. Bull. Math. 4 (1997), 185 – 192.

21. A. S. A. Noor & M. A. Latif, Properties of n-ideals of a lattice, SEA. Bull. Math. 24(2000), 1-7.

22. A. S. A. Noor & M. A. Latif, Two isomorphism theorem for standard n-ideals of a lattice, presented in the 5th west Bengal science congress, India. Math. 4 (1997), 185-192.

23. A. S. A. Noor & M. A. Latif, Permutability of standard n- congruence's, Rajshahi University Studies (Part-B), journal of science Vol. 23- 24, (1995- 96), 211- 214.

24. M.R. Talukder & A.S.A. Noor, Standard ideals of a join semilattices directed below, SEA. Bull. Math. 21(4) ,(1997), 435 – 438.

25. T. P. Speed- On Stone lattices, J. Austral. Math. Soc. 9 (1969), 297 – 307.

26. J. Verlet- On the characterization of Stone lattices, Acta Sci. Math. (Szeged) 27 (1996), 81 – 84.

2. Modular n-ideals of a lattice

Introduction: An *n-ideal M* of a *lattice L* is called a *modular n-ideal* if it is a *modular* element of the *lattice* $I_n(L)$. In other words is called *Modular* if for all $H,K \in I_n(L)$ with $K \subseteq I$,

$H \cap (M \vee K) = (H \cap M) \vee K.$

We know from [24] that a *lattice L* is *modular* if and only if its every element is *modular.* Also from [20]. We know that for a *neutral* element n of a *lattice L, L* is *modular* if and only if $I_n(L)$ is so.

Thus for a *neutral* element *n,* the *lattice L* is *modular* if and only if it every *n-ideal* is *modular*. Following result gives a characterization of *modular n-ideals* of a *lattice*.

Theorem :5.2.1: An *n-ideal M* of a *lattice L* is *modular* if and only if for any $J, K \in P_n(L)$ with $K \subseteq J, (J \cap M) \vee K = J \cap (M \vee K)$.

Proof: Suppose *M* is *modular lattice* of $I_n(L)$. The above relation obviously holds from the definition. Conversely, Suppose $(J \cap K) \vee K = J \cap (M \vee K)$ for all $J,K \in P_n(L)$ with $K \subseteq J$. Let $S.T \in I_n(L)$ with $T \subseteq S$.

We have to show that, $(S \cap M) \vee T = S \cap (M \vee T)$.

Clearly, $(S \cap M) \vee T \subseteq S \cap (M \vee T)$.

To prove the reverse inclusion let $x \in S \cap (M \vee T)$.

Then $x \in S$ and $x \in (M \vee T)$.

Then, $m \wedge t \leq x \leq m_1 \vee t_1$. for some $m_1 m_1 \in M, t, t_1 \in T$.

Thus, $x \vee n \leq x \leq m_1 \vee t_1 \vee n$.

Which implies $x \vee n \in \langle m_1 \vee n \rangle_n \vee \langle t_1 \vee n \rangle_n \subseteq M \vee \langle t_1 \vee n \rangle_n$

Moreover, $x \vee n \in \langle x \vee t_1 \vee n \rangle_n$ and $\langle x \vee t_1 \vee n \rangle_n \supseteq \langle t_1 \vee n \rangle_n$.