STEADY MHD FREE CONVECTION AND MASS TRANSFER FLOW WITH THERMAL DIFFUSION, DUFOUR EFFECT AND LARGE SUCTION

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Abstract

The numerical studies are performed to examine the MHD free convection and mass transfer flow with thermal diffusion and dufour effects past an infinite vertical porous plate. Method of Superposition and the Shooting method are used as main tools for the numerical approach. The studies of the flow feature mentioned above are made in different sections taking different aspects of the flow that are of practical importance. These studies are mainly based on the similarity approach. At first similarity solutions have been obtained for the Unsteady MHD free convection and mass transfer flow past an infinite vertical porous plate taking into account the thermal diffusion and Dufour effects. Impulsively started plate moving in its own plane is considered. Similarity equations of the corresponding momentum, energy and concentration equations are derived by introducing a time dependent length scale which infact plays the role of a similarity parameter. The suction velocity is taken to be inversely proportional to this parameter. The momentum, energy and concentration e quations are solved numerically by applying the method of superposition. The above flow problem has further been considered in a steady two dimensional problem of the MHD free convection and mass transfer flow past an infinite vertical porous plate taking into account the Soret and Dufour effects. The similarity solutions of the governing equations are obtained by employing the usual similarity technique based on large suction. The effects on the velocity, temperature, concentration, skin-friction, Nusselt number and the Sherwood number of the various important parameters entering into the problems separately are discussed for each problem with the help of graphs and tables.

Finally, a general dis cussion o n t he overall solutions o f the problems c onsidered in t he dissertation are sorted out.

Introduction

The aim of this dissertation is to make some numerical calculations on Magnetohydrodynamic free convection and mass transfer flows which have been of interest to the engineering community and to the investigators dealing with the problems in geophysics and astrophysics. The thermal diffusion and Dofour effects, which are often neglected in free convection and mass transfer processes, has been included in the analyses for the above mentioned calculations. The analyses so produced in fact arouse out of the natural tendency to investigate a subject that may be said to relate to some academic types of problems of solving the equations of the fluid mechanics. The results of this investigation may not have direct practical applications but are relevant to the problems mentioned above. It is however, to be mentioned that the thermal instability investigations of natural convection MHD flows have direct application to problems in geophysics and astrophysics. The natural convection processes involving the combined mechanism of heat and mass transfer are encountered in many natural processes, in many industrial applications and in many chemical processing systems. In our analyses the combined buoyancy effect arising from the simultaneous diffusion of thermal energy and chemical species are considered on the MHD flow of electrically conducting fluid under the action of a transversely applied magnetic field. Considering various aspects of an MHD free convection and mass transfer flow, the analyses presented here, as mentioned above, are classified mainly into two different numerical methods. one is superposition method for solving the linear ordinary coupled equations and the other is shooting method for solving the nonlinear ordinary coupled equations.

In chapter 1, available information regarding MHD heat and mass transfer flows along with various effects are summarized and discussed from both analytical and numerical point of view. In chapter 2, the basic governing equations related to the problems considered thereafter are shown in standard form. In chapter 3, the calculation techniques for different problems are discussed. In chapter 4, a specific problem of the Unsteady MHD free convection and mass transfer flow past an infinite vertical porous plate taking into account the thermal diffusion and Dufour effects are considered. In chapter 5, we have considered a steady two dimensional problem of MHD free convection and mass transfer flow past an infinite vertical porous plate taking into account the thermal diffusion and Dufour effects based on large suction. Finally a general discussion on all the problems dealt is produced with some conclusive remarks.

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Chapter 1

Available Information on MHD Flow

1.1. Magneto hydrodynamics (MHD)

Magneto hydrodynamics MHD is that branch of continuum mechanics, which deals with the flow of electrically conducting fluids in electric and magnetic fields. Probably, the largest advance towards an understanding of such phenomena comes from the field astrophysics. It has long been suspected that most of the matter in the universe is in the form of Plasma or highly ionized gaseous state, and much of the basic knowledge in the area of Electromagnetic fluid dynamics evolved from these studies.

As a branch of plasma physics, the field of Magneto-hydrodynamics (MHD) consists of the study of a continuous, electrically conducting fluid under the influence of electromagnetic fields. Originally, MHD included only the study of partially ionized gases as well as the other names have been suggested, such as magneto-fluid mechanics, or magneto-aerodynamics, but original nomenclature has persisted. The essential requirement for problems to be analyzed under the laws of MHD is that the continuum approach be applicable.

Many natural phenomena and engineering problems are susceptible to MHD analysis. It is conducting fields and magnetic fields that are present in and around heavenly bodies. Engineers employ MHD principles in the design of heat exchangers, pumps and flow meters, in space vehicle propulsion, control and re-entry, in creating novel power generating systems, and in developing confinement schemes for controlled fusion.

The most important application of MHD is in the generation of electrical power with the flow of an electrically conducting fluid through a transverse magnetic field. Recently, experiments with ionized gases have been performed with the hope of producing power on a large scale in stationary plants with large magnetic fields. Cryogenic and super conducting magnets are required to produce these very large magnetic fields. Generation of MHD power on a smaller scale is of interest for space applications.

It is generally known that, to convert the heat energy into electricity, several intermediate transformations are necessary. Each of these steps means a loss of energy. This naturally limits the overall efficiency, reliability and compactness of the conversion process. Methods for direct conversion to energy are now increasingly receiving attention. Of these, the fuel converts the chemical energy of fuel directly into electrical energy; fusion energy utilizes the energy released when two hydrogen molecule fuses into a heaver one, and thermoelectrically power generation uses a thermocouple. Magneto hydrodynamic power generation is another important new process that is receiving worldwide attention.

Faraday (1832) carried out experiments with the flow of mercury in glass tubes placed between poles of a magnet and discovered that a voltage was induced across the tube, due the motion of the mercury across the magnetic fields, perpendicular to the direction of flow and to the magnetic field. He observed that the current generated by the induced voltage interacted with the magnetic field to slow down the motion of the fluid and this current produced its own magnetic field that obeyed Ampere's right hand rule and thus, in turn distorted the magnetic field.

The first astronomical application of the MHD theory occurred in 1899 when Bigalow suggested that the sun was gigantic magnetic system. Alfven (1942) discovered MHD waves in the sun. These waves are produced by disturbances, which propagate simultaneously in the conducting fluid and the magnetic field.

The current trend for the application of magneto-fluid dynamics is toward a strong magnetic field (so that the influence of electromagnetic force is noticeable) and toward a low density of the gas (such as in space flight and in nuclear fusion research). Under this condition the Hall current and ion slip current become important.

1.2. The important dimensionless parameters

Reynolds number

It is the most important parameter of the fluid dynamics of a viscous fluid. It represents the ratio of the inertia force to the viscous force and is defined as

$$R_e = \frac{inertia\ force}{viscous\ force} = \frac{\rho U^2 L^2}{\mu U L} = \frac{UL}{\upsilon}$$

Where U, L, ρ and μ are the characteristic values of velocity, length, density and coefficient of viscosity of the fluid respectively. When the Reynolds number of the system is small the viscous force is predominant and the effect of viscosity is important in the whole velocity field. When the Reynolds number is large the inertial force is predominant, and the effects of viscosity is important only a narrow region, near the solid wall or other restricted region, which is known as boundary layer. If the Reynolds numbers is enormously large, the flow becomes turbulent.

Prandtl number P_r

The Prandtl number is the ratio of kinematic viscosity to thermal diffusivity and may be written as follows

$$P_r = \frac{Kinemetic\ viscisity}{Thermal\ diffusivity} = \frac{\upsilon}{k / C_p \rho}$$

The value of v shows the effect of viscosity of the fluid. The smaller the value of v is, the narrower is the region which is affected by viscosity and which is known as the boundary layer region. The value of $\frac{k}{C_p \rho}$ shows the thermal diffusivity due to heat conduction. The smaller the value of $\frac{k}{C_p \rho}$ is, the narrower is the region which is affected by the heat conduction and it is known as thermal boundary layer. Thus the Prandtl number shows the relative importance of heat conduction and viscosity of a fluid. For a gas the Prandtl number is of order of unity.

Magnetic Force Number M

This is obtained from the ratio of the magnetic force to the inertia force and is defined as

$$M = \frac{\mu_e B_0^2 \sigma' L}{U \rho}$$

Schmidt number Sc

This the ratio of the viscous diffusivity to the chemical molecular diffusivity and is defined as

$$S_c = \frac{Viscous\ diffusivity}{Chemical\ molecular\ diffusivity} = \frac{\upsilon}{D_M}$$

Grashof number G_r

This is defined as

$$G_r = \frac{g\beta \nabla T L^3}{\upsilon^2}$$

and is a measure of the relative importance of the buoyancy and viscous forces. The larger it is, the stronger is the convective current.

Modified Grashof number G_m

This is defined as

$$G_m = \frac{g\beta^{*} \nabla C L^3}{\upsilon^2}$$

Soret number S_0

This defined as

$$S_0 = \frac{D_m k_T (T_w - T_\infty)}{T_M \upsilon (C_w - C_\infty)}$$

Dufour Number D_f

This is defined as

$$D_f = \frac{D_M k_T (C_w - C_\infty)}{C_s C_p \upsilon (T_w - T_\infty)}$$

1.3. Suction and Injection

For boundary layer flows with adverse pressure gradients, the boundary layer will eventually separate from the surface. Separation of the flow causes many undesirable features over the whole field; for instance if separation occurs on the surface of an airfoil, the lift of the airfoil will decrease and the drag will enormously increase. In some problems we wish to maintain laminar flow without separation. Various means have been proposed to prevent the separation of boundary layer; suction and injection are two of them.

The stabilizing effect of the boundary layer development has been well known for several years and till to date suction is still the most of efficient, simple and common method of boundary layer control. Hence, the effect of suction on hydro-magnetic boundary layer is of great interest in astrophysics. It is often necessary to prevent separation of the boundary layer to reduce the drag and attain high lift values.

Many authors have made mathematical studies on these problems, especially in the case of steady flow. Among them the name of Cobble (1977) may be cited who obtained the conditions under which similarity solutions exist for hydro-magnetic boundary layer flow past a semi-infinite flat plate with or without suction. Following this, Soundalgekar & Ramanamurthy (1980) analyzed the thermal boundary layer. Then Singh (1980) studied this problem for large values of suction velocity employing asymptotic analysis in the spirit of Nanbu(1971). Singh & Dikshit(1988) have again adopted the asymptotic method to study the hydro-magnetic effect on the boundary layer development over a continuously moving plate. In a similar way Bestman (1990) studied the boundary layer flow past a semi-infinite heated porous plate for two-component plasma.

On the other hand, one of the important problems faced by the engineers engaged in high-speed flow is the cooling of the surface to avoid the structural failures as a result of frictional heating and other factors. In this respect the possibility of using injection at the surface is a measure to cool the body in the high temperature fluid. Injection of secondary fluid through porous walls is of practical importance in film cooling of turbine blades combustion chambers. In such application injection usually occurs normal to the surface and the injected fluid may be similar to or different from the primary fluid. In some recent applications, however, it has been recognized that the cooling efficiency can be enhanced by vectored injection at an angle other than 90^0 to the surface. A few workers including Inger & Swearn (1975) have theoretically proved this feature for a boundary layer. In addition, most previous calculations have been limited to injection rates ranging from small to moderate. Raptis et al. (1980) studied the free convection effects on the flow field of an incompressible, viscous dissipative fluid, past an infinite vertical porous plate, which is accelerated in its own plane. He considered that the fluid is subjected to a normal velocity of suction/injection proportional to $t^{-\frac{1}{2}}$ and the plate is perfectly insulated, i.e., there is no heat transfer between the fluid and the plate. Hasimoto (1957) studied the boundary layer growth on an infinite flat plate started at time t=0, with uniform suction or injection. Exact solutions of the Navier-Stokes equation of motion were derived for the case of uniform suction and injection, which was taken to be steady or proportional to $t^{-\frac{1}{2}}$. Numerical calculations are also made for the case of impulsive motion of the plate. In the case of injection, velocity profiles have injection points. The qualitative nature of the flow on both the suction and the cases are obtained form the result of the corresponding studies on steady boundary layer, so far obtained.

Large Suction

When the rate of suction is very high then it is called large suction. Singh (1988) studied the problem of Soundalgeker and Ramanamurthy(1980) for large value of suction parameter by making use of the perturbation technique, as has been done by Nanbu(1971). Later Singh and Dikshit(1988) studied the hydro-magnetic flow past a continuously moving semi-infinite porous plate employing the same perturbation technique. They also derived similarity solutions for large suction. The large suction in fact enabled them to obtain analytical solutions those are of immense value that compliment various numerical solutions. For the present p roblem s tudying on M HD f ree c onvection a nd m ass transfer f low w ith t hermal diffusion, Dufour effect and large suction we have to use the shooting method for getting the numerical solutions.

1.4. MHD Boundary layer and related transfer phenomena

Boundary layer phenomena occur when the influence of a physical quantity is restricted to small regions near confining boundaries. This phenomenon occurs when the non-dimensional diffusion parameters such as the Reynolds number and the Peclet number or the magnetic Reynolds number are large. The boundary layers are then the velocity and thermal or magnetic boundary layers, and each thickness is inversely proportional to the square root of the associated diffusion number. Prandtl fathered classical f luid dynamic boundary layer theory by observing, from experimental flows, that for large Reynold number, the viscosity and thermal conductivity appreciably influenced the flow only near a wall. When distant measurements in the flow direction are compared w ith a c haracteristic dim ension in that direction, transverse measurements compared with the boundary layer thickness, and velocities compared with the free stream velocity, the Navier Stiokes and energy equations can be considerably simplified by neglecting small quantities. The number of component equations is reduced to those in the flow direction and pressure is then only a function of the flow direction and can be determined from the inviscid flow solution. Also the number of viscous term is reduced to the dominant term and the heat conduction in the flow direction is negligible.

MHD boundary layer flows are separated in two types by considering the limiting cases of a very large o r a negligible small magnetic Reynolds num ber. W hen the magnetic field is oriented in an arbitrary direction relative to a confining surface and the magnetic Reynolds number is v ery s mall; t he f low di rection c omponent o f the magnetic i nteraction and t he corresponding Joule heating is only a function of the transverse magnetic field component and local velocity in the flow direction. Changes in the transverse magnetic boundary layer are negligible. The thickness of magnetic boundary layer is very large and the induced magnetic field is negligible. However, when the magnetic Reynolds numbers is large, the magnetic boundary layer thickness is small and is of nearly the same size as the viscous and thermal boundary layers and then the MHD boundary layer equations must be solved simultaneously. In this case, the magnetic field moves with the flow and is called frozen mass.

1.5. MHD and heat transfer

With the advent of hypersonic flight, the field of MHD, as defined above, which has been associated largely with liquid-metal pumping, has attracted the interest of aero dynamists. It is possible to alter the flow and the heat transfer around high-velocity vehicles provided that the air is sufficiently ionized. Further more, the invention of high temperature facilities such as the shock tube and plasma jet has provided laboratory sources of flowing ionized gas, which provide an insentive for the study of plasma accelerators and generators.

As a result of this, many of the classical problems of fluid mechanics have been reinvestigated. Some of these analyses arouse out of the natural tendency of scientists to investigate a new subject. In this case it was the academic problem of solving the equations of fluid mechanics with a new body force and another source of dissipation in the energy equation. Some t imes there w ere no practical a pplication for t hese r esults. F or example, natural convection MHD flows have been of interest to the engineering community only since the investigations, described later in section 1.1, are directly applicable to the problems in geophysics and astrophysics. But it was in the field of aerodynamic heating that the largest interest was aroused. Rossow (1957) presented the first paper on this subject. His result, for incompressible constant-property flat plate boundary layer flow, indicated that the skin friction and heat transfer were reduced substantially when a transverse magnetic field was applied to the fluid. This encouraged a multitude analyses for every conceivable type of aerodynamic flow, and most of the research centered on the stagnation point where, in hypersonic flight, the highest degree of ionization could be expected. The results of these studies were sometimes contradictory concerning the amount by which the heat transfer would be reduced (Some of this was due to misinterpretations and invalid comparisons). Eventually, however, it was concluded that the field strengths, necessary to provide sufficient shielding against heat fluxes during atmospheric flight, were not competitive (in terms of weight) with other methods of cooling (Sutton & Gloersen, 1961). However, the invention of new lightweight super conducting magnets has recently revived interests in the problem of providing heat protection during the very high velocity re-entry from orbital and super orbital flight (Levy & Petschek, 1962).

1.6. Free convection

In the studies related to heat transfer, considerable effort has been directed towards the convective mode, in which the relative motion of the fluid provides an additional mechanism for the transfer of energy and of material, the latter being a more important consideration in cases where mass transfer, due to a concentration difference, occurs. Convection is inevitably coupled with the conductive mechanisms, since, although the fluid motion modifies the transport process, the eventual transfer of energy from one fluid element to another in its neighborhood is through conduction. Also, at the surface the process is predominantly that of conduction because the relative fluid motion is brought to zero at the surface. A study of the convective heat transfer therefore involves the mechanisms of conduction and sometimes, those of radioactive processes as well, coupled with that fluid flow. This makes the study of this mode of heat or mass transfer very complex, although its importance in technology and in nature can hardly be exaggerated.

The convective mode of heat transfer is divided into two basic processes. If the motion of the fluid is caused by an external agent, such as the externally imposed flow of a fluid stream over a heated object, the process is termed *forced convection*. The fluid flow may be the result of, for instance, a fan, a blower, the wind or the motion of the heated object itself. Such problems are very frequently encountered in technology where the heat transfers to or from a body is often due to an imposed flow of a fluid at a different temperature from that of a body. If, on the other hand, no such externally induced flow is provided and arises naturally simply owing to the effect of a density difference, resulting from a temperature or concentration difference in a body force field, such as the gravitational field, the process is termed *natural convection*. The density difference gives rise to buoyancy effects, owing to which the flow is generated. A heated body cooling in ambient air generates such a flow in the region surrounding it. Similarly the buoyant flow arising from heat rejection to the atmosphere and to other ambient media, circulations arising in heated rooms, in the atmosphere, and in bodies of water, rise of buoyant flow to cause thermal stratification of the medium, as in temperature inversion, and many other such heat transfer process in our natural environment, as well as in many technological applications, are included in the area of natural convection. The flow may also arise owing to concentration differences, such as those caused by salinity differences in the sea and by composition differences in chemical processing unit, and these cause a natural convection mass transfer.

In many cases of practical interest, both processes are important and heat transfer is by mixed convection, in which neither mode is truly predominant. The main difference between the two really lies in the word external. A heated body lying in still air loses energy by natural convection. But it also generates a buoyant flow above it, and body placed in that flow is subjected to an external flow and it becomes necessary to determine the natural, as well as the forced, convection effects and the regime in which the heat transfer mechanisms lie.

When MHD became a popular subject, it was only normal that these flows would be investigated with the additional ponder motive body force as well as the buoyancy force. At the first glance there seems to be no practical application for these MHD solutions, for most heat exchangers utilize liquids whose conductivity is so small that prohibitively large magnetic fields are necessary to influence the flow. But some nuclear power plants employ heat exchangers with liquid metal coolants, so the application of moderate magnetic fields to change the convection pattern appears feasible. Another classical natural convection problem is the thermal instability that occurs in a liquid heated from below. This subject is of natural interest to geophysicists and astrophysicists, although some applications might arise in boiling heat transfer.

The basic concepts involved in employing the boundary layer approximation to natural convection flows are very similar to those in forced flows. The main difference lies in the fact that the pressure in the region beyond the boundary layer is hydrostatic, instead of being imposed by an external flow, and that the velocity outside the layer is zero. However, the basic treatment and analysis remain the same. The book by Schlichting (1968) is an excellent collection of the boundary layer analysis. There are several methods for the solution of the boundary layer equations namely the similarity variable method, the perturbation method, analytical method, numerical method etc. Details are available in the books by Rosenberg (1969), Gosman et. al. (1969), Patankar and Spalding (1970), Spalding (1977) and Jaluria (1969).

1.7. Heat and Mass transfer

Combined heat and mass transfer problems are of importance in many processes and have therefore received a considerable amount of attention (Jaluria, 1980). In many mass transfer processes, heat transfer considerations arise owing to chemical reaction and are often due to the nature of the process. In processes such as drying, evaporation at the surface of water body, energy transfer in a wet cooling tower and the flow in a desert cooler, heat and mass transfer occur simultaneously. In many of these processes, interest lies in the determination of the total energy transfer, although in processes such as drying, the interest lies mainly in the overall mass transfer for moisture removal. Natural convection processes involving the combined mechanisms are also encountered in many natural processes, such as evaporation, condensation and agricultural drying, in many industrial applications involving solution and mixtures in t he absence of a n externally induc ed f low and in many c hemical processing systems. In many processes such as the curing of plastics, cleaning and chemical processing of materials relevant to the manufacture of printed circuits, manufacture of pulp-insulated cables etc, the combined buoyancy mechanisms arise and the total energy and material transfer resulting from the combined mechanisms has to be determined.

The basic problem is governed by the combined buoyancy effects arising from the simultaneous diffusion of thermal energy and of chemical species. Therefore the equations of continuity, momentum, energy and mass diffusions are coupled through the buoyancy terms alone, if there are other effects, such as the Soret and Duffer effects, they are neglected. This would again be valid for low species concentration levels. These additional effects have also been considered in several investigations, for example, the work of the Caldwell (1974), Groots & Mozur (1962), Hurle & Jakeman (1971) and Legros, et al. (1968, 1970).

Somers (1956) considered combined buoyancy mechanisms for flow adjacent to a wet isothermal vertical surface in an unsaturated environment. Uniform temperature and uniform species concentration at the surface were assumed and an integral analysis was carried out to obtain the result which is expected to be valid for P_r and S_c values around 1.0 with one buoyancy effect being small compared with the other. Mathers et al. (1957) treated the problem as a boundary layer flow for low species concentration, neglecting inertia effects. Results were obtained numerically for P_r=1.0 and Sc varying from 0.5 to 10. Lowell and Adams (1967) and Gill et al. (1965) also considered this problem, including additional effects such as appreciable normal velocity at the surface and comparable species concentrations in the mixture. Similar solutions were investigated by Lowel & Adams (1967) and by Adams and Lowell (1968). Lightfoot (1968) and Saville and Churchill (1970) considered some asymptotic solutions. Adams and Mc Fadden (1966) presented experimental measurements of heat and mass transfer parameters, with opposed buoyancy effects. Gebhart and Pera (1971) studied laminar vertical natural convection flows resulting from the combined buoyancy mechanisms in terms of similarity solutions. Similar analyses have been carried out by Pera and Gebhart (1972) for flow over horizontal surfaces and by Mollendrof and Gebhart (1974) for axisymmetric flows, particularly for the axisymmetric plume.

Mollendrof and Gebhart (1974) carried out an analysis for axisymmetric flows. The governing equations were solved for the combined effects of thermal and mass diffusion in an axisymmetric plume flow. Boura and Gebhart (1976), Hubbel and Gebhart (1974) and Tenner and Gebhart (1971) have studied buoyant free boundary flows in a concentration-stratified medium. Agrawal el al. (1977) have studied the combined buoyancy effects on the thermal and mass diffusion on MHD natural flows, and it is observed that, for the fixed G_r and P_r the value of Xt (dimensionless length parameter) decreases as the strength of the magnetic parameter increases. G eorgantopoulos et al (1981) discussed the e ffects o f free convective and mass transfer flow in a conducting l iquid, when t he fluid is subject to a transverse magnetic field. Haldavnekar and Soundalgekar (1977) studied the effects of mass transfer on free convective flow of an electrically conducting viscous fluid past an infinite porous plate with constant suction and transversely applied magnetic field. An exact analysis was made by Soundalgekar et al. (1979) of the effects of mass transfer and the free convection currents on the MHD Stokes (Rayleigh) problem for the flow of an electrically conducting incompressible viscous fluid past an impulsively started vertical plate under the action of a transversely applied magnetic field. The heat due to viscous and Joule dissipation and induced magnetic field are neglected.

During the course of discussion, the effects of heating $G_r < 0$ of the plate by free convection currents, and G_m (modified Grashof number), S_c and M on the velocity and the skin friction are studied. Nunousis and Goudas (1979) have studied the effects of mass transfer on free convective problem in the Stokes problem for an infinite vertical limiting surface. Georgantopolous and Nanousis (1980) have considered the effects of the mass transfer on free convection flow of an electrically conducting viscous fluid (e.g. of a stellar atmosphere, of star) in the presence of transverse magnetic field. Solution for the velocity and skin friction in closed from are obtained with the help of the Laplace transform technique, and the results obtained for the various values of the parameters, S_c, P_r and M are given in graphical form. Raptis and Kafoussias (1982) presented the analysis of free convection and mass transfer steady hydro magnetic flow of an electrically conducting v iscous incompressible fluid, through a porous medium, occupying a semi infinite region of the space bounded by an infinite vertical and porous plate under the action of transverse magnetic field. Approximate solution has been obtained for the velocity, temperature, concentration field and the rate of heat transfer. The effects of different parameters on the velocity field and the rate of heat transfer are discussed for the case of air (Prandtl number $P_r = .71$) and the water vapor (Schmidt number $S_c = .60$), Raptis and Tzivanidis (1983) considered the effects of variable suction/injection on the unsteady two dimensional free convective flow with mass transfer of an electrically conducting fluid past vertical accelerated plate in the presence of transverse magnetic field. Solutions of the governing equations of the flow are obtained with the power series. An analysis of two dimensional steady free convective flow of a conducting fluid, in the presence of a magnetic field and a foreign mass, past an infinite vertical porous and unmoving surface is carried out by Raptis (1983), when the heat flux is constant at the limiting surface and the magnetic Reynolds number of the flow is not small. Assuming constant suction at the surface, approximate solutions of the coupled non-linear equations are derived for the velocity field, the temperature field, the magnetic field and for their related quantities. Agrawal et al. (1987) considered the steady laminar free convection flow with mass transfer of an electrically conducting liquid along a plane wall with periodic suction. The considered sinusoidal suction velocity distribution is of the form $v' = v_0\left\{1 + \varepsilon \cos\frac{\pi z'}{l}\right\}$, where v_0>0, is the wavelength of the periodic suction velocity distribution, and ε is the amplitude of the suction velocity variation which is assumed to be small quantity. It is observed that near the plate the velocity is a maximum and decreases as y increases. Also, an increase in the magnetic parameter the velocity decreases. Agrawal et al. (1983) have investigated the effect of Hall current on the combined effect of thermal and mass diffusion of an electrically conducting liquid past an infinite vertical porous plate, when the free stream oscillates about constant nonzero mean. The velocity and temperature distributions are shown on graphs for different values of parameters. The value of P_r is chosen as 0.71 for air. In selecting the values of S_c, the Schimidt number, the diffusing chemical species of most common interest in air are considered. From the figures it is seen that, with the increase in Hall parameter, the mean primary velocity decreases, where as the mean secondary velocity increases for a fixed magnetic parameter M and S_c. However, for a fixed m, and increase in magnetic parameter M or S_c, leads to a decrease in both the primary and the secondary velocities. The mean shear stresses at the plate due to primary and secondary velocity and the mean rate of heat transfer from the plate are also given. To study the behavior of the oscillatory and transient part of the velocity and temperature distribution, curves are drawn for various values of parameters that describe the flow at $wt = \frac{\pi}{2}$. The non-dimensional shear stress and the rate of heat transfer are obtained. The above problem has been extended by the same authors (Agrawal et al. (1982)) when the plate temperature oscillates in time about a constant nonzero mean, while the free stream is isothermal. The velocity, temperature and concentration distribution, together with the heat and mass transfer results, have been computed for different values of Pr, Gr, M and m.

1.8. Soret and Dufour Effect

In the above-mentioned studies, heat and mass transfer occur simultaneously in a moving fluid where the relations between the fluxes and the driving potentials are of more complicated nature. In general the thermal-diffusion effects is of a smaller order of magnitude than the effects described by Fourier or Flick's laws and is often neglected in heat and mass transfer process. Mass fluxes can also be created by temperature gradients and this is Soret or Thermal diffusion effect. There are, however, exceptions. The thermal-diffusion effect, (commonly known as Soret effect) for instance, has been utilized for isotope separation and in mixtures between gases with very light molecular weight (H_2, He) and of medium molecular weight (N_2, air). The diffusion thermo effect was found to be of such a magnitude that it could not be neglected (Eckert and Drake, 1972). In view of the importance of the diffusion thermo effect, Jha and Singh (1990) presented an analytical study for free convection and mass transfer flow for an infinite vertical plate moving impulsively in its own plane, taking into account the Soret effect. Kaffoussias (1992) studied the MHD free convection and mass transfer flow, past an infinite vertical plate moving on its own plane, taken into account the thermal diffusion when (1) the boundary surface is impulsively started moving in its own plane (ISP) and (ii) it is uniformly accelerated (UAP). The problem is solved with the help of Laplace transfer method and analytical expressions are given for the velocity field as well as for the skin friction for the above mentioned two different cases. The effects of the velocity and skin friction of the various dimensionless parameters entering into the problem are discussed with the help of graphs. For the I.S.P and U.A.P. cases, it is seen from the figures that the effect of magnetic parameters M is to decrease the fluid (water) velocity inside the boundary layer. This influence of the magnetic field on the velocity field is more evident in the presence of thermal diffusion. From the same figures it is also concluded that the fluid velocity rises due to greater thermal diffusion. Hence, the velocity field is considerably affected by the magnetic field and the thermal diffusion. Nanousis(1992) extended the work of Kafoussias (1992) to the case of rotating fluid taking into account the Soret effect. The plate is assumed to be moving on its own plane with arbitrary velocity $U_0 f(t')$ where U_0 is a constant velocity and $f(t')$ a non-dimensional function of the time t'. The solution of the problem is obtained with the help of Laplace transform technique. Analytical expression is given for the velocity field and for skin friction for two different cases, e.g., when the plate is impulsively started, moving on its own plane (case I) and when it is uniformly accelerated (*case* II). The effects on the velocity field and skin friction, of various dimensionless parameters entering into the problem, especially of the Soret number S_0, are discussed with the help of graphs. In case of an impulsively started plate and uniformly accelerated plate (*case* I and *case* II), it is seen that the primary velocity increase with the increase of S_0 and the magnetic parameter M. It has been observed that energy can be generated not only by temperature gradients but also by composition gradients. The energy flux caused by composition gradients is called the Dufour or diffusion thermo effect. On the other hand, mass fluxes can also be created by temperature gradients and this is the Soret or thermal diffusion effect.

Chapter 2

The Basic Governing Equation

The Navier-Stokes equation and energy equations, together with the Maxwell equations, for the basis for studying Magnetofluid Dynamics (MFD). In MFD, we consider a conducting fluid that is approximately grossly neutral; the charge density in the Maxwell equations must then be interpreted as an excess charge density which is not large. If we disregard the excess charge density, then we must disregard the displacement current. In most problems, the displacement current, excess charge density, excess charge body force and the current due to convection of the excess charge are small. The electrodynamics equations to be used are then pre-Maxwell equations and the complete set becomes

$$\nabla.\mathbf{D} = 0 \tag{2.1}$$

$$\nabla.\mathbf{J} = 0 \tag{2.2}$$

$$\nabla.\mathbf{B} = 0 \tag{2.3}$$

$$\nabla \wedge \mathbf{H} = \mathbf{J} \tag{2.4}$$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.5}$$

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{2.6}$$

$$\mathbf{B} = \mu_e \mathbf{H} \tag{2.7}$$

$$\mathbf{J} = \sigma'(\mathbf{E} + \mathbf{q} \wedge \mathbf{B}) \tag{2.8}$$

where $\mathbf{D}$ is the electric displacement, $\mathbf{J}$ is the current density, $\mathbf{B}$ is the magnetic induction, $\mathbf{H}$ is the magnetic field strength, $\mathbf{E}$ is the electrostatic field, ε is the electrical permeability, σ' is the electrical conductivity, $\mathbf{q}$ is the velocity, μ_e is the magnetic permeability.

The continuity equation for a viscous compressible electrically conducting fluid in vector form is

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{q}) = 0 \tag{2.9}$$

where ρ is the density of the fluid and $\mathbf{q}$ is the fluid velocity.

For incompressible fluid, the equation (2.9) becomes

$$\nabla.\mathbf{q} = 0 \tag{2.10}$$

In three-dimensional Cartesian coordinate system the equation (2.10) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2.11}$$

where u, v and w are the velocity components in the x, y and z direction respectively.

The Momentum equation for a viscous compressible electrically conducting fluid in vector form is

$$\frac{d\mathbf{q}}{dt}=\mathbf{F}-\frac{1}{\rho}\nabla P+\frac{\upsilon}{3}\nabla(\nabla.\mathbf{q})+\upsilon\nabla^2\mathbf{q} \tag{2.12}$$

where $\mathbf{F}$ is the body force per unit volume, P is the fluid pressure and υ is the kinematic viscosity.

For incompressible fluid, the equation (2.12) becomes

$$\frac{d\mathbf{q}}{dt}=\mathbf{F}-\frac{1}{\rho}\nabla P+\upsilon\nabla^2\mathbf{q} \tag{2.13}$$

In the absence of pressure gradient, the equation (2.13) becomes

$$\frac{d\mathbf{q}}{dt}=\mathbf{F}+\upsilon\nabla^2\mathbf{q} \tag{2.14}$$

When the fluid moves through a magnetic field, then the equation (2.14) becomes as a MHD equation

$$\frac{d\mathbf{q}}{dt}=\mathbf{F}+\upsilon\nabla^2\mathbf{q}+\frac{1}{\rho}\mathbf{J}\wedge\mathbf{B} \tag{2.15}$$

where $\mathbf{J}\wedge\mathbf{B}$ is the force on the fluid per unit volume produced by the interaction of the electric and magnetic field (called Lorentz force).

We have $\frac{d\mathbf{q}}{dt}=\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\mathbf{q}$ (2.16)

Then the equation (2.16) becomes:

$$\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\mathbf{q}=\mathbf{F}+\upsilon\nabla^2\mathbf{q}+\frac{\mu_e}{\rho}\mathbf{J}\wedge\mathbf{H}\,.\ [\because \mathbf{B}=\mu_e\mathbf{H}] \tag{2.17}$$

The generalized Ohm's law in the absence of electric field (Mayer, 1958), is of the form

$$\mathbf{J}+\frac{\omega_e\tau_e}{H_0}\mathbf{J}\wedge\mathbf{H}=\sigma'\left(\mu_e\mathbf{q}\wedge\mathbf{H}+\frac{1}{en_e}\nabla P_e\right) \tag{2.18}$$

where ω_e is the cyclotron frequency, τ_e is the electron collision, e is the electric charge, n_e is the number density electron.

Neglecting the Hall-current, we have from equation (2.18)

$$\mathbf{J} = \sigma'(\mu_e \mathbf{q} \wedge \mathbf{H}) \; [\Theta \; \omega_e \tau_e = 0 \, and \, P_e = 0] \tag{2.19}$$

Let $\mathbf{H} = (H_x, H_y, H_z)$. (2.20)

$$\text{Then } \mathbf{q} \wedge \mathbf{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ H_x & H_y & H_z \end{vmatrix} = (vH_z - wH_y)\hat{i} + (wH_x - uH_z)\hat{j} + (uH_y - vH_x)\hat{k} \tag{2.21}$$

Therefore the equation (2.19) becomes

$$\mathbf{J} = \sigma'\mu_e(vH_z - wH_y)\hat{i} + \sigma'\mu_e(wH_x - uH_z)\hat{j} + \sigma'\mu_e(uH_y - vH_x)\hat{k} \tag{2.22}$$

$$\text{Then } \mathbf{J} \wedge \mathbf{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sigma'\mu_e(vH_z - wH_y) & \sigma'\mu_e(wH_x - uH_z) & \sigma'\mu_e(uH_y - vH_x) \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \sigma'\mu_e\{(wH_xH_z - uH_z^2) - (uH_y^2 - vH_xH_y)\}\hat{i} + \sigma'\mu_e\{(uH_xH_y - vH_x^2) - (vH_z^2 - wH_yH_z)\}\hat{j}$$

$$+ \sigma'\mu_e\{(vH_yH_z - wH_y^2) - (wH_x^2 - uH_xH_z)\}\hat{k}$$

$$= (\sigma'\mu_e wH_xH_z - \sigma'\mu_e uH_z^2 - \sigma'\mu_e uH_y^2 + \sigma'\mu_e vH_xH_y)\hat{i}$$

$$+ (\sigma'\mu_e uH_xH_y - \sigma'\mu_e vH_x^2 - \sigma'\mu_e vH_z^2 - \sigma'\mu_e wH_yH_z)\hat{j}$$

$$+ (\sigma'\mu_e vH_yH_z - \sigma'\mu_e wH_y^2 - \sigma'\mu_e wH_x^2 + \sigma'\mu_e uH_xH_z)\hat{k} \tag{2.23}$$

Let $\mathbf{F} = (F_x, F_y, F_z)$. (2.24)

In three-dimensional Cartesian coordinate system the equation (2.15) with the help of the equations (2.17), (2.23) and (2.24) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$- \frac{\sigma'\mu_e^2}{\rho}\{(wH_xH_z - uH_z^2) - (uH_y^2 - vH_xH_y)\} \tag{2.25}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$- \frac{\sigma'\mu_e^2}{\rho}\{(uH_xH_y - vH_x^2) - (vH_z^2 - wH_yH_z)\} \tag{2.26}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \upsilon\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$-\frac{\sigma'\mu_e^2}{\rho}\left\{\left(vH_yH_z - wH_y^2\right) - \left(wH_x^2 - uH_xH_z\right)\right\} \tag{2.27}$$

The MHD Energy equation for a viscous incompressible electrically conducting fluid in vector form (in the absence of heat source, viscous dissipation term and Joule heating term) is

$$\frac{\partial T}{\partial t} + (\mathbf{q}.\nabla)T = \frac{k}{\rho c_p}\nabla^2 T + \frac{D_m k_T}{c_s c_p}\nabla^2 C \tag{2.28}$$

where k is the thermal conductivity of the medium, ρ is the density of the fluid, c_p is the specific heat at constant pressure, D_m is the coefficient of mass diffusivity, k_T is the thermal diffusion ratio and c_s is the concentration susceptibility.

In three-dimensional Cartesian coordinate system the equation (2.28) becomes

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{D_m k_T}{c_s c_p}\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) \tag{2.29}$$

The MHD Concentration equation for a viscous incompressible electrically conducting fluid in vector form (in the absence of heat source, viscous dissipation term and Joule heating term) is

$$\frac{\partial C}{\partial t} + (\mathbf{q}.\nabla)C = D_m\nabla^2 C + \frac{D_m k_T}{T_m}\nabla^2 T \tag{2.30}$$

where T_m is the mean fluid temperature.

In three-dimensional Cartesian coordinate system the equation (2.30) becomes

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D_m\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) + \frac{D_m k_T}{T_m}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{2.31}$$

Thus in three-dimensional Cartesian coordinate system the Continuity equation, the Momentum equation, the Energy equation and the Concentration equation become

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.32}$$

Momentum equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$-\frac{\sigma'\mu_e^{\,2}}{\rho}\left\{\left(wH_xH_z-uH_z^{\,2}\right)-\left(uH_y^{\,2}-vH_xH_y\right)\right\} \tag{2.33}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$-\frac{\sigma'\mu_e^{\,2}}{\rho}\left\{\left(uH_xH_y-vH_x^{\,2}\right)-\left(vH_z^{\,2}-wH_yH_z\right)\right\} \tag{2.34}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$-\frac{\sigma'\mu_e^{\,2}}{\rho}\left\{\left(vH_yH_z-wH_y^{\,2}\right)-\left(wH_x^{\,2}-uH_xH_z\right)\right\} \tag{2.35}$$

Energy equation

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{D_m k_T}{c_s c_p}\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}\right) \tag{2.36}$$

Concentration equation

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=D_m\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}\right)+\frac{D_m k_T}{T_m}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right) \tag{2.37}$$

The next section deals with the specific problem.

Case-I:

Let us consider an unsteady MHD free convection and mass transfer flow of an electrically conducting viscous fluid past an infinite vertical porous plate $y = 0$. The flow is also assumed to be in the x - direction which is taken along the plate in the upward direction and y -axis is normal to it. The temperature and the species concentration at the plate are instantly raised from T_w and C_w to T_∞ and C_∞ respectively, which are thereafter maintained as constant, where T_∞ and C_∞ are the temperature and species concentration of the uniform flow respectively. A uniform magnetic field of strength **B** is imposed to the plate, to be acting along the $y-axis$, which is assumed to be electrically non-conducting. We assume that the magnetic Reynolds number of the flow be small enough so that the induced magnetic field is negligible in comparison with applied one (Pai, 1962), so that **B**=(0,B$_0$,0) and the magnetic lines of force are fixed relative to the fluid. The equation of conservation of charge $\nabla.\mathbf{J} = 0$ gives $J_y =$ constant, where the current density $\mathbf{J} = (J_{x,} J_{y,} J_z)$. Since the plate is electrically non-conducting, this constant is zero and hence $J_y = 0$ at the plate and hence zero everywhere.

In the free-convection and mass transfer flow along the vertical plate, the body force along the flow direction is

$$F_x = g\beta(T - T_\infty) + g\beta^*(C - C_\infty) \tag{2.1.1}$$

where g is the acceleration due to gravitation, β is the coefficient of volume expansion, β^* is the volumetric coefficient of expansion with concentration.

With reference to the above assumptions, the continuity equation (2.32), the momentum equations (2.33)-(2.35), the energy equation (2.36) and the concentration equation (2.37) become

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1.2}$$

Momentum equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g\beta(T - T_\infty) + g\beta^*(C - C_\infty) + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{\sigma' B_0{}^2 u}{\rho} \tag{2.1.3}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{2.1.4}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \upsilon\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{\sigma' B_0{}^2 w}{\rho} \tag{2.1.5}$$

Energy equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{D_m k_T}{c_s c_p}\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) \tag{2.1.6}$$

Concentration equation

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D_m\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) + \frac{D_m k_T}{T_m}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{2.1.7}$$

Since the plate occupying the plane $y=0$ is of infinite extent and the fluid motion is unsteady, **all physical quantities will depend only upon y and t**. Thus mathematically the problem reduces to a one dimensional problem. Then the equations (2.1.2)-(2.1.7) become

$$\frac{\partial v}{\partial y}=0 \tag{2.1.8}$$

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=g\beta(T-T_{\infty})+g\beta^{*}(C-C_{\infty})+\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0{}^2 u}{\rho} \tag{2.1.9}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}=\upsilon\frac{\partial^2 v}{\partial y^2} \tag{2.1.10}$$

$$\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}=\upsilon\frac{\partial^2 w}{\partial y^2}-\frac{\sigma B_0{}^2 w}{\rho} \tag{2.1.11}$$

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{D_m k_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2} \tag{2.1.12}$$

$$\frac{\partial C}{\partial t}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}+\frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2} \tag{2.1.13}$$

Let viscosity of the fluid be small and let δ be small thickness of the boundary layer. Let $\varepsilon \ll 1$ be the order of magnitude of δ, i.e., $O(\delta)=\varepsilon$. Let the order of magnitude of u and w are one, i.e. $O(u)=1$ and $O(w)=1$. Then the order of magnitude of v and y are ε and the order of magnitude of t is one, i.e., $O(v)=\varepsilon$, $O(y)=\varepsilon$ and $O(t)=1$.

Hence $O\left(\frac{\partial u}{\partial t}\right)=1$, $O\left(\frac{\partial u}{\partial y}\right)=\frac{1}{\varepsilon}$, $O\left(\frac{\partial^2 u}{\partial y^2}\right)=\frac{1}{\varepsilon^2}$ and $O\left(\frac{\partial v}{\partial t}\right)=\varepsilon$, $O\left(\frac{\partial v}{\partial y}\right)=1$, $O\left(\frac{\partial^2 v}{\partial y^2}\right)=\frac{1}{\varepsilon}$

within the boundary layer.

Then the equations (2.1.8)-(2.1.11) with order become

$$\frac{\partial v}{\partial y}=0 \tag{2.1.14}$$

1

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=g\beta(T-T_{\infty})+g\beta^{*}(C-C_{\infty})+\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0{}^2 u}{\rho} \tag{2.1.15}$$

$1 \quad \varepsilon \quad \frac{1}{\varepsilon} \qquad\qquad \frac{1}{\varepsilon^2} \quad 1$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}=\upsilon\frac{\partial^2 v}{\partial y^2} \tag{2.1.16}$$

$$\varepsilon \quad \varepsilon \; 1 \qquad \frac{1}{\varepsilon}$$

$$\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}=\upsilon\frac{\partial^2 w}{\partial y^2}-\frac{\sigma' B_0^{\ 2} w}{\rho} \tag{2.1.17}$$

$$1 \quad \varepsilon \; \frac{1}{\varepsilon} \qquad \frac{1}{\varepsilon^2} \qquad 1$$

Again, let δ_T be the thermal boundary layer thickness and let $\varepsilon << 1$ be also the order of magnitude of δ_T, i.e., $O(\delta_T)=\varepsilon$. Let the order of magnitude of T and C be one, i.e., $O(T)=1$ and $O(C)=1$.

Hence $O\left(\frac{\partial T}{\partial t}\right)=1$, $O\left(\frac{\partial T}{\partial y}\right)=\frac{1}{\varepsilon}$, $O\left(\frac{\partial^2 T}{\partial y^2}\right)=\frac{1}{\varepsilon^2}$ and $O\left(\frac{\partial C}{\partial t}\right)=1$, $O\left(\frac{\partial C}{\partial y}\right)=\frac{1}{\varepsilon}$, $O\left(\frac{\partial^2 C}{\partial y^2}\right)=\frac{1}{\varepsilon^2}$ within the boundary layer.

Then the equations (2.1.12) and (2.1.13) with order become

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{D_m k_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2} \tag{2.1.18}$$

$$1 \quad \varepsilon \; \frac{1}{\varepsilon} \qquad \frac{1}{\varepsilon^2} \qquad \frac{1}{\varepsilon^2}$$

$$\frac{\partial C}{\partial t}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}+\frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2} \tag{2.1.19}$$

$$1 \quad \varepsilon \; \frac{1}{\varepsilon} \qquad \frac{1}{\varepsilon^2} \qquad \frac{1}{\varepsilon^2}$$

Equations (2.1.14)-(2.1.19) requires that $O(g\beta(T-T_\infty))=1$, $O(g\beta^*(C-C_\infty))=1$, $O(\frac{\sigma' B_0^{\ 2}}{\rho})=1$, $O(\frac{k}{\rho c_p})=1$, $O(\frac{D_m k_T}{c_s c_p})=1$, $O(D_m)=1$, $O(\frac{D_m k_T}{T_m})=1$ and $O(\upsilon)=\varepsilon^2$.

Since the viscosity is very small, so neglecting the small order terms, we have from equations (2.1.14)-(2.1.19)

$$\frac{\partial v}{\partial y}=0 \tag{2.1.20}$$

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+g\beta^*(C-C_\infty)+\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma' B_0^{\ 2} u}{\rho} \tag{2.1.21}$$

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{D_m k_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2} \tag{2.1.22}$$

$$\frac{\partial C}{\partial t}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}+\frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2} \tag{2.1.23}$$

The boundary conditions for the problem are

$t \le 0, u=0, v=0,\ T=T_\infty, C=C_\infty$ for all values of y

$$\left.\begin{aligned} &t>0, u=U_0(t), v=v(t), T=T_w, C=C_w \ at\ y=0\\ &t>0, u=0, v=0, T\to T_\infty, C\to C_\infty \ at\ y\to\infty\end{aligned}\right\} \tag{2.1.24}$$

Case-II:

Let us consider a steady MHD free convection and mass transfer flow of an electrically conducting viscous fluid past a semi-infinite vertical porous plate $y=0$. The detail descriptions of the present problem are similar to those of **Case-I**.

In the free-convection and mass transfer flow along the vertical plate, the body force along the flow direction is

$$F_x=g\beta(T-T_\infty)+g\beta^*(C-C_\infty) \tag{2.2.1}$$

where g is the acceleration due to gravitation, β is the coefficient of volume expansion, β^* is the volumetric coefficient of expansion with concentration.

With reference to the above assumptions, the continuity equation (2.32), the momentum equations (2.33)-(2.35) the energy equation (2.36) and the concentration equation (2.37) become

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2.2.2}$$

Momentum equations

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=g\beta(T-T_\infty)+g\beta^*(C-C_\infty)+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\sigma B_0{}^2 u}{\rho} \tag{2.2.3}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{2.2.4}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\frac{\sigma B_0{}^2 w}{\rho} \tag{2.2.5}$$

Energy equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{D_m k_T}{c_s c_p}\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}\right) \tag{2.2.6}$$

Concentration equation

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=D_m\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}\right)+\frac{D_m k_T}{T_m}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right) \tag{2.2.7}$$

Since the plate occupying the plane $y=0$ is of semi-infinite extent and the motion is steady, **all physical quantities will depend only upon x and y**. Thus mathematically the problem reduces to a two dimensional problem.

Then the equations (2.2.2)-(2.2.7) become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.2.8}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+g\beta^*(C-C_\infty)+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma B_0{}^2 u}{\rho} \tag{2.2.9}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{2.2.10}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=\upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\frac{\sigma B_0{}^2 w}{\rho} \tag{2.2.11}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{D_m k_T}{c_s c_p}\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right) \tag{2.2.12}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_m\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)+\frac{D_m k_T}{T_m}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{2.2.13}$$

Let the viscosity of the fluid be small and let δ be small thickness of the boundary layer. Let $\varepsilon << 1$ be the order of magnitude of δ, i.e., $O(\delta)=\varepsilon$. Let the order of magnitude of u, w and x are one, i.e. $O(u)=1$, $O(w)=1$ and $O(x)=1$. Then the order of magnitude of v and y are ε, i.e., $O(v)=\varepsilon$ and $O(y)=\varepsilon$.

Hence $O\left(\frac{\partial u}{\partial x}\right)=1$, $O\left(\frac{\partial^2 u}{\partial x^2}\right)=1$, $O\left(\frac{\partial u}{\partial y}\right)=\frac{1}{\varepsilon}$, $O\left(\frac{\partial^2 u}{\partial y^2}\right)=\frac{1}{\varepsilon^2}$ and $O\left(\frac{\partial v}{\partial x}\right)=\varepsilon$, $O\left(\frac{\partial^2 v}{\partial x^2}\right)=\varepsilon$, $O\left(\frac{\partial v}{\partial y}\right)=1$, $O\left(\frac{\partial^2 v}{\partial y^2}\right)=\frac{1}{\varepsilon}$ within the boundary layer.

Then the order of the equations (2.2.8)-(2.2.11) become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.2.14}$$

$$1 \quad 1$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+g\beta^*(C-C_\infty)+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma' B_0^{\,2}u}{\rho} \tag{2.2.15}$$

$$1\ 1 \quad \varepsilon\ \frac{1}{\varepsilon} \qquad\qquad 1 \quad \frac{1}{\varepsilon^2} \qquad 1$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{2.2.16}$$

$$1\ \varepsilon \quad \varepsilon\ 1 \qquad \varepsilon \quad \frac{1}{\varepsilon}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=\upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\frac{\sigma' B_0^{\,2}w}{\rho} \tag{2.2.17}$$

$$1\ \ 1 \quad \varepsilon\ \frac{1}{\varepsilon} \qquad 1 \quad \frac{1}{\varepsilon^2} \qquad 1$$

Again, let δ_T be the thermal boundary layer thickness and let $\varepsilon \ll 1$ be also the order of magnitude of δ_T, i.e., $O(\delta_T)=\varepsilon$. Let the order of magnitude of T and C be one, i.e., $O(T)=1$ and $O(C)=1$.

Hence $O\left(\frac{\partial T}{\partial x}\right)=1$, $O\left(\frac{\partial^2 T}{\partial x^2}\right)=1$, $O\left(\frac{\partial T}{\partial y}\right)=\frac{1}{\varepsilon}$, $O\left(\frac{\partial^2 T}{\partial y^2}\right)=\frac{1}{\varepsilon^2}$ and $O\left(\frac{\partial C}{\partial x}\right)=1$, $O\left(\frac{\partial^2 C}{\partial x^2}\right)=1$, $O\left(\frac{\partial C}{\partial y}\right)=\frac{1}{\varepsilon}$, $O\left(\frac{\partial^2 C}{\partial y^2}\right)=\frac{1}{\varepsilon^2}$ within the boundary layer.

Then the order of the equations (2.2.12) and (2.2.13) become

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{D_m k_T}{c_s c_p}\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right) \tag{2.2.18}$$

$$1 \quad 1 \quad \varepsilon \frac{1}{\varepsilon} \qquad 1 \quad \frac{1}{\varepsilon^2} \qquad 1 \quad \frac{1}{\varepsilon^2}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_m\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)+\frac{D_m k_T}{T_m}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{2.2.19}$$

$$1 \quad 1 \quad \varepsilon \frac{1}{\varepsilon} \qquad 1 \quad \frac{1}{\varepsilon^2} \qquad 1 \quad \frac{1}{\varepsilon^2}$$

Equations (2.2.14)-(2.2.19) requires that $O(g\beta(T-T_\infty))=1$, $O(g\beta^*(C-C_\infty))=1$, $O(\frac{\sigma' B_0^{\,2}}{\rho})=1$, $O(\frac{k}{\rho c_p})=1$, $O(\frac{D_m k_T}{c_s c_p})=1$, $O(D_m)=1$, $O(\frac{D_m k_T}{T_m})=1$ and $O(\upsilon)=\varepsilon^2$.

Since the viscosity is very small, so neglecting the small order terms, we have from equations (2.2.14)-(2.2.19)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.2.20}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+g\beta^*(C-C_\infty)+\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma' B_0^{\,2} u}{\rho} \tag{2.2.21}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{D_m k_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2} \tag{2.2.22}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}+\frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2} \tag{2.2.23}$$

The boundary conditions for the problem are

$$\left.\begin{aligned} &u=U_0, v=v_0(x), T=T_w, C=C_w \; at \; y=0 \\ &u=0, v=0, T\to T_\infty, C\to C_\infty \; at \; y\to\infty \end{aligned}\right\} \tag{2.2.24}$$

Chapter 3

The Calculation Technique

3.1. The Method of Superposition

The theoretical treatment of MHD heat and mass transfer flow has so far been made mostly analytically, applying Laplace transform and perturbation methods. In some cases asymptotic method has been applied. There are, however, few numerical solutions regarding MHD heat and mass transfer boundary layer flows. Here our solutions would be based mainly on numerical methods. For this purpose method of superposition (*Na*, 1979) and shooting method will be used for solving the problems for which similarity solutions of ordinary differential equations are sought. The detail description of the shooting method will be sought in the next section. As for the superposition method, which will be used in this dissertation, it is necessary to discuss the method in brief. The basic idea, of the superposition method is to reduce the boundary value problem into an initial value problem which then can be easily solved by any initial value solver such as the Runge-Kutta, method. The transformation of linear ordinary differential equations from boundary value to initial value problems by the method of superposition is well known. For linear ordinary differential equations it is in general possible to reduce the boundary value to initial value problems. Thus combination of these solutions then give the solution of the original boundary value problem. For example, let us consider a second order linear ordinary differential equation

$$\frac{d^2 y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = p(x) \tag{3.1.1}$$

Subject to the boundary conditions

$$y(a) = y_a,\ y(b) = y_b \tag{3.1.2}$$

To transfer the boundary value problem into an initial value problem, we consider that

$$y(x) = y_1(x) + \lambda y_2(x) \tag{3.1.3}$$

where λ is a constant, to be determined.

Substituting equation (3.1.3) into equation (3.1.1) we get

$$\left[\frac{d^2y_1}{dx^2}+f_1(x)\frac{dy_1}{dx}+f_2(x)y_1-p(x)\right]+\lambda\left[\frac{d^2y_2}{dx^2}+f_1(x)\frac{dy_2}{dx}+f_2(x)y_2\right]=0 \quad (3.1.4)$$

From the above equation, we obtain the following two equations

$$\frac{d^2y_1}{dx^2}+f_1(x)\frac{dy_1}{dx}+f_2(x)y_1=p(x) \quad (3.1.5)$$

$$\frac{d^2y_2}{dx^2}+f_1(x)\frac{dy_2}{dx}+f_2(x)y_2=0 \quad (3.1.6)$$

Applying the first boundary condition from (3.1.2), the equation (3.1.3) is next transformed to

$$y\ (x)=y_1(x)+\lambda y_2(x) \quad (3.1.7)$$

from which we have

$$y_1(a)=y_a,\ y_2(a)=0 \quad (3.1.8)$$

Further differentiating equation (3.1.3) we have

$$\frac{dy(x)}{dx}=\frac{dy_1(x)}{dx}+\lambda\frac{dy_2(x)}{dx} \quad (3.1.9)$$

Setting x equal to a in (3.1.9), we have

$$\frac{dy(a)}{dx}=\frac{dy_1(a)}{dx}+\lambda\frac{dy_2(a)}{dx} \quad (3.1.10)$$

For the transformed equations (3.1.5) and (3.1.6) we require two more boundary conditions in addition to (3.1.8). These unknown boundary conditions are taken to be

$$\frac{dy_1(a)}{dx}=0,\frac{dy_2(a)}{dx}=1 \quad (3.1.11)$$

such that from (3.1.10)

$$\frac{dy(a)}{dx}=\lambda \quad (3.1.12)$$

Hence the unknown constant λ is identified as the missing initial slope. As a final step, the boundary condition at the second point is transformed to

$$y_b=y_1(b)+\lambda y_2(b) \quad (3.1.13)$$

From which we have

$$\lambda=\frac{[y_b-y_1(b)]}{y_2(b)} \quad (3.1.14)$$

From the equation (3.1.14) λ can be calculated, which according to the equation (3.1.3), is the missing initial slope.

3.2. The Shooting Method

To solve the boundary layer equations by using shooting method technique, there are two asymptotic boundary condition and hence two unknown surface conditions $f''(0)$ and $\theta'(0)$. Within the context of initial value method and Nachtscheim-Swigert iteration technique the outer boundary conditions may be functionally represented as

$$f'(\eta_{\max}) = f'(f''(0), \theta'(0)) = \delta_1 \tag{3.2.1}$$

$$\theta(\eta_{\max}) = \theta(f''(0), \theta'(0)) = \delta_2 . \tag{3.2.2}$$

With asymptotic convergence criteria given by

$$f''(\eta_{\max}) = f''(f''(0), \theta'(0)) = \delta_3 \tag{3.2.3}$$

$$\theta'(\eta_{\max}) = \theta'(f''(0), \theta'(0)) = \delta_4 . \tag{3.2.4}$$

Let us choose $f''(0) = g_1$, $\theta'(0) = g_2$. Expanding first order Taylor series expansion after using the above equations (3.2.1)-(3.2.4) yields

$$f'(\eta_{\max}) = f'_c(\eta_{\max}) + \frac{\partial f'}{\partial g_1}\Delta g_1 + \frac{\partial f'}{\partial g_2}\Delta g_2 = \delta_1 \tag{3.2.5}$$

$$\theta(\eta_{\max}) = \theta_c(\eta_{\max}) + \frac{\partial \theta}{\partial g_1}\Delta g_1 + \frac{\partial \theta}{\partial g_2}\Delta g_2 = \delta_2 \tag{3.2.6}$$

$$f''(\eta_{\max}) = f''_c(\eta_{\max}) + \frac{\partial f''}{\partial g_1}\Delta g_1 + \frac{\partial f''}{\partial g_2}\Delta g_2 = \delta_3 \tag{3.2.7}$$

$$\theta'(\eta_{\max}) = \theta'_c(\eta_{\max}) + \frac{\partial \theta'}{\partial g_1}\Delta g_1 + \frac{\partial \theta'}{\partial g_2}\Delta g_2 = \delta_4 \tag{3.2.8}$$

where the subscript 'c' indicates the value of the function at $\eta_{\max}$ determined from the trial integration.

Solution of these equations in a least squares sense requires determining the minimum value of the error E as,

$$E = \delta_1^{\,2} + \delta_2^{\,2} + \delta_3^{\,2} + \delta_4^{\,2} \tag{3.2.9}$$

With respect to g_1 and g_2.

Now diferentiating E with respect to g_1 and g_2 we get,

$$\delta_1 \frac{\partial \delta_1}{\partial g_1} + \delta_2 \frac{\partial \delta_2}{\partial g_1} + \delta_3 \frac{\partial \delta_3}{\partial g_1} + \delta_4 \frac{\partial \delta_4}{\partial g_1} = 0 \tag{3.2.10}$$

$$\delta_1 \frac{\partial \delta_1}{\partial g_2} + \delta_2 \frac{\partial \delta_2}{\partial g_2} + \delta_3 \frac{\partial \delta_3}{\partial g_2} + \delta_4 \frac{\partial \delta_4}{\partial g_2} = 0. \tag{3.2.11}$$

Applying equations (3.2.5)-(3.2.8) in equations (3.2.10), we obntain

$$\left(f_c' + \frac{\partial f'}{\partial g_1}\Delta g_1 + \frac{\partial f'}{\partial g_2}\Delta g_2\right)\frac{\partial f'}{\partial g_1} + \left(\theta_c + \frac{\partial \theta}{\partial g_1}\Delta g_1 + \frac{\partial \theta}{\partial g_2}\Delta g_2\right)\frac{\partial \theta}{\partial g_1}$$
$$+ \left(f_c'' + \frac{\partial f''}{\partial g_1}\Delta g_1 + \frac{\partial f''}{\partial g_2}\Delta g_2\right)\frac{\partial f''}{\partial g_1} + \left(\theta_c' + \frac{\partial \theta'}{\partial g_1}\Delta g_1 + \frac{\partial \theta'}{\partial g_2}\Delta g_2\right)\frac{\partial \theta'}{\partial g_1} = 0$$

i.e. $$\left[\left(\frac{\partial f'}{\partial g_1}\right)^2 + \left(\frac{\partial \theta}{\partial g_1}\right)^2 + \left(\frac{\partial f''}{\partial g_1}\right)^2 + \left(\frac{\partial \theta'}{\partial g_1}\right)^2\right]\Delta g_1 + \left(\frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_2} + \frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2} + \frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2} + \frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2}\right)\Delta g_2$$
$$= -\left[f_c'\frac{\partial f'}{\partial g_1} + \theta_c\frac{\partial \theta}{\partial g_1} + f_c''\frac{\partial f''}{\partial g_1} + \theta_c'\frac{\partial \theta'}{\partial g_1}\right]. \tag{3.2.12}$$

Again applying equations (3.2.5)-(3.2.8) in equations (3.2.11), similarly we obntain

$$\left[\left(\frac{\partial f'}{\partial g_2}\right)^2 + \left(\frac{\partial \theta}{\partial g_2}\right)^2 + \left(\frac{\partial f''}{\partial g_2}\right)^2 + \left(\frac{\partial \theta'}{\partial g_2}\right)^2\right]\Delta g_2 + \left(\frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_2} + \frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2} + \frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2} + \frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2}\right)\Delta g_1$$
$$= -\left[f_c'\frac{\partial f'}{\partial g_2} + \theta_c\frac{\partial \theta}{\partial g_2} + f_c''\frac{\partial f''}{\partial g_2} + \theta_c'\frac{\partial \theta'}{\partial g_2}\right]. \tag{3.2.13}$$

From the equations (3.2.12) and (3.2.13), we have

$$a_{11}\Delta g_1 + a_{12}\Delta g_2 = b_1 \tag{3.2.14}$$

$$a_{21}\Delta g_1 + a_{22}\Delta g_2 = b_2, \tag{3.2.15}$$

where

$$a_{11} = \left(\frac{\partial f'}{\partial g_1}\right)^2 + \left(\frac{\partial \theta}{\partial g_1}\right)^2 + \left(\frac{\partial f''}{\partial g_1}\right)^2 + \left(\frac{\partial \theta'}{\partial g_1}\right)^2 \tag{3.2.16}$$

$$a_{12} = \frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_2} + \frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2} + \frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2} + \frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2} \tag{3.2.17}$$

$$a_{21} = \frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_2} + \frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2} + \frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2} + \frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2} \tag{3.2.18}$$

$$a_{22} = \left(\frac{\partial f'}{\partial g_2}\right)^2 + \left(\frac{\partial \theta}{\partial g_2}\right)^2 + \left(\frac{\partial f''}{\partial g_2}\right)^2 + \left(\frac{\partial \theta'}{\partial g_2}\right)^2 \tag{3.2.19}$$

$$b_1 = -\left[f_c'\frac{\partial f'}{\partial g_1} + \theta_c\frac{\partial \theta}{\partial g_1} + f_c''\frac{\partial f''}{\partial g_1} + \theta_c'\frac{\partial \theta'}{\partial g_1}\right] \tag{3.2.20}$$

$$b_2 = -\left[f_c' \frac{\partial f'}{\partial g_2} + \theta_c \frac{\partial \theta}{\partial g_2} + f_c'' \frac{\partial f''}{\partial g_2} + \theta_c' \frac{\partial \theta'}{\partial g_2} \right]. \tag{3.2.21}$$

In matrix form, equations (3.2.14) and (3.2.15) can be writen as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \Delta g_1 \\ \Delta g_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{3.2.22}$$

Now to solve the system of linear equations (3.2.22) by Cramers rule we have

$$\Delta g_1 = \frac{\det A_1}{\det A}, \ \Delta g_2 = \frac{\det A_2}{\det A}, \tag{3.2.23}$$

where

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21}) \tag{3.2.24}$$

$$\det A_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = (b_1 a_{22} - b_2 a_{12}) \tag{3.2.25}$$

$$\det A_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = (b_2 a_{11} - b_2 a_{21}). \tag{3.2.26}$$

Then we obtain the (unspecified) missing values g_1 and g_2 as

$$g_1 \leftarrow g_1 + \Delta g_1 \tag{3.2.27}$$

$$g_2 \leftarrow g_2 + \Delta g_2. \tag{3.2.28}$$

Thus adopting this type of numerical technique as described above, a computer program was setup for the solution of the basic nonlinear differential equations of our problem where the integration technique was adopted as the sixth order Runge-Kutta method of integration. Based on the integrations done with the above numerical technique, the results obtained are presented in the appropriate section.

Chapter 4

Unsteady MHD free convection and mass transfer flow past an infinite vertical porous plate with thermal diffusion and Dufour effects

4.1.1. Introduction

From technological point of view, MHD free-convection flows have great significance in the fields of stellar and planetary magnetospheres, aeronautics, chemical engineering, and electronics (Alfven, 1950; Cramer and Pai, 1973; Lui, 1987). Model studies of MHD free convection flows have thus been made by many, some of them are Georgantopoulos (1979) and Raptis and Singh (1985). The effect of mass transfer on MHD free convection flows have also been considered by many of whom the names of Haldavnekar and Soundalgekar(1977), Soundalgekar and Gupta (1979), Nanousis and Goudas (1979) and Georgantopoulos and Nanousis(1980) are worth mentioning.

However, in the above studies the thermal diffusion effect was ignored under the assumption that the concentration level is very low. In view of the importance of this diffusion thermo effect, Jha and Singh (1990) studied the free convection and mass transfer flow in an infinite vertical plate moving impulsively in its own plane, taking into account the Soret effect. Kafoussias (1992) studied the same problem to the case of MHD flow. They made analytical studies based on Laplace transform technique. The objective of the present study is to consider the above problem past an infinite vertical porous plate taking into account the thermal diffusion as well as Dufour effects. For this purpose impulsively started plate moving in its own plane is considered. Similarity counterpart of the momentum, energy and concentration equations is derived by introducing a time dependent length scale. The suction velocity is taken to be inversely proportional to the above length scale. The momentum, energy and concentration equations are solved numerically by the method of superposition as introduced by Na (1979).

4.1.2. The Governing Equation

Let us consider an unsteady MHD free convection and mass transfer flow of an electrically conducting viscous fluid past an infinite vertical porous plate *y=0*. The flow is also assumed to be in the *x*-direction which is taken along the plate in the upward direction and *y*-axis is normal to it. The temperature and the species concentration at the plate are instantaneously raised from T_w and C_w to T_∞ and C_∞ respectively, which are thereafter maintained as constant, where T_∞ and C_∞ are the temperature and species concentration of the uniform flow respectively. A uniform magnetic field **B** is imposed to the plate to be acting along the *y*-axis, which is assumed to be electrically non-conducting. We assumed that the magnetic Reynolds number of the flow be small enough so that the induced magnetic field is negligible in comparison with applied one (Pai, 1962), so that **B**=(0,B_0,0) and the magnetic lines of force are fixed relative to the fluid. The equation of conservation of charge $\nabla.\mathbf{J}=0$ gives J_y= constant, where the current density **J**=(J_x, J_y., J_z)Since the plate is electrically non-conducting, this constant is zero and hence J_y =0 at the plate and hence zero everywhere.

With reference to the generalized equations described in **case-I of Chapter 2**, the one dimensional problem under the above assumptions and Boussinesq approximation can be put in the following form:

$$\frac{\partial v}{\partial y}=0 \tag{4.1.1}$$

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+g\beta^*(C-C_\infty)+\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma' B_0{}^2 u}{\rho} \tag{4.1.2}$$

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{D_m k_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2} \tag{4.1.3}$$

$$\frac{\partial C}{\partial t}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}+\frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2} \tag{4.1.4}$$

and the boundary conditions for the problem are

$$\left.\begin{array}{l} t>0, u=U_0(t), v=v(t), T=T_w, C=C_w\ at\ y=0 \\ t>0, u=0, v=0, T\to T_\infty, C\to C_\infty\ at\ y\to\infty \end{array}\right\} \tag{4.1.5}$$

where *u*, *v* are the velocity components in the *x* ,*y* direction respectively, υ is the kinematic viscosity, g is the acceleration due to gravity, ρ is the density, β is the coefficient of volume expansion, β^* is the volumetric coefficient of expansion with concentration. T, T_w and T_∞ are the temperature of the fluid inside the thermal boundary layer, the plate temperature and the fluid temperature in the free stream, respectively, while C, C_w, C_∞ are the corresponding concentrations. k is the thermal conductivity of the medium, D_m is the coefficient of mass diffusivity, C_p is the specific heat at constant pressure, T_m is the mean fluid temperature, k_T is the thermal diffusion ratio, C_s is the concentration susceptibility and other symbols have their usual meaning.

4.1.3. Mathematical Formulation

In order to obtain similarity solutions we introduce a similarity parameter σ as

$$\sigma = \sigma(t) \qquad (4.1.6)$$

such that σ is the time dependent length scale. In terms of this length scale, a convenient solution of equation (4.1.1) is considered to be

$$v = -v_0 \frac{\upsilon}{\sigma} \qquad (4.1.7)$$

Here the constant v_0 represents a dimensionless normal velocity at the plate, which is positive for suction and negative for blowing.
We now introduce the following dimensionless variables

$$\eta = \frac{y}{\sigma} \qquad (4.1.8)$$

$$f(\eta) = \frac{u}{U_0} \qquad (4.1.9)$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \qquad (4.1.10)$$

$$\phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \qquad (4.1.11)$$

From the equation (4.1.9), we have

$$u = U_0 f(\eta) \qquad (4.1.12)$$

From the equation (4.1.12), we have the following derivatives

$$\frac{\partial u}{\partial t} = -\frac{U_0}{\sigma}\frac{\partial \sigma}{\partial t}\eta f'(\eta) \qquad (4.1.13)$$

$$\frac{\partial u}{\partial y} = \frac{U_0}{\sigma} f'(\eta) \qquad (4.1.14)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_0}{\sigma^2} f''(\eta) \tag{4.1.15}$$

Again from the equation (4.1.10), we have

$$(T - T_\infty) = (T_w - T_\infty)\theta(\eta) \tag{4.1.16}$$

Also from the equation (4.1.11), we have

$$(C - C_\infty) = (C_w - C_\infty)\phi(\eta) \tag{4.1.17}$$

Substituting the equations (4.1.7), (4.1.10), (4.1.12) and (4.1.13)-(4.1.17) into the equation (4.1.2), we get

$$-\frac{\sigma}{\upsilon}\frac{\partial \sigma}{\partial t}\eta f' - v_0 f' = f'' + G_r\theta + G_m\phi - Mf \tag{4.1.18}$$

where $G_r = \frac{g\beta(T_w - T_\infty)\sigma^2}{U_0\upsilon}$ is the Grashof number, $G_m = \frac{g\beta^*(C_w - C_\infty)\sigma^2}{U_0\upsilon}$ is the modified Grashof number, $M = \frac{\sigma' B_0^{\,2}\sigma^2}{\rho\upsilon}$ is the Magnetic parameter.

Again, from the equation (4.1.10), we have

$$T = T_\infty + (T_w - T_\infty)\theta(\eta) \tag{4.1.19}$$

From the equation (4.1.19), we have the following derivatives

$$\frac{\partial T}{\partial t} = -\frac{(T_w - T_\infty)}{\sigma}\frac{\partial \sigma}{\partial t}\eta\theta'(\eta) \tag{4.1.20}$$

$$\frac{\partial T}{\partial y} = \frac{(T_w - T_\infty)}{\sigma}\theta'(\eta) \tag{4.1.21}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{(T_w - T_\infty)}{\sigma^2}\theta''(\eta) \tag{4.1.22}$$

Further, from the equation (4.1.11), we have

$$C_w = C_\infty + \bar{x}(C_0 - C_\infty) \tag{4.1.23}$$

From the equation (4.1.23), we have the following derivatives

$$\frac{\partial C}{\partial t} = -\frac{(C_w - C_\infty)}{\sigma}\frac{\partial \sigma}{\partial t}\eta\phi'(\eta) \tag{4.1.24}$$

$$\frac{\partial C}{\partial y} = \frac{(C_w - C_\infty)}{\sigma}\phi'(\eta) \tag{4.1.25}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{(C_w - C_\infty)}{\sigma^2}\phi''(\eta) \tag{4.1.26}$$

Substituting the equations (4.1.7), (4.1.20)-(4.1.22) and (4.1.26) into the equation (4.1.3), we have

$$-\frac{\sigma}{\upsilon}\frac{\partial\sigma}{\partial t}\eta\vartheta' - v_0\theta' = \frac{1}{P_r}\theta'' + D_f\phi'' \tag{4.1.27}$$

where $P_r = \frac{\rho\upsilon c_p}{k}$ is the Prandlt number and $D_f = \frac{D_m k_T}{c_s c_p \upsilon}\frac{(C_w - C_\infty)}{(T_w - T_\infty)}$ is the Dafour number.

Again substituting the equations (4.1.7), (4.1.22), (4.1.24)-(4.1.26) into the equation (4.1.4), we get

$$-\frac{\sigma}{\upsilon}\frac{\partial\sigma}{\partial t}\eta\phi' - v_0\phi' = \frac{1}{S_c}\phi'' + S_o\theta'' \tag{4.1.28}$$

where $S_c = \frac{\upsilon}{D_m}$ is the Schmidt number and $S_r = \frac{D_m k_T}{\upsilon T_m}\frac{(T_w - T_\infty)}{(C_w - C_\infty)}$ is the Soret number.

The corresponding boundary conditions for the above mentioned problem are

$$\left.\begin{aligned} f = 1, \theta = 1, \phi = 1 at \eta = 0 \\ f = 0, \theta = 0, \phi = 0 as \eta \to \infty \end{aligned}\right\} \tag{4.1.29}$$

The equations (4.1.18), (4.1.27) and (4.1.28) are similar except for the term $\frac{\sigma}{\upsilon}\frac{\partial\sigma}{\partial t}$ where time t appears explicitly. Thus the similarity condition requires that $\frac{\sigma}{\upsilon}\frac{\partial\sigma}{\partial t}$ in the equations (4.1.18), (4.1.27) and (4.1.28) must be constant quantity. Hence following the works of Sattar and Alam (1994) one can try a class of solutions of the equations (4.1.18), (4.1.27) and (4.1.28) by assuming that

$$\frac{\sigma}{\upsilon}\frac{\partial\sigma}{\partial t} = c \text{ (a constant)} \tag{4.1.30}$$

Now integrating (4.1.30) one obtains

$$\sigma = \sqrt{2c\upsilon t} \tag{4.1.31}$$

where the constant of integration is determined through the condition that $\sigma = 0$ when *t=0*. It thus appears from (4.1.30) that, by making a realistic choice of c to be equal to 2 in (4.1.31) the length scale σ becomes equal to $\sigma = 2\sqrt{\upsilon t}$ which exactly corresponds to the usual scaling factor considered for various unsteady boundary layer flows (Schlichting, 1968). Since σ is a scaling factor as well as a similarity parameter, any value of c in (4.1.31) would not change the nature of the solution except that the scale would be different. Finally, introducing (4.1.31) with $c = 2$ in equations (4.1.18), (4.1.27) and (4.1.28) we respectively have the following dimensionless ordinary differential equations

$$f'' + 2\xi f' + G_r\theta + G_m\phi - Mf = 0 \tag{4.1.32}$$

$$\theta'' + 2\xi P_r\theta' + P_r D_f\phi'' = 0 \tag{4.1.33}$$

$$\phi'' + 2\xi S_c\phi' + S_c S_o\theta'' = 0 \tag{4.1.34}$$

where $\xi = \eta + \frac{v_0}{2}$.

The corresponding boundary conditions are

$$\left.\begin{array}{l} f = 1, \theta = 1, \phi = 1 \, at \, \eta = 0 \\ f = 0, \theta = 0, \phi = 0 \, as \, \eta \to \infty \end{array}\right\} \tag{4.1.35}$$

In all above equations primes denote the differentiation with respect to η. The section **4.1.5.** deals with the solution of the problem.

4.1.4. Skin friction Co-efficient, Nusselt number and Sherwood number

The quantities of chief physical interest are the local Skin friction Co-efficient, Nusselt number and Sherwood number.

The equation defining the skin friction is

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$\tau \propto f'$$

The Nusselt number, denoted by N_u, is proportional to $-\left(\frac{\partial T}{\partial y}\right)_{y=0}$ and it can be expressed as

$$N_u \propto -\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

That is, $N_u \propto -\theta'$

The Sherwood number, denoted by S_h, is proportional to $-\left(\frac{\partial C}{\partial y}\right)_{y=0}$ and it can be expressed as

$$S_h \propto -\left(\frac{\partial C}{\partial y}\right)_{y=0}$$

That is, $S_h \propto -\phi'$

Thus the values proportional to the skin friction co-efficient, the Nusselt number and the Sherwood number are respectively obtained numerically. These values are sorted in the tables 4.1.1-4.1.3

4.1.5. Numerical Solution

The solutions of equations (4.1.32)-(4.1.34) with the boundary conditions (4.1.35) are obtained by the method of superposition (Na, 1979). This method is used to reduce the boundary value problem to an initial value problem that can easily be integrated out by an initial value solver. Thus to reduce equations (4.1.32)-(4.1.34) to an initial value problem the function $f(\eta), \theta(\eta)$ and $\phi(\eta)$ are respectively decomposed to

$$f(\eta) = f_1(\eta) + \mu f_2(\eta) + \lambda f_3(\eta) + \delta f_4(\eta) \tag{4.1.36}$$

$$\theta(\eta) = \theta_1(\eta) + \mu\theta_2(\eta) + \lambda\theta_3(\eta) + \delta\theta_4(\eta) \tag{4.1.37}$$

$$\phi(\eta) = \phi_1(\eta) + \mu\phi_2(\eta) + \lambda\phi_3(\eta) + \delta\phi_4(\eta) \tag{4.1.38}$$

where μ, λ and δ are constants, the physical significance of which is obtained later. Now substituting the equations 4.1.36-4.1.38 in equations 4.1.32-4.1.34 and then equating the different coefficients to zero we obtain the following differential equations

$$f_1'' + 2\zeta f_1' - Mf_1 + G_r\theta_1 + G_m\phi_1 = 0 \tag{4.1.39}$$

$$f_2'' + 2\zeta f_2' - Mf_2 + G_r\theta_2 + G_m\phi_2 = 0 \tag{4.1.40}$$

$$f_3'' + 2\zeta f_3' - Mf_3 + G_r\theta_3 + G_m\phi_3 = 0 \tag{4.1.41}$$

$$f_4'' + 2\zeta f_4' - Mf_4 + G_r\theta_4 + G_m\phi_4 = 0 \tag{4.1.42}$$

$$\theta_1'' + 2\zeta P_r\theta_1' + D_f P_r\phi_1'' = 0 \tag{4.1.43}$$

$$\theta_2'' + 2\zeta P_r\theta_2' + D_f P_r\phi_2'' = 0 \tag{4.1.44}$$

$$\theta_3'' + 2\zeta P_r\theta_3' + D_f P_r\phi_3'' = 0 \tag{4.1.45}$$

$$\theta_4'' + 2\zeta P_r\theta_4' + D_f P_r\phi_4'' = 0 \tag{4.1.46}$$

$$\phi_1'' + 2\zeta S_c\phi_1' + S_c S_0\theta_1'' = 0 \tag{4.1.47}$$

$$\phi_2'' + 2\zeta S_c\phi_2' + S_c S_0\theta_2'' = 0 \tag{4.1.48}$$

$$\phi_3'' + 2\zeta S_c\phi_3' + S_c S_0\theta_3'' = 0 \tag{4.1.49}$$

$$\phi_4'' + 2\zeta S_c\phi_4' + S_c S_0\theta_4'' = 0 \tag{4.1.50}$$

The initial values of the decomposed functions $f_1(\eta), f_2(\eta), f_3(\eta), f_4(\eta), \ldots$ etc. are now obtained through the boundary conditions (4.1.35) as

$$\left.\begin{aligned} & f_1(\eta)=1.0, f_2(\eta)=0, f_3(\eta)=0, f_4(\eta)=0 \\ & \theta_1(\eta)=1.0, \theta_2(\eta)=0, \theta_3(\eta)=0, \theta_4(\eta)=0 \\ & \phi_1(\eta)=1.0, \phi_2(\eta)=0, \phi_3(\eta)=0, \phi_4(\eta)=0 \end{aligned}\right\} \tag{4.1.51}$$

Again as $\eta \to \infty$, applying the boundary conditions (4.1.35) in (4.1.36)-(4.1.38) we get

$$\mu=-\frac{f_1(\theta_3\phi_4-\theta_4\phi_3)+\theta_1(f_4\phi_3-f_3\phi_4)+\phi_1(f_1\theta_4-f_4\theta_1)}{f_2(\theta_3\phi_4-\theta_4\phi_3)+f_1(f_4\phi_3-f_3\phi_4)+\phi_1(f_1\theta_4-f_4\theta_1)} \tag{4.1.52}$$

$$\lambda=-\frac{f_1(\theta_4\phi_2-\theta_2\phi_4)+\theta_1(f_2\phi_4-f_4\phi_2)+\phi_1(\theta_2 f_4-\theta_4 f_2)}{f_2(\theta_3\phi_4-\theta_4\phi_3)+f_1(f_4\phi_3-f_3\phi_4)+\phi_1(f_1\theta_4-f_4\theta_1)} \tag{4.1.53}$$

$$\delta=-\frac{f_1(\theta_2\phi_3-\theta_3\phi_2)+\theta_1(f_3\phi_2-f_2\phi_3)+\phi_1(\theta_3 f_2-\theta_2 f_3)}{f_2(\theta_3\phi_4-\theta_4\phi_3)+f_1(f_4\phi_3-f_3\phi_4)+\phi_1(f_1\theta_4-f_4\theta_1)} \tag{4.1.54}$$

In (4.1.36)-(4.1.38) all the functional values are obtained as

$$\frac{\partial f(\eta)}{\partial\eta}=\frac{\partial f_1(\eta)}{\partial\eta}+\mu\frac{\partial f_2(\eta)}{\partial\eta}+\lambda\frac{\partial f_3(\eta)}{\partial\eta}+\delta\frac{\partial f_4(\eta)}{\partial\eta} \tag{4.1.55}$$

$$\frac{\partial\theta(\eta)}{\partial\eta}=\frac{\partial\theta_1(\eta)}{\partial\eta}+\mu\frac{\partial\theta_2(\eta)}{\partial\eta}+\lambda\frac{\partial\theta_3(\eta)}{\partial\eta}+\delta\frac{\partial\theta_4(\eta)}{\partial\eta} \tag{4.1.56}$$

$$\frac{\partial\phi(\eta)}{\partial\eta}=\frac{\partial\phi_1(\eta)}{\partial\eta}+\mu\frac{\partial\phi_2(\eta)}{\partial\eta}+\lambda\frac{\partial\phi_3(\eta)}{\partial\eta}+\delta\frac{\partial\phi_4(\eta)}{\partial\eta} \tag{4.1.57}$$

Then by setting the missing slopes

$\frac{\partial f(0)}{\partial\eta}, \frac{\partial\theta(0)}{\partial\eta}$ and $\frac{\partial\phi(0)}{\partial\eta}$ as $\frac{\partial f(0)}{\partial\eta}=\mu$, $\frac{\partial\theta(0)}{\partial\eta}=\lambda$ and $\frac{\partial\phi(0)}{\partial\eta}=\delta$

the initial conditions for the slopes of the decomposed functions are obtained easily. The well known Runge-Kutta Merson integration scheme has been used as an initial value solver to integrate the above mentioned equations and to obtain converged solutions which are presented graphically in Figs. 4.1.2 – 4.1.16. If now τ , N_u and S_h are respectively denoted as the local values of the skin friction, Nusselt number and Sherwood number, they are respectively proportional to $\frac{\partial f(0)}{\partial\eta}$, $-\frac{\partial\theta(0)}{\partial\eta}$ and $-\frac{\partial\phi(0)}{\partial\eta}$. The numerical values of the skin friction, Nusselt number and Sherwood number are sorted in Tables 4.1.1-4.1.3.

4.1.6. Results and Discussions

The velocity profiles are shown in Figs. 4.1.2-4.1.7 for different values of suction parameter, V_0, the magnetic parameter, M, the Prandtl number, P_r, the Soret number, S_0, the Schmidt number, S_c and the Dufour number, D_f and for fixed values of local Grashof number, G_r and modified local Grashof number G_m. The value of G_r is taken to be large, since this value corresponds to a cooling problem that is generally encountered in nuclear engineering in connection with the cooling of reactors. For P_r, three values 0.71,1.0 and 7.0 are considered (0.71 represents air at 20^0 C, 1.0 corresponds to electrolyte solutions such as salt water and 7.0 corresponds to water). The values 0.22,0.60 and 0.75 of S_c are also considered as they represent specific conditions of the flow. In particular, 0.22 corresponds to Hydrogen while 0.60 corresponds to water vapor that represents a diffusivity chemical species of most common interest in air and the value 0.75 represents Oxygen. The values of V_0, M, S_0, D_f and G_m are however chosen arbitrarily.

With the above mentioned parameters, the velocity profiles are presented in Figs. 4.1.2-4.1.7, the temperature profiles are presented in Figs. 4.1.8-4.1.11 and the concentration profiles are presented in Figs. 4.1.12-4.1.16. The effects of the suction parameter V_0 on the velocity is shown in Fig. 4.1.2. It is observed from this figure that an increase in V_0 leads to a decrease in the velocity. The usual stabilizing effect of the suction parameter on the boundary layer growth is also evident from this figure. In Figs. 4.1.3 and 4.1.4, the effects of Soret number S_0 and Dufour number D_f on the velocity field are shown respectively. It is observed from these figures that the velocity increases with the increase of S_0 and D_f. In Figs. 4.1.5, 4.1.6 and 4.1.7, the effects of magnetic force number M, Schmidt number S_c and Prandtl number P_r on the velocity field are shown respectively. From Fig. 4.1.5, it is observed that the velocity decreases with the increase of magnetic force number. The same effect is observed from Figs. 4.1.6 in case of Schmidt number. From Fig. 4.1.7, it is seen that the Prandtl number has quite a larger decreasing effect on the velocity field with its increase.

The effects of suction parameter V_0 on the temperature profiles is shown in Fig. 4.1.8. It is observed from this figure that the temperature decreases as the suction parameter increase. In Figs. 4.1.9 and 4.1.10, the effects of Soret number and Dufour number on the temperature profiles are shown respectively. It is observed from these figures that the temperature decreases as the Soret number increases (Fig. 4.1.9) while the temperature increase as the Dufour number increase (Fig. 4.1.10). In Fig. 4.1.11, the effects of Prandtl number on the temperature field is shown. It is observed from this figure that with the increase of the Prandtl number the temperature decreases at a particular position of the boundary layer (Fig.

4.1.11). This decrease is very large in case of water (P_r=7.0). We also observe that for P_r=7.0 the field temperature remains less than the uniform flow temperature for most part of the boundary layer.

The effects of suction parameter V_0 on the concentration field is displayed in Fig. 4.1.12, which shows that the concentration decreases as the V_0 increase. In Figs. 4.1.13 and 4.1.14, the effects of Soret number S_0 and Dufour number D_f on the concentration profiles are displayed respectively. It is observed from these figures that the Soret number has a large increasing effect on concentration (Fig. 4.1.13) while the Dufour number has a minor decreasing effect on concentration (Fig. 4.1.14) with their increase. In Figs. 4.1.15 and 4.1.16, the effect of Prandtl number P_r and Schmidt number S_c on the concentration field are shown respectively. It is observed from these figures that the concentration increases as the Prandtl number P_r increase (Fig. 4.1.15). It is also seen from these figures that the Schmidt number S_c has a major decreasing effect on the concentration (Fig. 4.1.16) with its increase.

Finally, the effects of various parameters on the skin friction τ , the Nusselt number N_u and the Sherwood number S_h are shown in the Tables 4.1.1-4.1.3. From Table 4.1.1, we observe that the skin friction τ decreases while the Nusselt number N_u and the Sherwood number S_h increases with the increase of the suction parameter V_o. It is also seen from this table that the skin friction τ and the Nusselt number N_u increases while the Sherwood number S_h decreases with the increase of Dufour number D_f. From Table 4.1.2, it is seen that the Nusselt number N_u and the Sherwood number S_h increases while the skin friction τ decreases with the increase of Magnetic force number M. Also, from Table 4.1.2, we observe that the skin friction τ and the Nusselt number N_u increases whereas the Sherwood number S_h decreases with increase of Soret number S_0. From table 4.1.3, we see that the skin friction τ and the Sherwood number S_h decreases while the Nusselt number N_u increases the with increase of Prandtl number P_r. Further, from Table 4.1.3, we observe that the skin friction τ and the Sherwood number S_h decreases with the increase of Schmidt number S_c, but the Nusselt number N_u increases.

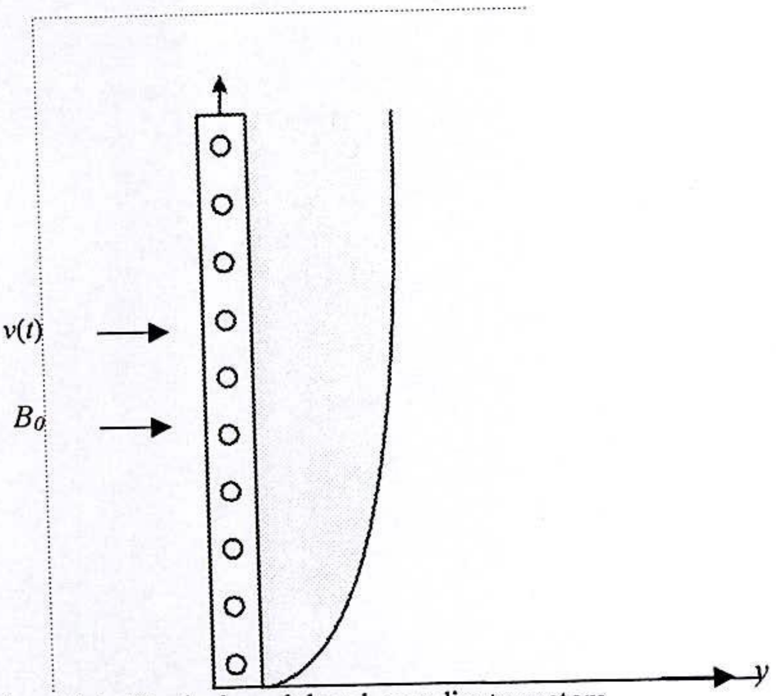

Fig. 4.1.1. Physical model and coordinate system.

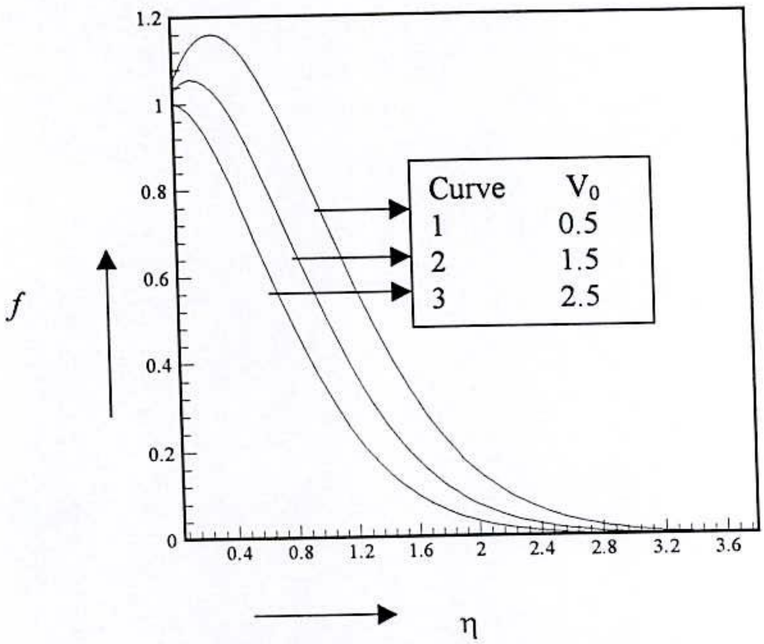

Fig.4.1.2. Velocity Profiles for different values of V_0, taking $M = 0.50$, $S_0 = 2.00$, $D_f = 0.20$, $G_r = 10.00$, $G_m = 5.00$, $P_r = 0.71$, $S_c = 0.60$ as fixed.

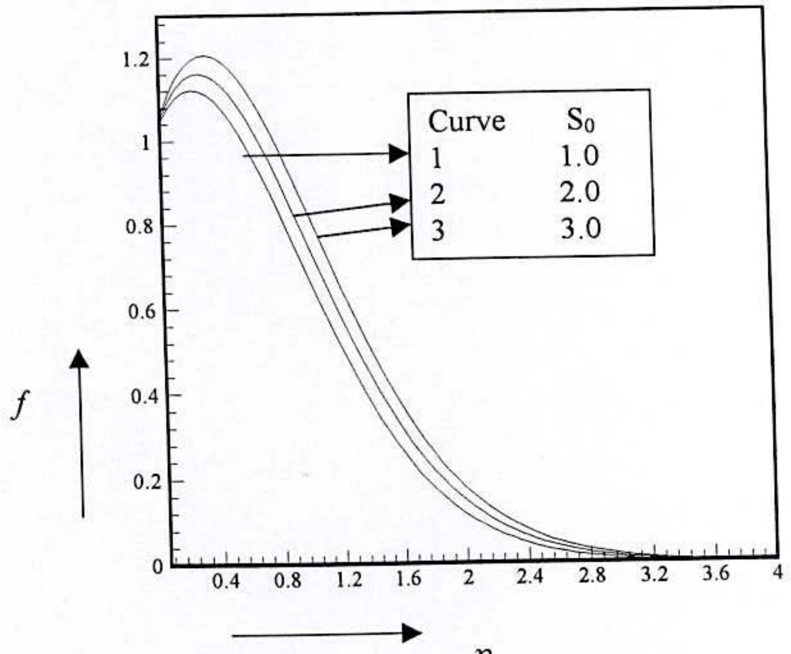

Fig.4.1.3. Velocity Profiles for different values of S_0, taking $V_0 = 0.50$, $P_r = 0.71$, $M = 0.5$, $D_f = 0.2$, $G_r = 10.00$, $G_m = 5.00$, $S_c = 0.60$ as fixed.

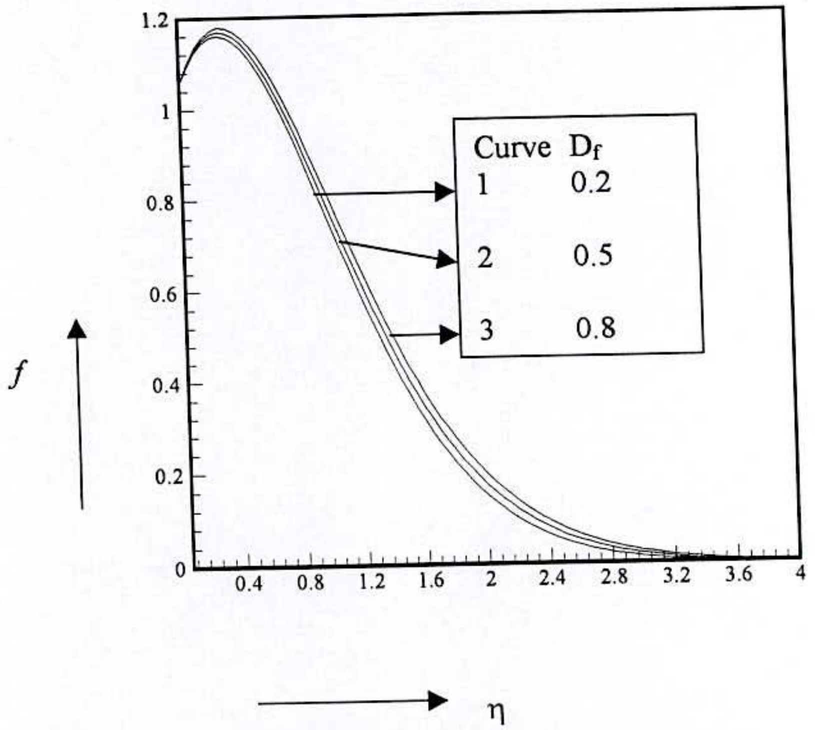

Fig.4.1.4. Velocity Profiles for different values of D_f, taking $V_0 = 0.50$, $S_0 = 2.00$, $M = 0.5$, $G_r = 10.00$, $G_m = 5.00$, $P_r = 0.71$, $S_c = 0.60$ as fixed.

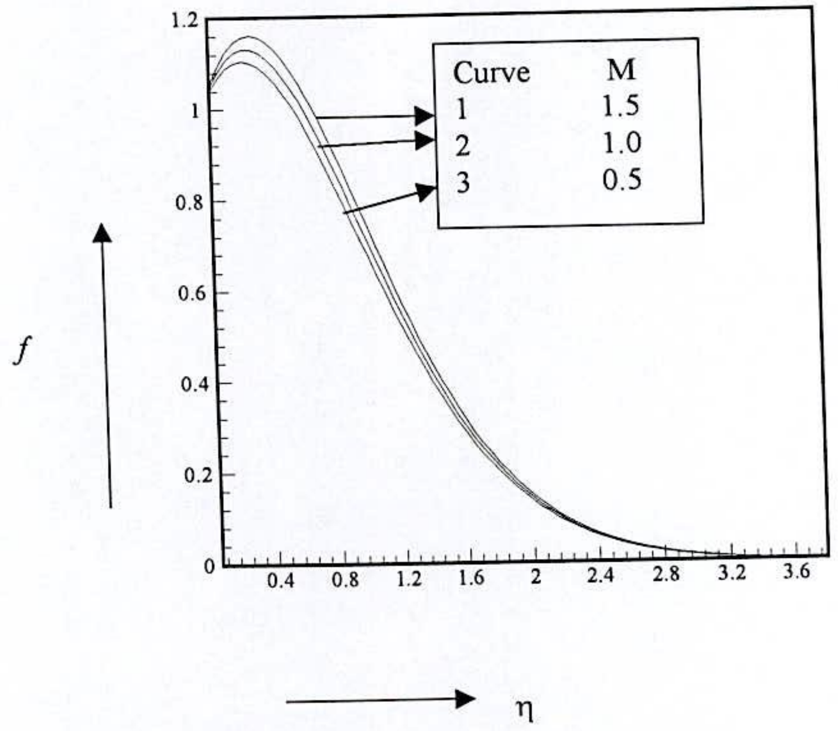

Fig.4.1.5. Velocity Profiles for different values of M, taking $V_0 = 0.50$, $S_0 = 2.00$, $D_f = 0.20$, $G_r = 10.00$, $G_m = 5.00$, $P_r = 0.71$, $S_c = 0.60$ as fixed.

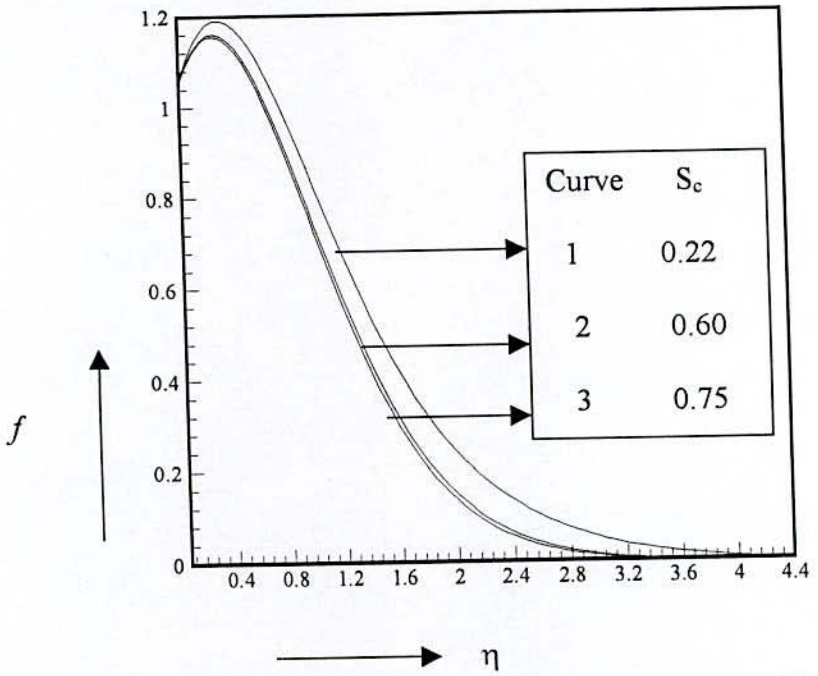

Fig.4.1.6. Velocity Profiles for different values of S_c, taking $V_0 = 0.50$, $P_r = 0.71$, $M = 0.5$, $D_f = 0.2$, $G_r = 10.00$, $G_m = 5.00$, $S_0 = 2.0$ as fixed.

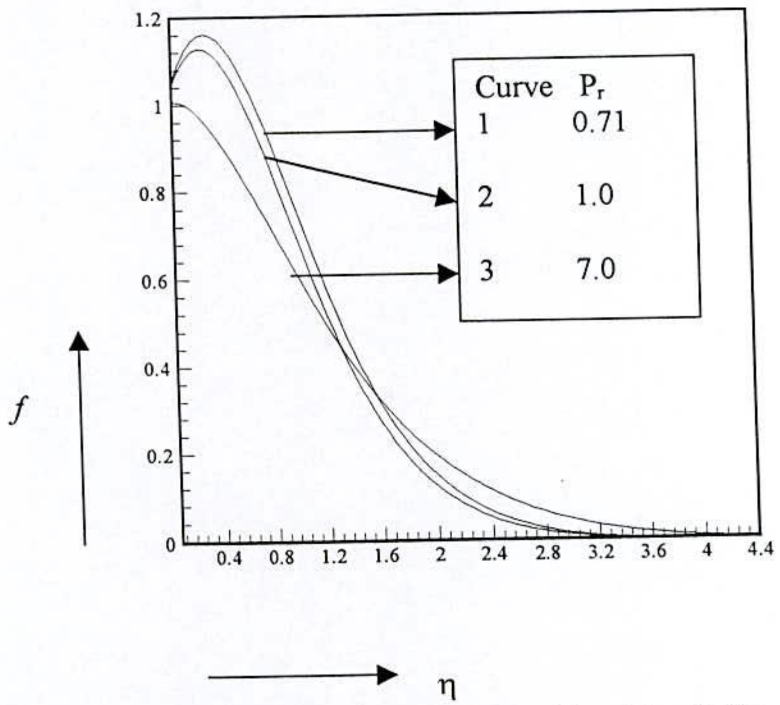

Fig.4.1.7. Velocity Profiles for different values of P_r, taking $V_0 = 0.50$, $S_0 = 2.00$,

$M = 0.5$, $D_f = 0.2$, $G_r = 10.00$, $G_m = 5.00$, $S_c = 0.22$ as fixed.

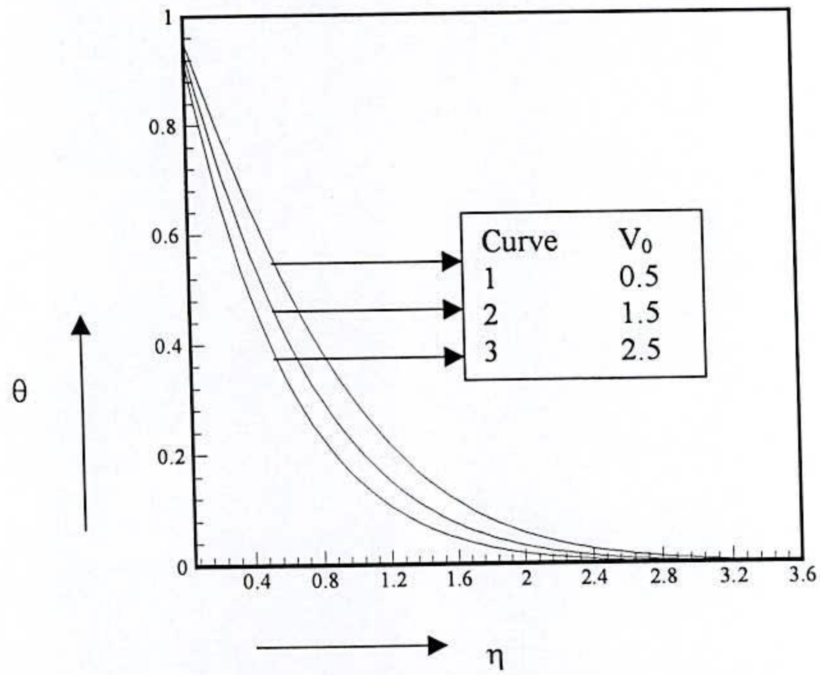

Fig.4.1.8. Temperature Profiles for different values of V_0, taking $M = 0.50$, $S_0 = 2.00$, $D_f = 0.20$, $G_r = 10.00$, $G_m = 5.00$, $P_r = 0.71$, $S_c = 0.60$ as fixed.

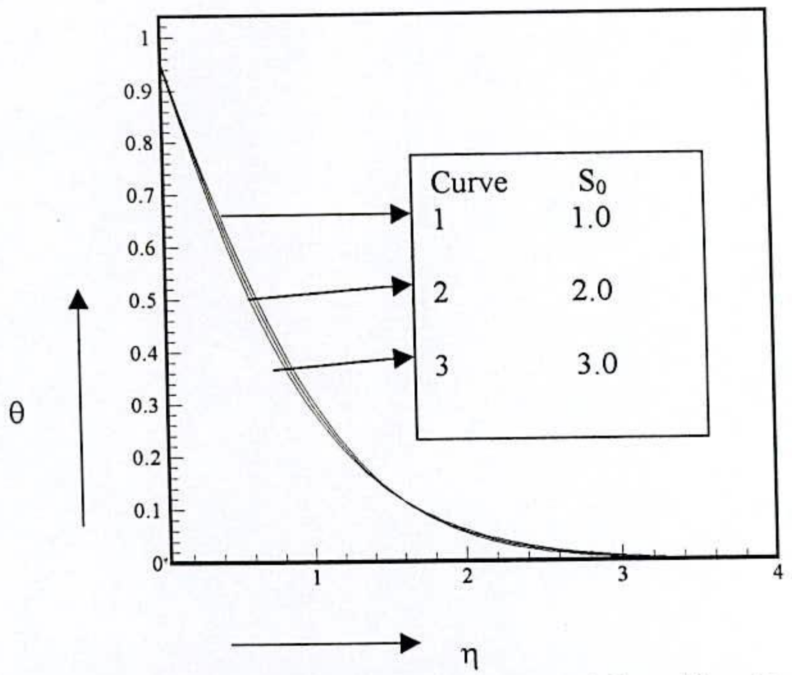

Fig.4.1.9. Temperature Profiles for different values of S_0, taking $V_0 = 0.50$, $P_r = 0.71$, $M = 0.5$, $D_f = 0.2$, $G_r = 10.00$, $G_m = 5.00$, $S_c = 0.60$ as fixed.

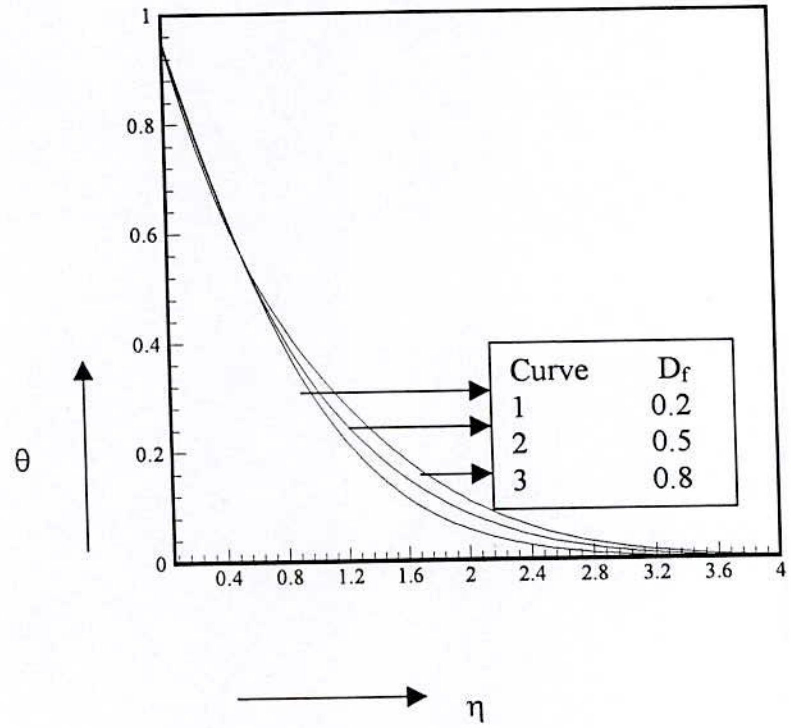

Fig.4.1.10. Temperature Profiles for different values of D_f, taking $V_0 = 0.50$, $S_0 = 2.00$, $M = 0.5$, $G_r = 10.00$, $G_m = 5.00$, $P_r = 0.71$, $S_c = 0.60$ as fixed.

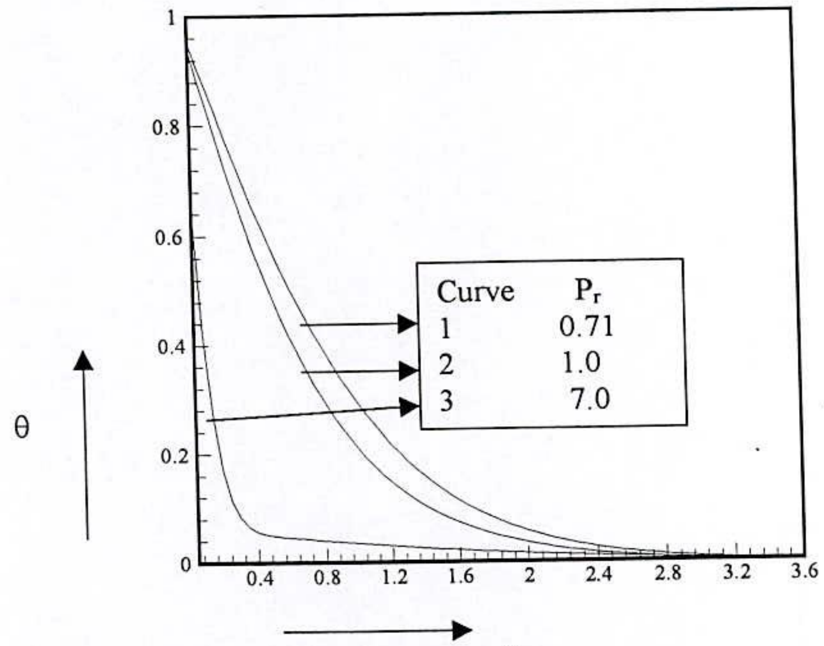

Fig.4.1.11. Temperature Profiles for different values of P_r, taking $V_0 = 0.50$, $S_0 = 2.00$, $M = 0.5$, $D_f = 0.2$, $G_r = 10.00$, $G_m = 5.00$, $S_c = 0.22$ as fixed.

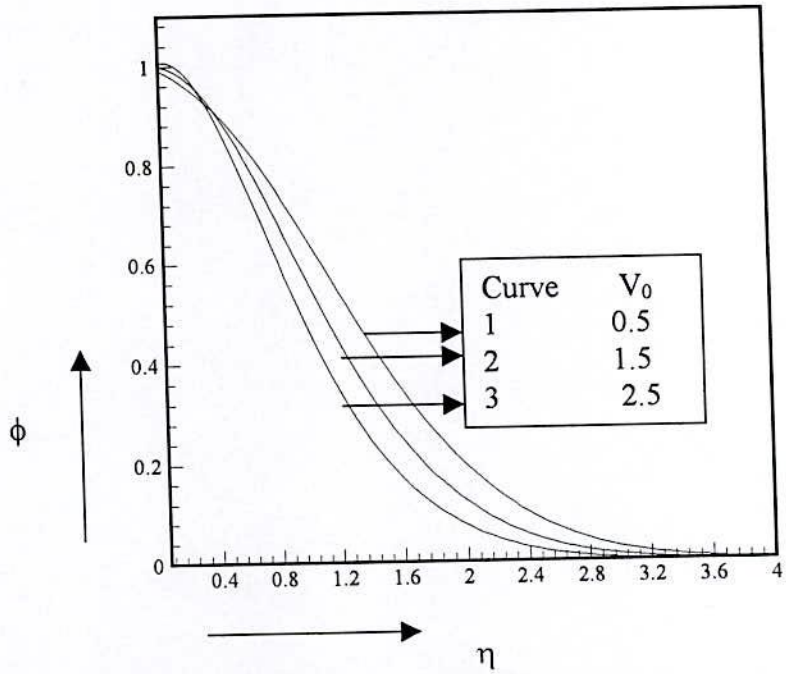

Fig.4.1.12. Concentration Profiles for different values of V_0, taking M = 0 .50 , S_0=2.00 , D_f= 0 .20, G_r =10.00 , G_m = 5.00 , P_r= 0.71 , S_c = 0.60 as fixed.

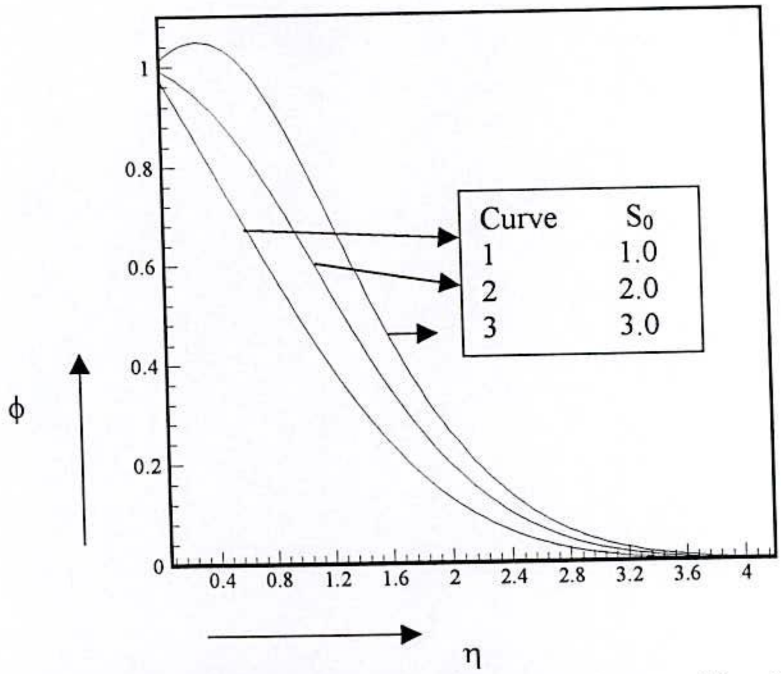

Fig.4.1.13. Concentration Profiles for different values of S_0, taking V_0 = 0 .50, Pr=0.71 ,M = 0 .5, D_f =0.2 , G_r = 10.00, G_m = 5.00 , S_c = 0.60 as fixed.

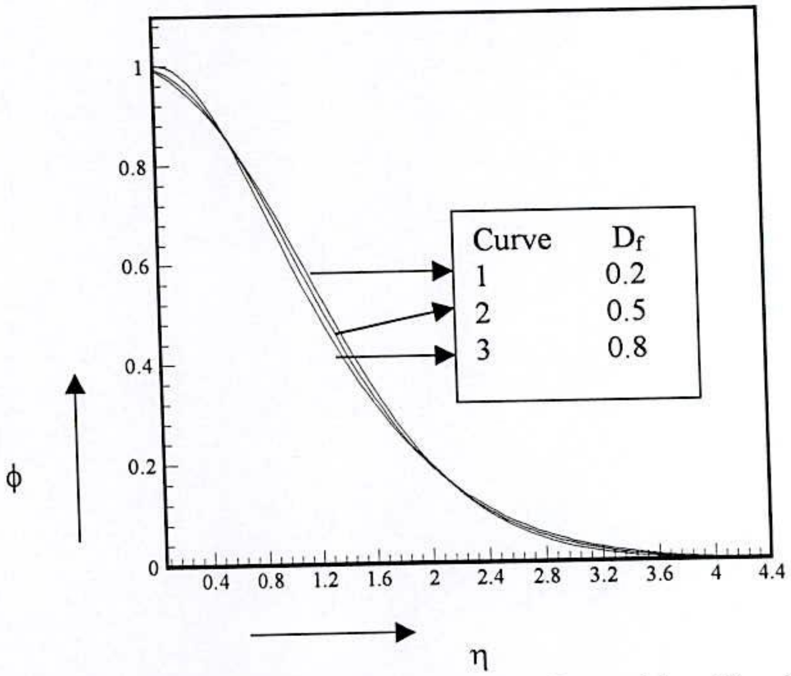

Fig.4.1.14. Concentration Profiles for different values of D_f, taking $V_0 = 0.50$, $S_0 = 2.00$, $M = 0.5$, $G_r = 10.00$, $G_m = 5.00$, $P_r = 0.71$, $S_c = 0.60$ as fixed.

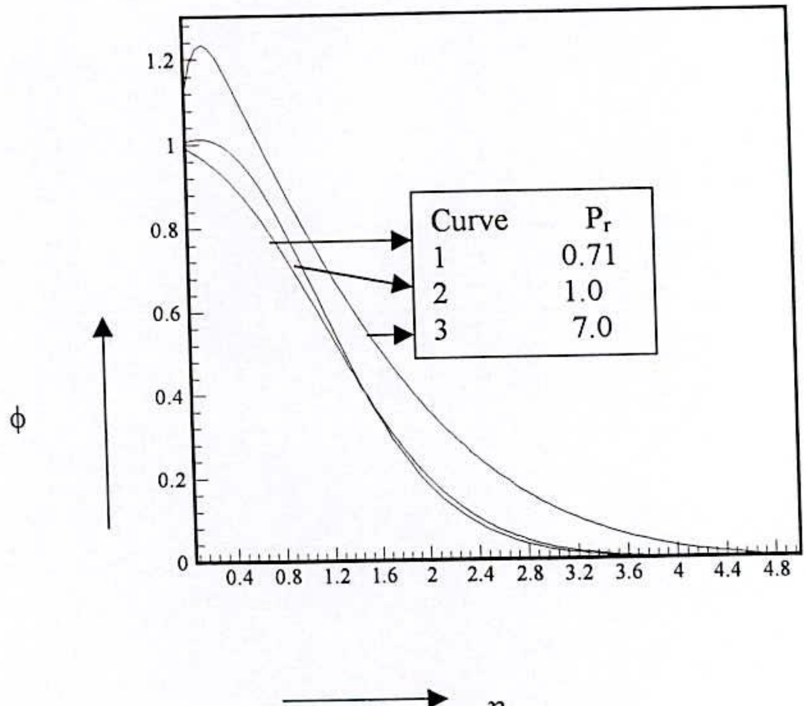

Fig.4.1.15. Concentration Profiles for different values of P_r, taking $V_0 = 0.50$, $S_0 = 2.00$, $M = 0.5$, $D_f = 0.2$, $G_r = 10.00$, $G_m = 5.00$, $S_c = 0.22$ as fixed.

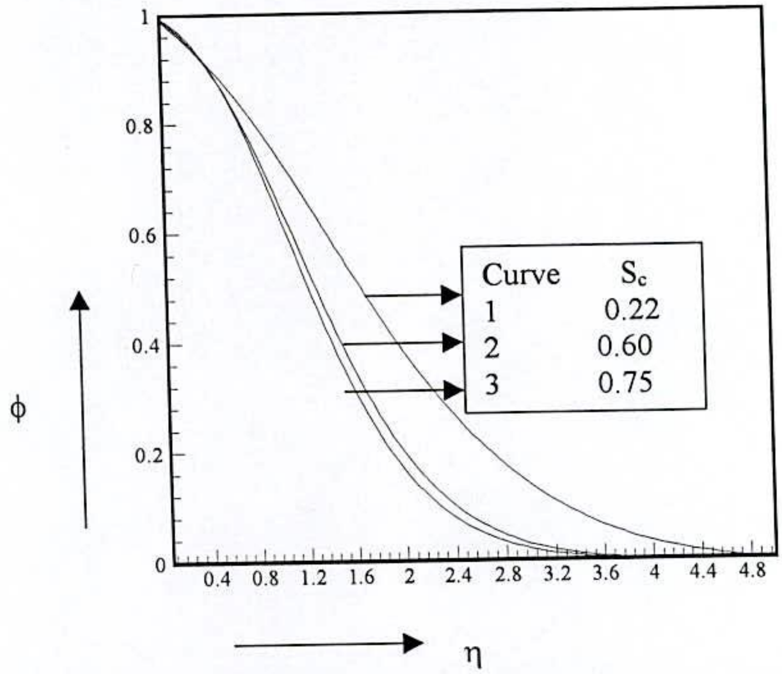

Fig.4.1.16. Concentration Profiles for different values of S_c, taking $V_0 = 0.50$, P_r=0.71, M= 0 .5, D_f=0.2 , G_r = 10.00, G_m = 5.00 , S_0 = 2.0 as fixed.

Table: 4.1.1 Numerical values of Skin Friction, Nusselt Number and Sherwood Number proportional to $f', -\theta'\, and -\phi'$ respectively, for different values of V_0 and D_f ,taking G_r =10.00,G_m = 5.00, M=0.5,S_0= 2.0, P_r=0.71,S_c=0.60 as fixed.

V_0	D_f	τ	$-\theta'$	$-\phi'$
0.5	0.2	1.784958492	1.41983732	0.22187363
1.5	0.2	1.107044722	1.973658468	0.3951249
2.5	0.2	0.25258435	2.5808135	0.397388
0.5	0.5	1.828867473	1.48232346	0.13702065
0.5	0.8	1.87162634	1.65752702	-0.82259632

Table: 4.1.2 Numerical values of Skin Friction, Nusselt Number and Sherwood Number proportional to $f', -\theta' and -\phi'$ respectively, for different values of M and S_0 ,taking G_r =10.00,G_m = 5.00, V_0 =0.5 , D_f =0.20, P_r=0.71,S_c=0.60 as fixed.

M	S_0	τ	$-\theta'$	$-\phi'$
0.5	1.0	1.784958492	1.41983732	0.22187363
1.0	1.0	1.5721742	1.4198413397	0.22188526
1.5	1.0	1.3711217	1.419846436	0.22190479
0.5	2.0	1.55512162	1.35394784	0.78080973
0.5	3.0	2.028048486	1.4981375278	-0.43033981

Table: 4.1.3 Numerical values of Skin Friction, Nusselt Number and Sherwood Number proportional to $f', -\theta' and -\phi'$ respectively, for different values of P_r and S_c ,taking G_r =10.00,G_m = 5.00, M=0.5,S_0= 2.0, D_f=0.20, V_0=0.5 as fixed.

P_r	S_c	τ	$-\theta'$	$-\phi'$
0.71	0.22	1.9344219	1.403163282	0.289813788
1.0	0.22	1.5900065	1.822974568	-0.20471111
7.0	0.22	0.386611	12.07647156	4.342624045
0.71	0.60	1.784958492	1.41983732	0.22187363
0.71	0.75	1.75809629	1.42843432	0.1710122012

Chapter 5

Steady MHD free convection and mass transfer flow past a semi-infinite vertical porous plate with thermal diffusion, Dufour effect and large suction.

5.1.1. Introduction

Cobble(1977) showed the condition under which similarity solutions exist to a hydromagnetic flow over a semi-infinite plate in presence of a magnetic field and pressure gradient with or without injection and suction. The heat-transfer aspect of this problem has been studied by Soundalgekar and Ramanamurthy(1980) taking the effects of suction and injection. Following the above studies, Singh (1985) studied the problem of Soundalgekar and Ramanamurthy(1980) for large values of suction parameter by making use of the perturbation technique as has been done by Nanbu (1971). Later Singh (1985) and Singh and Dikshit (1988) s tudied the hydromagnetic f low past a c ontinuously moving s emi-infinite porous plate employing the same perturbation technique. They also derived similarity solutions for large suction. This large suction, infact, enabled them to obtain analytical solutions and indeed these analytical solutions are of immense value and compliment various numerical solutions. However, in the above studies the thermal diffusion effect was ignored.
In view of the above studies we aim to make a study of the steady two dimensional problem of the MHD free convection and mass transfer flow past a semi-infinite vertical porous plate taking into account the Soret as well as Dufour effects. Similarity counterparts of the momentum, energy and concentration equations are derived by introducing usually employed technique. The momentum, energy and concentration equations are then solved numerically by the method of shooting based on large suction as suggested by Na (1979)

5.1.2. The Governing Equation

Let us consider a steady MHD free convective and mass transfer flow of an electrically conducting viscous fluid past a porous medium along a semi-infinite vertical porous plate y=0. The detailed descriptions of the present problem are similar to those of chapter 4.

With reference to the generalized equations described in **case-II of Chapter 2**, the two dimensional problem under the above assumptions and Boussinesq approximation can be put in the following form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5.1.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty)+g\beta^*(C-C_\infty)-\frac{\sigma' B_0{}^2 u}{\rho} \tag{5.1.2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{D_m k_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2} \tag{5.1.4}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 C}{\partial y^2}+\frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2} \tag{5.1.5}$$

and the boundary conditions for the problem are:

$$\left.\begin{array}{l} u=U_0, v=v_0(x), T=T_w, C=C_w \; at\; y=0 \\ u=0, v=0, T\to T_\infty, C\to C_\infty \; at\; y\to\infty \end{array}\right\} \tag{5.1.6}$$

where u, v are the velocity components in the x, y direction respectively, υ is the kinematic viscosity, g is the acceleration due to gravity, ρ is the density, β is the coefficient of volume expansion, β^* is the volumetric coefficient of expansion with concentration T, T_w, T_∞, are the temperatures of the fluid inside the thermal boundary layer, at the plate and in the free stream, respectively, while C, C_w, C_∞, are the corresponding concentrations. Also, k is the thermal conductivity of the medium, D_M is the coefficient of mass diffusivity, C_p is the specific heat at constant pressure, T_M is the mean fluid temperature, k_T is the thermal diffusion ratio, C_S is the concentration susceptibility and other symbols have their usual meaning.

5.1.3. Mathematical Formulations

To solve the above system of equations (5.1.2)-(5.1.5) under the boundary conditions (5.1.6), we adopt the well defined similarity analysis to attain similarity solutions.

For this purpose, we introduce the similarity variables

$$\eta=y\sqrt{\frac{U_0}{2\upsilon x}} \tag{5.1.7}$$

$$f'(\eta) = \frac{u}{U_0} \tag{5.1.8}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \tag{5.1.9}$$

$$\phi(\eta) = \frac{C - C_\infty}{\bar{x}(C_0 - C_\infty)} \tag{5.1.10}$$

The concentration at the plate is given by

$$C_w = C_\infty + \bar{x}(C_0 - C_\infty) \tag{5.1.11}$$

Where C_0 is considered to be mean concentration and $\bar{x} = \dfrac{xU_0}{\upsilon}$.

From the equation (5.1.8), we have

$$u = U_0 f'(\eta) \tag{5.1.12}$$

In order to satisfy the continuity equation, we introduce (5.1.12) in the equation (5.1.1) to obtain

$$\frac{\partial v}{\partial \eta} = \sqrt{\frac{\upsilon U_0}{2x}}\,[\eta f''(\eta)] \tag{5.1.13}$$

Integrating both sides with respect to η, we have

$$v(\eta) = \sqrt{\frac{\upsilon U_0}{2x}}\,[\eta f'(\eta) - f(\eta)] \tag{5.1.14}$$

From the equation (5.1.12), we have the following derivatives

$$\frac{\partial u}{\partial x} = -\frac{\eta}{2x} U_0 f''(\eta) \tag{5.1.15}$$

$$\frac{\partial u}{\partial y} = U_0 \sqrt{\frac{U_0}{2\upsilon x}} f''(\eta) \tag{5.1.16}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_0^{\,2}}{2\upsilon x} f'''(\eta) \tag{5.1.17}$$

Again from the equation (5.1.9), we have

$$(T - T_\infty) = (T_W - T_\infty)\theta(\eta) \tag{5.1.18}$$

Also from the equation (5.1.10) and (5.1.11), we have

$$(C - C_\infty) = (C_W - C_\infty)\,\square(\eta) \tag{5.1.19}$$

Substituting the equations (5.1.12) and (5.1.14) - (5.1.19) into the equation (5.1.2), we get

$$f''' + ff'' + G_r\theta + G_m\phi - Mf' = 0 \tag{5.1.20}$$

where $G_r = \frac{g\beta(T_w - T_\infty)}{U_0^2}2x$ is the local Grashof number, $G_m = \frac{g\beta^*(C_w - C_\infty)}{U_0^2}2x$ is the modified local Grashof number and $M = \frac{\sigma B_0^2 2x}{\rho U_0}$ is the Magnetic force number.

Now from the equation (5.1.9), we have

$$T = T_\infty + (T_W - T_\infty)\theta(\eta) \tag{5.1.21}$$

From the equation (5.1.21), we have the following derivatives

$$\frac{\partial T}{\partial x} = -\frac{\eta}{2x}(\mathrm{T_w} - T_\infty)\theta'(\eta) \tag{5.1.22}$$

$$\frac{\partial T}{\partial y} = \sqrt{\frac{U_0}{2\upsilon x}}(\mathrm{T_w} - T_\infty)\theta'(\eta) \tag{5.1.23}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{U_0}{2\upsilon x}(\mathrm{T_w} - T_\infty)\theta''(\eta) \tag{5.1.24}$$

Again from the equation (5.1.10), we have

$$C = C_\infty + (C_0 - C_\infty)\bar{x}\phi(\eta) \tag{5.1.25}$$

From the equation (5.1.25), we have the following derivatives

$$\frac{\partial C}{\partial x} = (\mathrm{C_w} - C_\infty)\frac{1}{x}\phi'(\eta) - (\mathrm{C_w} - C_\infty)\frac{1}{2x}\phi'(\eta) \tag{5.1.26}$$

$$\frac{\partial C}{\partial y} = \sqrt{\frac{U_0}{2\upsilon x}}(\mathrm{C_w} - C_\infty)\phi'(\eta) \tag{5.1.27}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{U_0}{2\upsilon x}(\mathrm{C_w} - C_\infty)\phi''(\eta) \tag{5.1.28}$$

Substituting the equations (5.1.12), (5.1.14), (5.1.22)-(5.1.24) and (5.1.28) into the equation (5.1.4), we get

$$\theta'' + P_r f\theta' + P_r D_f \phi'' = 0 \tag{5.1.29}$$

where $P_r = \frac{\rho\upsilon c_p}{k}$ is the Prandlt number and $D_f = \frac{D_m k_T}{c_s c_p \upsilon}\frac{(C_w - C_\infty)}{(T_w - T_\infty)}$ is the Dafour number.

Again substituting the equations (5.1.12), (5.1.14), (5.1.2) and (5.1.26)-(5.1.28) into the equation (5.1.5), we get

$$\phi'' - 2S_c f'\phi + S_c f\phi' + S_c S_r \theta'' = 0 \tag{5.1.30}$$

where $S_c = \frac{\upsilon}{D_m}$ is the Schmidt number and $S_0 = \frac{D_m k_T}{\upsilon T_m}\frac{(T_w - T_\infty)}{(C_w - C_\infty)}$ is the Soret number.

Thus the equations (5.1.2)-(5.1.5), reduces to the following dimensionless differential equations

$$f''' + ff'' + G_r\theta + G_m\phi - Mf' = 0 \tag{5.1.31}$$

$$\theta'' + P_r f\theta' + P_r D_f \phi'' = 0 \tag{5.1.32}$$

$$\phi'' - 2S_c f'\phi + S_c f\phi' + S_c S_r \theta'' = 0 \tag{5.1.33}$$

where G_r is the local Grashof number, G_m is the modified local Grashof number, M is the magnetic force number, P_r is the Prandtl number, S_c is the Schmidt number and S_0 is the Soret number.

The corresponding boundary conditions are

$$\left.\begin{array}{l} f = f_w, f' = 1, \theta = 1, \phi = 1\, at\, \eta = 0 \\ f' = 0, \theta = 0, \phi = 0\, as\, \eta \to \infty \end{array}\right\} \tag{5.1.34}$$

where $f_w = -v_0(x)\sqrt{\dfrac{2x}{\upsilon U_0}}$ is taken as transpiration/ suction parameter.

In section (5.1.5) the solution of the problem is given.

5.1.4. Skin friction Co-efficient, Nusselt number and Sherwood number

The quantities of chief physical interest are the local skin friction co-efficient, local Nusset number and local Sherwood number.

The equation defining the local skin friction is

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

i.e. $\tau \propto f''$

The local Nusset number, denoted by N_u, is proportional to $-\left(\dfrac{\partial T}{\partial y}\right)_{y=0}$ and it can be expressed as

$$N_u \propto -\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

That is, $N_u \propto -\theta'$

The local Sherwood number, denoted by S_h, is proportional to $-\left(\frac{\partial C}{\partial y}\right)_{y=0}$ and it can be expressed as

$$S_h \propto -\left(\frac{\partial C}{\partial y}\right)_{y=0}$$

That is, $S_h \propto -\phi'$

The values proportional to the skin friction co-efficient, the Nusselt number and the Sherwood number are respectively obtained numerically. These values are sorted in the Tables 5.1.1-5.1.3

5.1.5. Numerical Solution

Equations (5.1.31)-(5.1.33) with boundary conditions (5.1.34) are solved numerically using a standard initial value solver, we have chosen the shooting method. For the purpose of this method, we applied the Nachtscheim-Swigert iteration technique.

In a shooting method, the missing (unspecified) initial condition at the initial point of the interval is assumed and the differential equation is integrated numerically as an initial value problem to the terminal point. The accuracy of the assumed missing initial condition is then checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. If a difference exists, another value of the missing initial condition must be assumed and the process is repeated. This process is continued until the agreement between the calculated and the given condition at the terminal point is, within the specified degree of accuracy, reached. For this type of iterative approach, one naturally inquires whether or not there is a systematic way of finding each succeeding (assumed) value of the missing initial condition.

The boundary conditions (5.1.34) associated with the nonlinear ordinary differential equations (5.1.31)-(5.1.33) of the boundary layer type is of the two-point asymptotic class. Two-point boundary conditions have values of the dependent variable specified at two different values of the independent variable. Specification of an asymptotic boundary condition implies the value of velocity approaches to unity and the value of temperature approaches to zero as the outer specified value of the independent variable is approached. The method of numerical integration of two-point asymptotic boundary value problem of the boundary layer type, the initial value method, requires that the problem be recast as an initial value problem. Thus it is necessary to set up as many boundary conditions at the surface as there are at infinity. The governing differential equations are then integrated with these assumed surface boundary conditions. If the required outer boundary condition is satisfied, a solution has been achieved. However, this is not generally the case. Hence a method must be devised to logically estimate the new surface boundary conditions for the next trial integration. Asymptotic boundary value problems, such as those governing the boundary layer equations, become more complicated by the fact that the outer boundary condition is specified at infinity. In the trial integration infinity is numerically approximated by some large value of the independent variable. There is no general method of estimating this value. Selection of too small a maximum value for the independent variable may not allow the solution to asymptotically converge to the required accuracy. Selecting a large value may result in divergence of the trial integration or slow convergence. Selecting too large a value of the independent variable is expensive in terms of computer time. Nachtscheim-Swigert developed an iteration method, which overcomes these difficulties. Extension of the Nachtscheim-Swigert iteration to above system of differential equations (5.1.31)-(5.1.33) is straightforward. In equation (5.1.34) there are three asymptotic boundary conditions and hence three unknown surface conditions $f''(0)$, $\theta'(0)$. and $\phi'(0)$.

5.1.5. Results and Discussion

For the purpose of discussing the effects of various parameters on the flow behavior, some numerical calculations have been carried out for different values of f_w, M, P_r, S_0, S_c, D_f and for fixed values of G_r and G_m. The value of G_r is taken to be large (G_r =10) throughout the calculations. For Prandtl number P_r, three values 0.71, 1.0 and 7.0 are considered and three values 0.22, 0.60, and 0.75 of the Schmidt number S_c are also considered for those which represent specific conditions of the flow. In the calculations f_w, M, S_0, D_f and G_m are chosen arbitrarily. In this case we consider large suction.

The effects of various parameters on the velocity are shown in Figs. 5.1.2-5.1.7. With the above mentioned parameters, it is observed from Fig. 5.1.2 that the suction parameter f_w has a decreasing effect on the velocity with its increase. In Figs. 5.1.3 and 5.1.4, the variations of the velocity field for different values of Soret number S_0 and Dufour number D_f are shown respectively. From these figures it is observed that the primary velocity increases with the increase of Soret number S_0. The same effect is observed in the case of Dufour number D_f. In

Figs. 5.1.5, 5.1.6 and 5.1.7, the variations of the velocity field for different values of magnetic force number M, Prandtl number P_r and Schmidt number S_c are shown respectively. It is observed from these figures that the magnetic force number M has a decreasing effect on the velocity (Fig. 5.1.5). It is also seen from these figures that the velocity decreases with the increase of Prandtl number. The same effect is observed in the case of Schmidt number. The effects of various parameters on temperature field are shown in Figs. 5.1.8-5.1.11. It is observed from Fig. 5.1.8 that the temperature decreases as the suction parameter f_w increase. In Figs. 5.1.9 and 5.1.10, the temperature profiles for different values of Soret number S_0 and Dufour number D_f are shown respectively. It is observed from these figures that the Soret number S_0 has a minor decreasing effect while the Dufour number D_f has an increasing effect. In Figs. 5.1.11, the temperature field for different values of Prandtl number P_r is shown respectively. It is observed from this figure that the Prandtl number P_r has a large decreasing effect on temperature.

The effects of various parameters on the concentration field are shown in Figs. 5.1.12-5.1.16. It is observed from Fig. 5.1.12 that the concentration decreases as the suction parameter f_w increase.

In Figs. 5.1.13 and 5.1.14, the concentration profiles for different values of Soret number S_0 and Dufour number D_f are shown respectively. It is observed from these figures that the concentration increases as the Soret number S_0 increases while the concentration decreases as the Dufour number D_f increases. In Figs. 5.1.15 and 5.1.16, the concentration profiles for different values of Prandtl number P_r and Schmidt number S_c are shown respectively. It is observed from these figures that the concentration increases as the Prandtl number P_r increases while the concentration decreases as the Schmidt number S_c increases.

Finally, the effects of various parameters on the skin friction τ, the Nusselt number N_u and the Sherwood number S_h are shown in the Tables 5.1.1-5.1.3. From Table 5.1.1, we observe that the skin friction τ decrease while the Nusselt number N_u and the Sherwood number S_h increases with increase of the suction parameter f_w. It is also seen from this table that the skin friction τ and the Sherwood number S_h increase while the Nusselt number N_u decreases with the increase of Dufour number D_f. From Table 5.1.2, it is seen that the skin friction τ , the Sherwood number S_h and the Nusselt number N_u decreases with increase of magnetic force number M. Also, from Table 5.1.2, we observe that the skin friction τ and the Nusselt number N_u increases with the increase of Soret number S_0, but the Sherwood number S_h

decreases. From table 5.1.3, we see that the skin friction τ decreases while the Sherwood number S_h and the Nusselt number N_u increases with the increase of Prandtl number P_r. Further, from Table 5.1.3, we observe that the skin friction τ and the Nusselt number N_u decrease with increase of Schmidt number S_c, but the Sherwood number S_h increases.

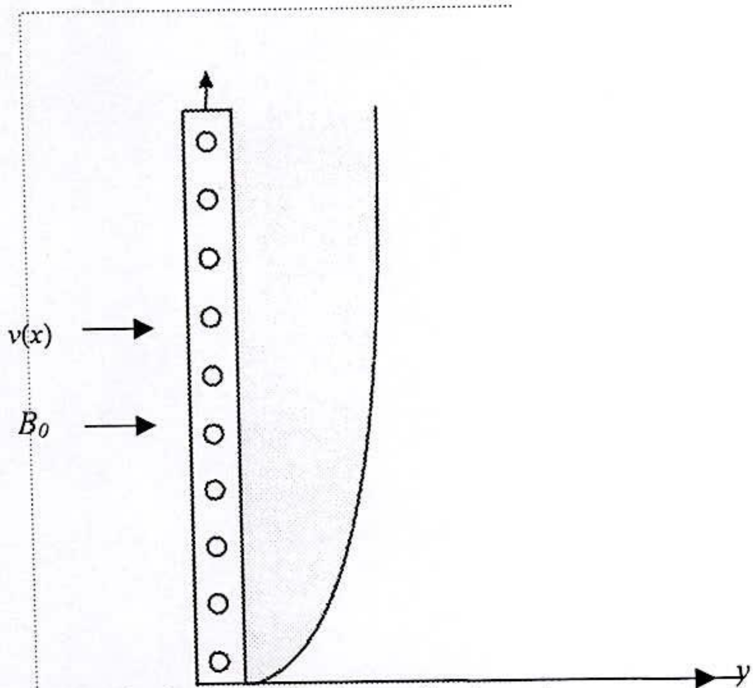

Fig. 5.1.1. Physical model and coordinate system.

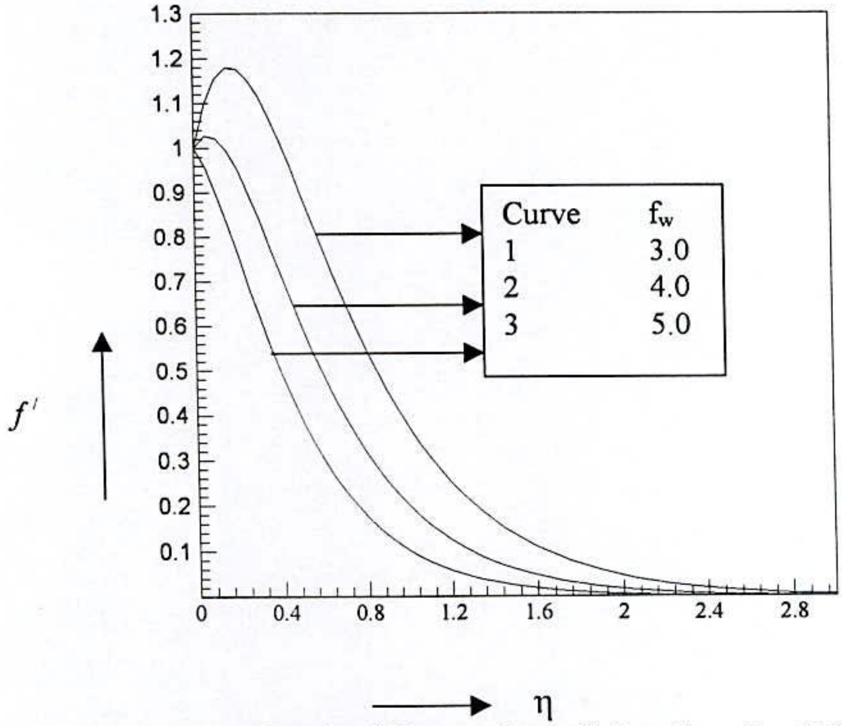

Fig.5.1.2. Velocity Profiles for different values of f_w, taking G_r =10.00 , G_m = 4.00 , M = 0 .50 , P_r= 0.71 , S_0=1.00 , S_c = 0.60 , D_f= 0 .20 as fixed.

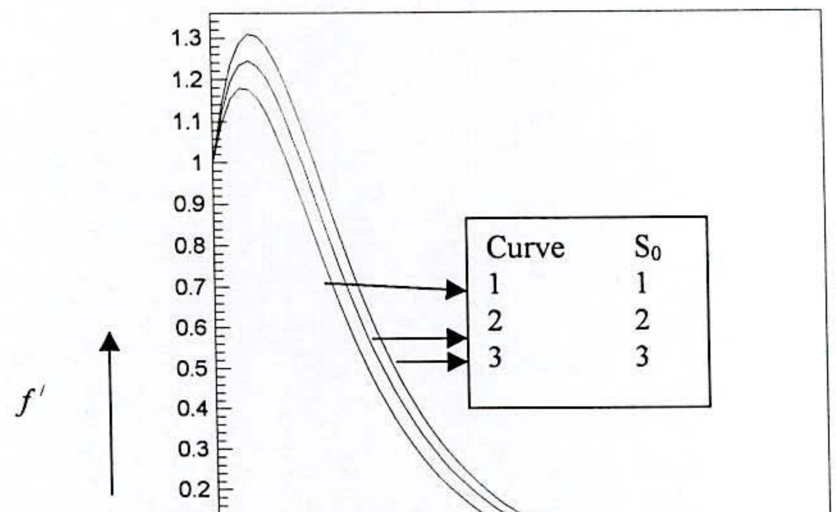

Fig.5.1.3. Velocity Profiles for different values of S_0, taking G_r =10.00 , G_m = 4.00 , f_w = 3 .00 , P_r= 0.71 , S_c=0.60 , D_f = 0.20 , M = 0 .50 as fixed.

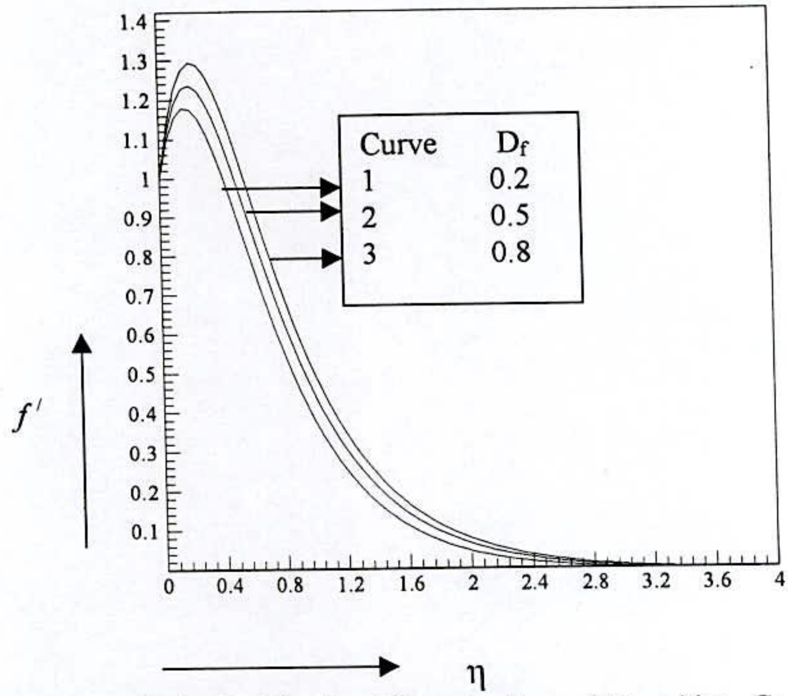

Fig.5.1.4. Velocity Profiles for different values of D_f, taking $G_r = 10.00$, $G_m = 4.00$, $f_w = 3.00$, $P_r = 0.71$, $S_0 = 1.00$, $S_c = 0.60$, $M = 0.50$, as fixed.

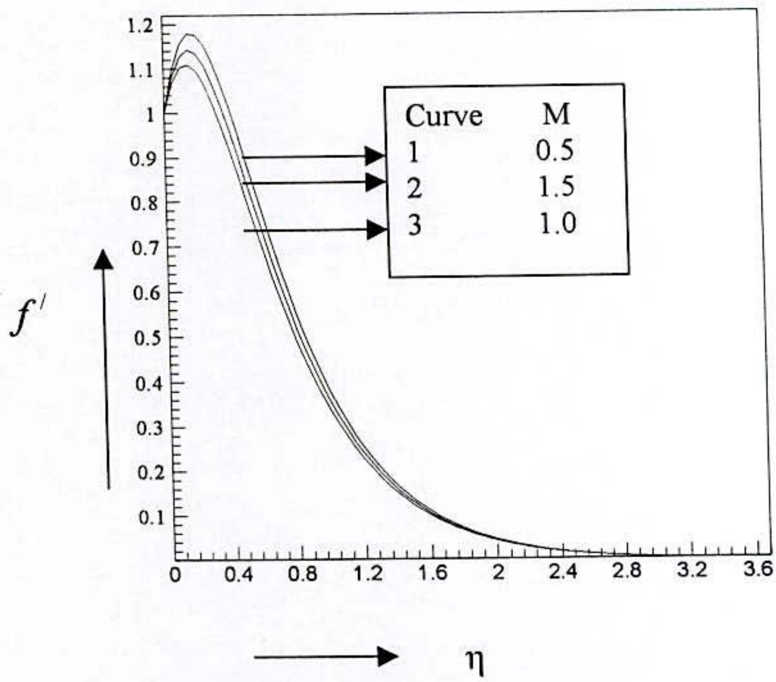

Fig.5.1.5. Velocity Profiles for different values of M, taking $G_r = 10.00$, $G_m = 4.00$, $f_w = 3.00$, $P_r = 0.71$, $S_0 = 1.00$, $S_c = 0.60$, $D_f = 0.20$ as fixed.

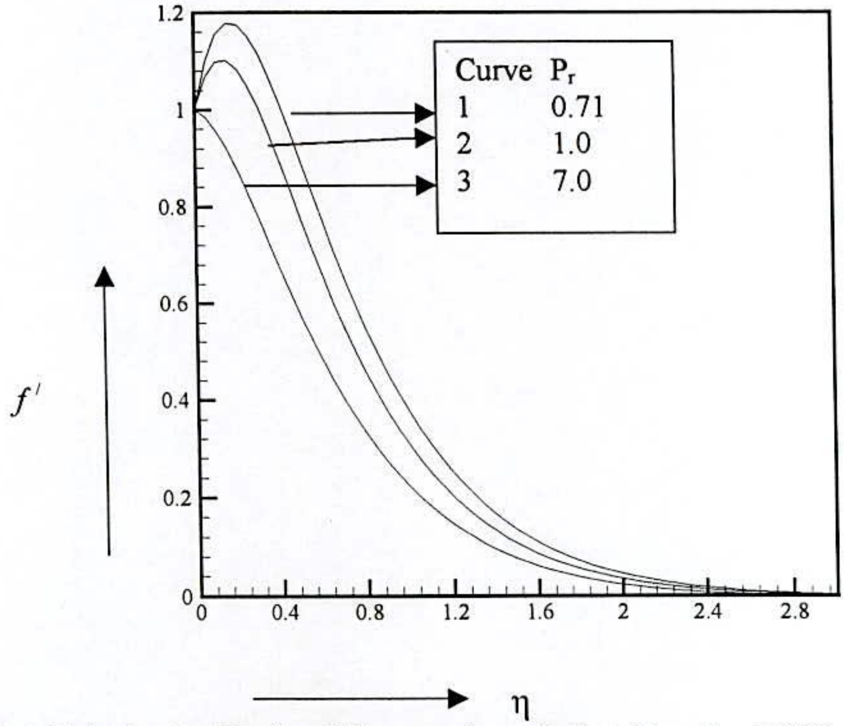

Fig.5.1.6. Velocity Profiles for different values of P_r, taking G_r =10.00 ,G_m = 4.00, f_w =3 .00 , S_c = 0.60 , S_0 =1.00 , D_f = 0.20 , M = 0 .50 as fixed.

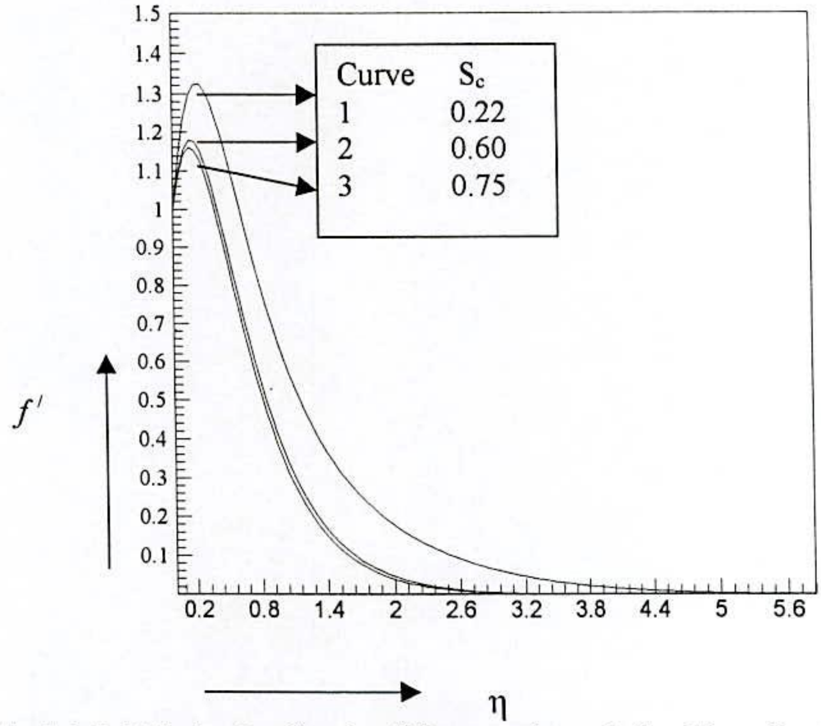

Fig.5.1.7. Velocity Profiles for different values of S_c, taking G_r =10.00 , G_m = 4.00 , f_w = 3 .00 , P_r = 0.71 , S_0 =1.00 , D_f = 0.20 , M = 0 .50 , as fixed.

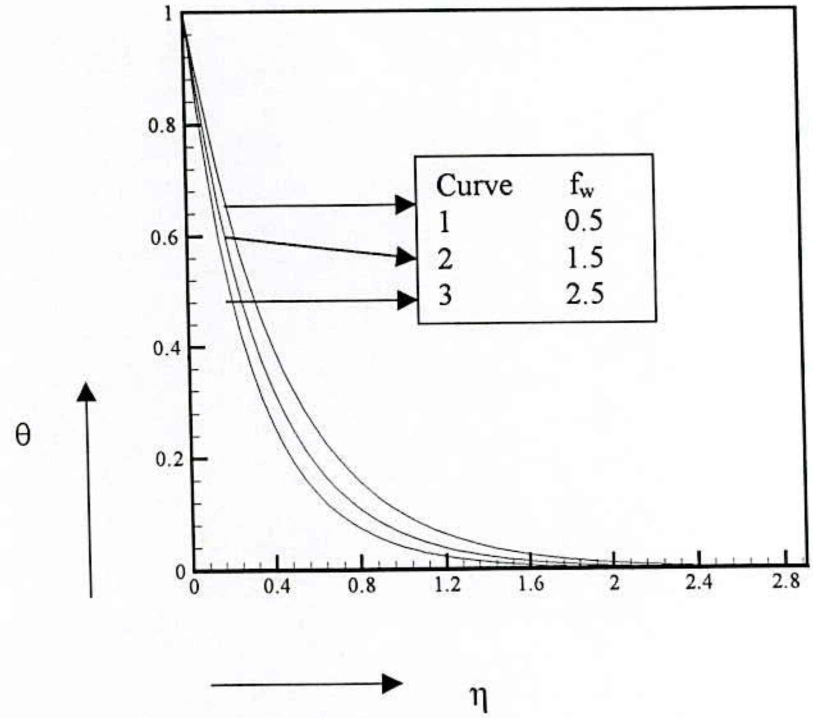

Fig.5.1.8. Temperature Profiles for different values of f_w, taking G_r =10.00 , G_m = 4.00 , M = 0 .50 , P_r = 0.71 , S_0 =1.00 , S_c = 0.60 , D_f = 0 .20 as fixed.

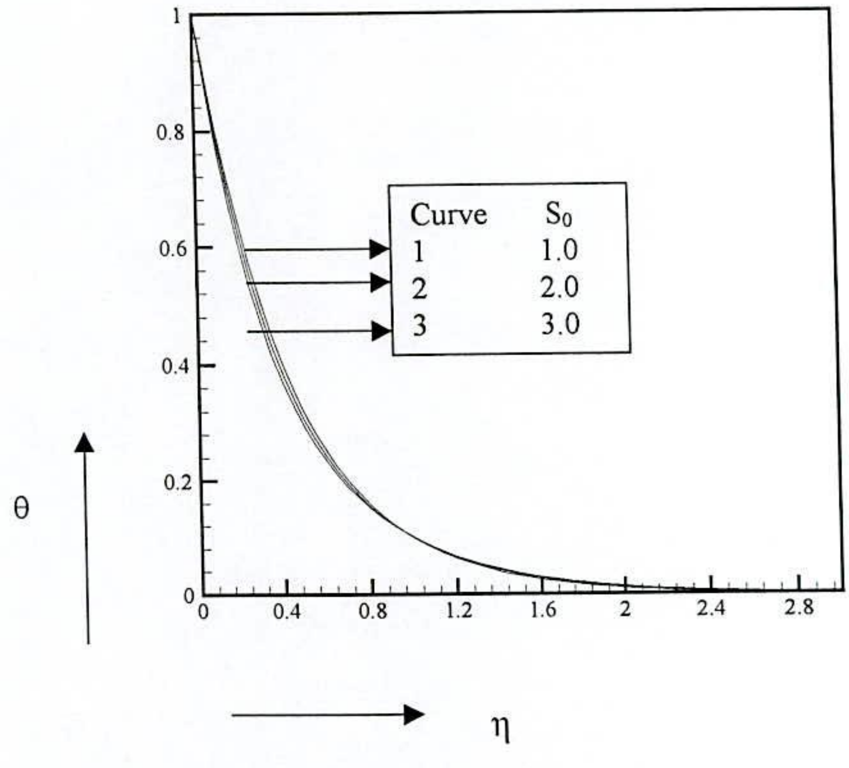

Fig.5.1.9 Temperature Profiles for different values of S_0, taking G_r =10.00 , G_m = 4.00 , f_w = 3 .00 , P_r = 0.71 , S_c =0.60 , D_f = 0.20 , M = 0 .50 as fixed.

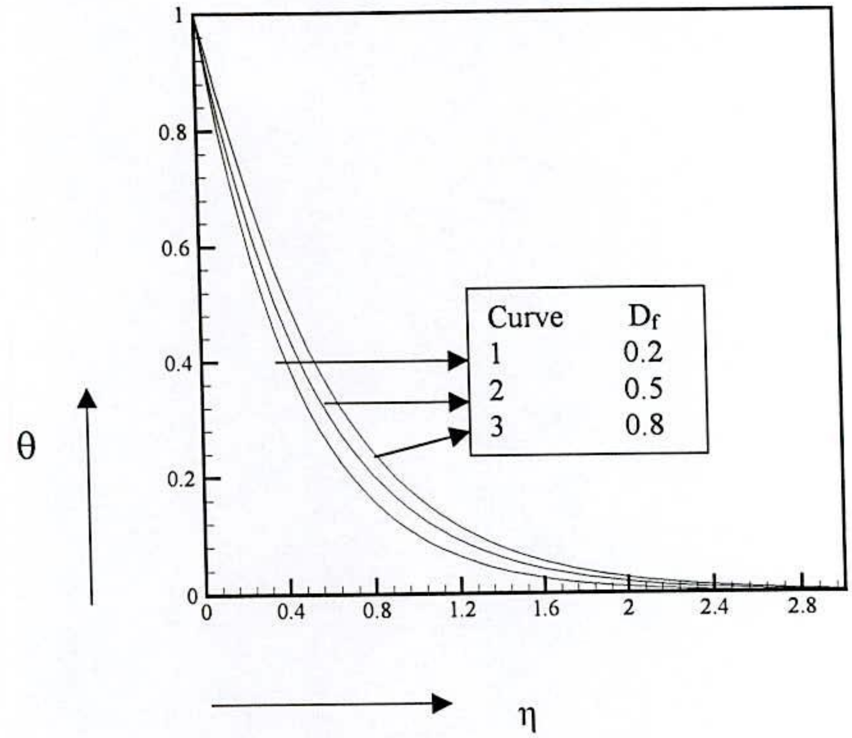

Fig.5.1.10. Temperature Profiles for different values of D_f, taking $G_r = 10.00$, $G_m = 4.00$, $f_w = 3.00$, $P_r = 0.71$, $S_0 = 1.00$, $S_c = 0.60$, $M = 0.50$, as fixed.

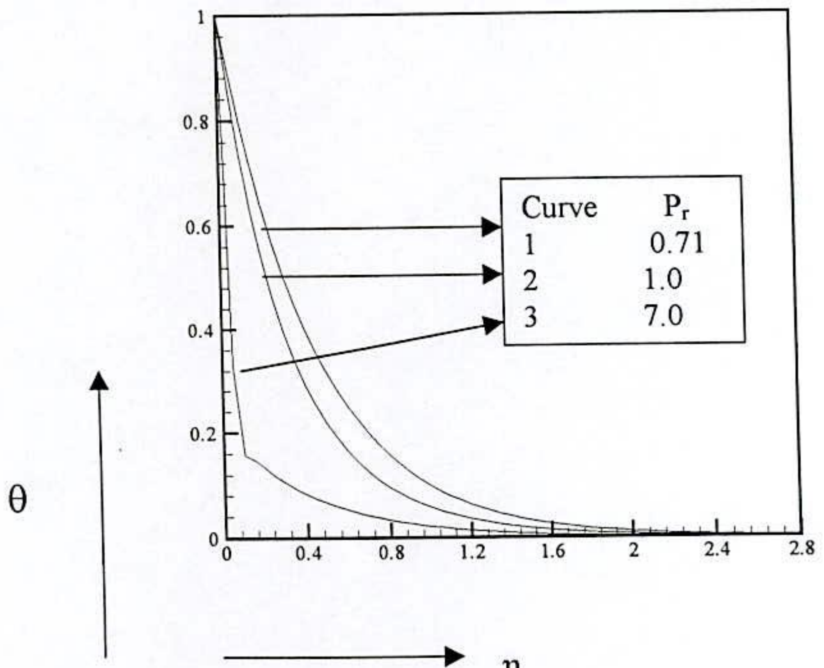

Fig.5.1.11. Temperature Profiles for different values of P_r, taking $G_r = 10.00$, $G_m = 4.00$, $f_w = 3.00$, $S_c = 0.60$, $S_0 = 1.00$, $D_f = 0.20$, $M = 0.50$ as fixed

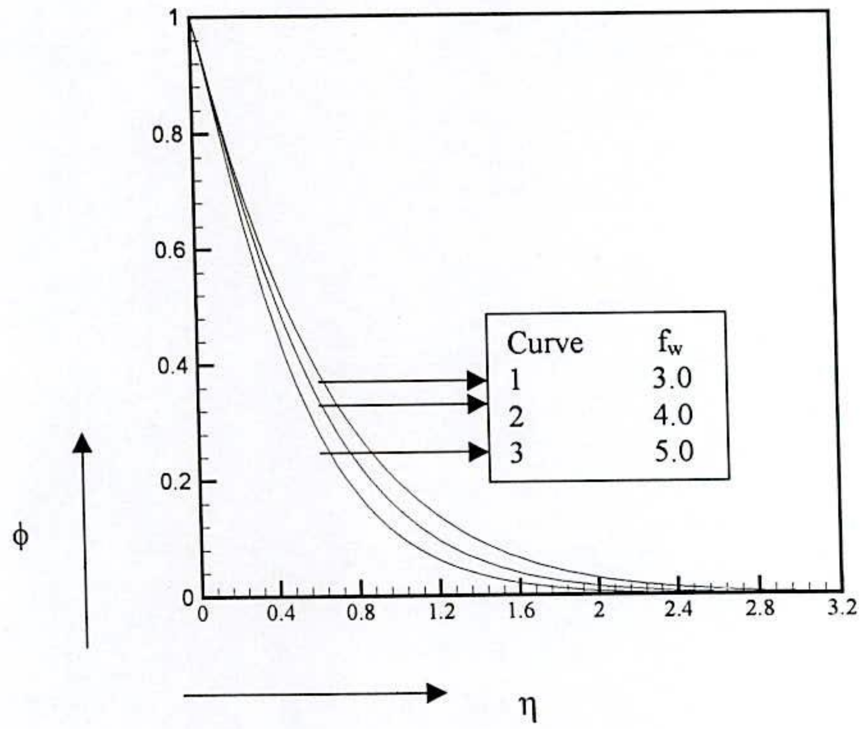

Fig.5.1.12. Concentration Profiles for different values of f_w, taking G_r =10.00 G_m = 4.00 , M = 0 .50 , P_r = 0.71 , S_0 =1.00 , S_c = 0.60 , D_f = 0 .20 as fixed.

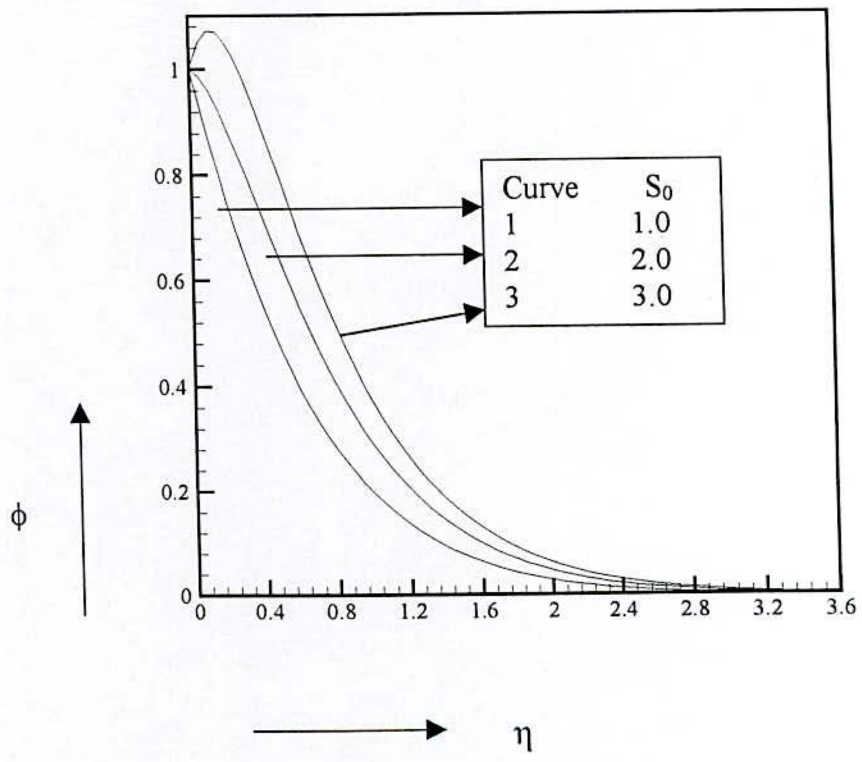

Fig.5.1.13. Concentration Profiles for different values of S_0, taking G_r =10.00,

$G_m = 4.00$, $f_w = 3.00$, $P_r = 0.71$, $S_c = 0.60$, $D_f = 0.20$, $M = 0.50$ as fixed.

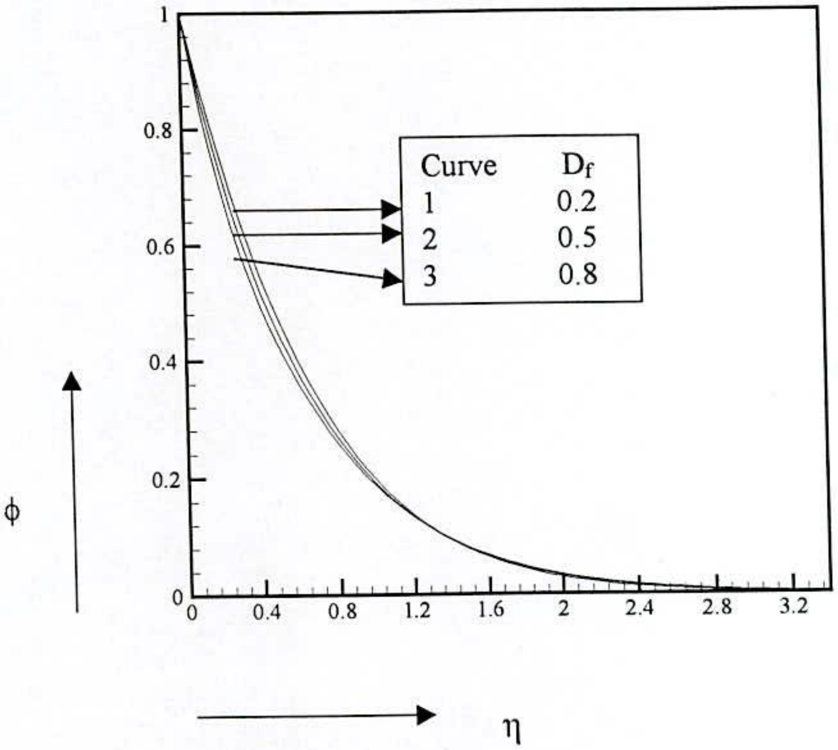

Fig.5.1.14. Concentration Profiles for different values of D_f, taking $G_r = 10.00$, $G_m = 4.00$, $f_w = 3.00$, $P_r = 0.71$, $S_0 = 1.00$, $S_c = 0.60$, $M = 0.50$, as fixed.

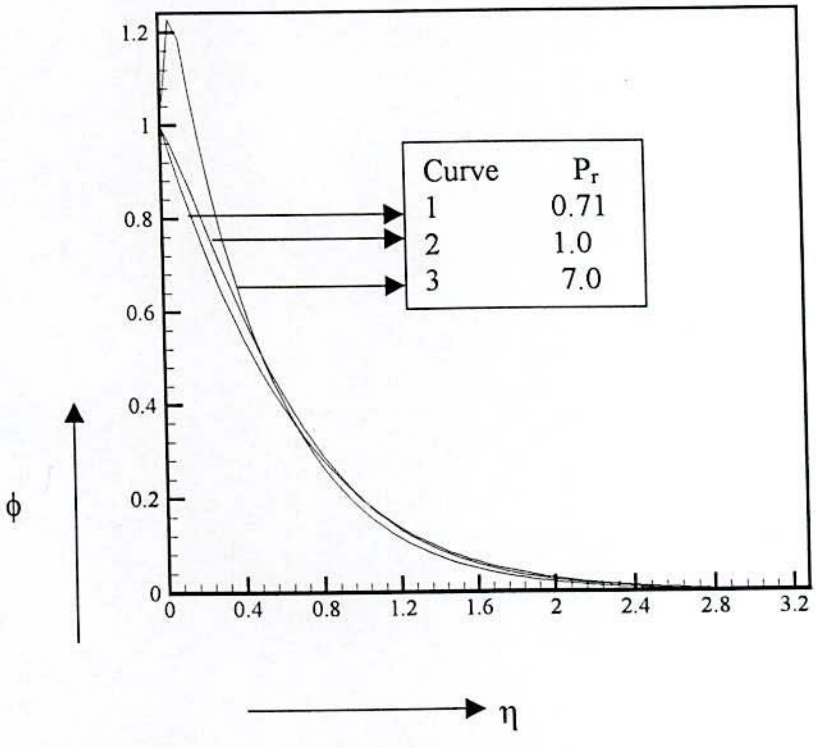

Fig 5.1.15. Concentration Profiles for different values of P_r, taking $G_r = 10.00$,

$G_m = 4.00$, $f_w = 3.00$, $S_c = 0.60$, $S_0 = 1.00$, $D_f = 0.20$, $M = 0.50$ as fixed.

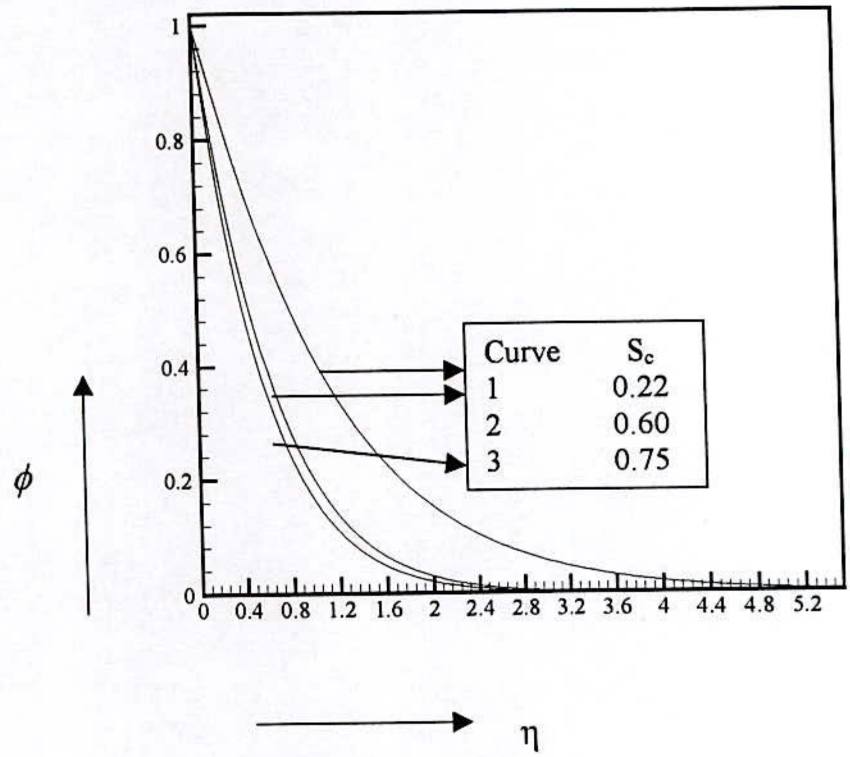

Fig.5.1.16. Concentration Profiles for different values of S_c, taking $G_r = 10.00$, $G_m = 4.00$, $f_w = 3.00$, $P_r = 0.71$, $S_0 = 1.00$, $D_f = 0.20$, $M = 0.50$, as fixed.

Table: 5.1.1 Numerical values of Skin Friction, Nusselt Number and Sherwood Number proportional to $f'',-\theta' and -\phi'$ respectively, for different values of f_w and D_f ,taking G_r =10.00,G_m = 4.00, M=0.5,S_0= 1.0, P_r=0.71,S_c=0.60 as fixed.

f_w	D_f	τ	$-\theta'$	$-\phi'$
3.00	0.20	2.4593537	2.21977383	1.384951
4.00	0.20	.9303722928	2.845368	1.359548
5.00	0.20	-.63798719	3.49663641	1.393932
3.00	0.50	2.87896333959	1.892243032	1.6117786
3.00	0.80	3.28386396	1.432133258	1.91051521

Table: 5.1.2 Numerical values of Skin Friction, Nusselt Number and Sherwood Number proportional to $f'',-\theta' and -\phi'$ respectively, for different values of M and S_0 ,taking G_r =10.00,G_m = 4.00, D_f=0.20,f_w= 3.00, P_r=0.71,S_c=0.60 as fixed.

M	S_0	τ	$-\theta'$	$-\phi'$
0.5	1.00	2.4593537	2.21977383	1.384951
1.00	1.00	2.11764846	2.2140913	1.3606401
1.50	1.00	1.802440554	2.208901627	1.33807262
0.5	2.00	2.93868726	2.409968647	0.144219291
0.5	3.00	3.38549656	2.635870087	-0.135753897

Table: 5.1.3 Numerical values of Skin Friction, Nusselt Number and Sherwood Number proportional to $f',-\theta' and -\phi'$ respectively, for different values of P_r and S_c ,taking G_r =10.00,G_m = 4.00, M=0.5,f_w= 3.0, S_0=1.0,D_f=0.20 as fixed.

P_r	S_c	τ	$-\theta'$	$-\phi'$
0.71	0.60	2.4593537	2.21977383	1.384951
1.0	0.60	1.798839883	3.146054229	7.8857607
7.0	0.60	-0.20454209	137.323342503	79.76484544
0.71	0.22	3.452743603	2.350056672	0.711242531
0.71	0.75	2.306378539	2.176246469	1.652005523

General Discussions

Model studies on various aspects of the magnetohydrodynamic free convection and mass transfer flows have been made considering some physical viability of the flows. In natural processes or in engineering problems, the types of flows that arise are of similar nature to the model studies made here in. The well recognized mathematical approach, numerical method, has been adopted to analyze the equations being constructed separately. The model flows were considered into two parts, one comparatively simple one dimensional model flows, and the second one is a relatively difficult two-dimensional model flows. In the 1st case the problem was considered to be unsteady while in the 2nd case the steady problem was taken into account. As for the unsteady one dimensional problem, similarity solutions have been obtained by introducing a similarity parameter $\sigma(t)$, the functional value of which has been obtained during the process of analyses. This functional value was found to correspond exactly with the usual similarity length scale considered prior to the analyses adopted in unsteady problem. The advantage of taking this similarity parameter $\sigma(t)$ is that one can easily obtain the similarity equation of a governing equation as has been found in chapter 4. The results obtained of the problem have been discussed and analyzed in the respective chapters. Since no experimental results of the corresponding studies are available, comparison of our results could not be made with experimental results. However, in chapter 4 and chapter 5 qualitative agreement of our results with other results due to numerical solution is very good.

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