STUDY ON DISTRIBUTIVE LATTICES AND BOOLEAN ALGEBRAS.

A THESIS PRESENTED FOR THE DEGREE OF MASTER OF PHILOSOPHY

By

MD. ZAIDUR RAHMAN

B.Sc. Hons, M.Sc(Pure Math) (D.U.)

in the

Department of Mathematics

Khulna University of Engineering & Thchnology.

Khulna-9203, Bangladesh.

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We hereby recommend that the thesis prepared by

MD. Zaidur Rahman

Roll No. 0051501, Registration No. 106, Session 2000-2001

entitled "**STUDY ON DISTRIBUTIVE LATTICES AND BOOLEAN ALGEBRAS**" be accepted as fulfilling this part of the requirements for the degree of Master of Philosophy in the Department of Mathematics

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To my parents,

who have profoundly

influenced my life

STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

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Md. Zaidur Rahman.

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Contents

Summary

In this thesis we have studied the nature of distributive Lattice and Boolean Algebra. Lattice theory is branch of Mathematics. A poset $(L,\leq)$ is said to be form a Lattice if for every $a,b \in L$, $a \vee b$ and $a \wedge b$ exist in L. where $\vee$, $\wedge$ are two binary operation. A letter L is called lattice, if it is distributive lattice then we have shown that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a,b,c \in L$. In this thesis we give several results on distributive Lattice, Boolean algebra and Boolean ring which are certainly extend and generalized many results in Lattice theory. The material of this thesis has been divided into five Chapters. A brief scenario of which we present as below.

Chapter one we have discussed basic definition of set, Lattice, convex sub lattice, meet semi-lattice and joint semi-lattices which are the basic to this thesis. We also prove that if A and B are two Lattices, that the product of A and B is a Lattice. In this Chapter we have also discussed the definition of ideals, bounded lattice, finite lattice, Complemented lattice and relatively complemented lattice. We have established the relations among them. Also we studied some other properties of these concepts. We have prove that two bounded Lattice are complemented iff the cartesian product of the two Lattice is complemented.

In Chapter two we have discussed Modular lattice, Distributive lattice. We include some characterization of modular and distributive Lattices. We have also proved a modular lattice is distributive lattice if and only if it has no sublattice isomorphic M_5.

In Chapter three we discuss Pseudocomplemented lattice, Stone lattice, Stone algebra are discussed. We have proved the theorem let L be a

pseudocomplementd distributive lattice and P be a prime ideal of L. Then the following conditions are equivalent.

(i) P is minimal.

(ii) $x \in P$ implies $x* \notin P$.

(iii) $x \in P$ implies $x ** \in P$.

(iv) $P \cap D(L) = \Phi$

In Chapter four Boolean algebra has discussed here. Since Boolean Lattice, Boolean subalgebra have been studied by several authors including Cornish [9] and A. Monteiro [33]. We have established the relation among them. Also we have studied some other properties of this concept. We also proved that in a Boolean algebra, the following result are holds

(i) $(a')' = a$

(ii) $(a \wedge b)' = a' \vee b'$ [De Morgan's Law]

(iii) $(a \vee b)' = a' \wedge b'$ [De Morgan's Law]

(iv) $a \leq b \Leftrightarrow a' \geq b'$

(v) $a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = u$

In Chapter five Boolean ring, Disjunctive Normal form, Conjunctive Normal form are expressed here. We also have showed every Boolean ring with unity is a Boolean algebra.

Last section in this chapter we should try to discussed the switching circuit system. The simplest example of such switch being on ordinary ON-OFF. These are two basic way in which switches are generally interconnected. These are referred to as in series and parallel. We have also explained with figure the circuit represented by the Boolean function $f = a \wedge (b \vee c)$.

"Lattice and convex sublattice"

1.1 Introduction

In this chapter we recall some definitions and known results on Lattice, convex sublattice and ideals. Some more definitions and result are included in the relevant chapters. We consider this chapter as the base and background for the study of subsequent chapters. The intention of this chapter is to outline and fixed the notation for some of the concepts of ideals, convex sublattice, meet and joint semi-lattice of a Lattice which are the basic of this thesis.

The ideal, meet and joint semi-lattice all are introduced by Gratzer (15), Cornish (9), Noor (35) in their several papers. The ideals have also been used for improving some results J. Nieminen (34) . The meet and joint semi-lattices have been studied extensively by Noor and Latif (36).

Cornish and Hickman (10) has defined meet semi-lattices and joint semi-lattices by introducing upper bound property.

A sublattice of a Lattice L is a convex sublattice if and only if for all $x, y \in K,\ (x \leq y),\ [x, y] \subseteq K$.

Definition (Set) : Any collection of objects which are related to each other

Example 1.1.1 : $A = \{1, 2, 3\}$ is a set.

Definition (Finite set) : A set is finite if it consists of a specific number of different elements ie. if in counting the different members of the set the counting process can come to end.

Example 1.1.2: Let M be the set of days of week. Then M is finite.

Definition (Infinite set): A set is infinite if it does not consist of a specific number of different elements ie. if in counting the different members of the set the counting process can not come to end.

Example 1.1.3 : Let $A = \{1, 5, 10, 15, \cdots\cdots\}$. Then A is infinite.

Definition (Comparable) : Two sets A and B are said to be comparable if

$$A \subset B \quad \text{or } B \subset A$$

ie if one set is a subset of the other.

Example 1.1.4 : Let $A = \{1, 5, 10, 15\}$ and $B = \{1, 5, 10, 15, \cdots\cdots\}$. Then $A \subset B$

ie A and B are comparable.

Definition (Line diagrams): If $A \subset B$, then we write B on a higher level then A and connect them by a line;

Fig. 1.1

If $A \subset B$ and $B \subset C$, we write

Fig. 1.2

Example 1.1.5 : Let $A = \{1\}, B = \{2\}$ and $C = \{1, 2\}$ Then the line diagram of A, B and C is

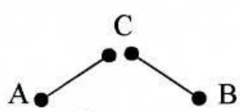


Fig. 1.3

Example 1.1.6 : Let $X = \{x\}$, $Y = \{x, y\}$, $Z = \{x, y, z\}$ and $W = \{x, y, w\}$

Then the line diagram of X, Y, Z and W is

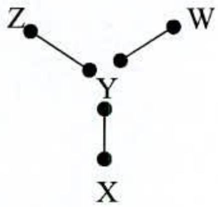


Fig. 1.4

Definition (Power set) : The family of all the subsets of any set S is called the power set of S. We denoted the power set of S by 2^S.

Example 1.1.7 : Let $M = \{x, y\}$. Then $2^M = \{\{x, y\}, \{x\}, \{y\}, \phi\}$

Definition (Disjoint set): If sets A and B have no elements in common. ie. no element of A is in B and no element of B is in A, then we say A and B are disjoint.

Example 1.1.8 : Let $Y = \{x, y\}$, $Z = \{x, y, z\}$, then Y and Z are not disjoint since x, y in both sets ie $x, y \in Y$ and $x, y \in Z$

Example 1.1.9: Let $E = \{x, y, z\}$and $F = \{r, s, t\}$. Then E and F are disjoint.

Theorem 1.1.1: Let A and B be two sets which are not comparable. Construct the line diagram of A, B and $A \cap B$.

Proof: $A \cap B$ is a subset of both A and B that is $A \cap B \subset A$ and $A \cap B \subset B$. Accordingly, we have the following diagram

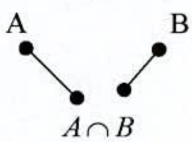


Fig. 1.5

Definition (Function) : Let A and B be two sets , a relation $R : A \to B$ is called a function if each element of A is assigned to a unique element of B.

Definition (Domain and co-domain): If the relation $R : A \to B$ is a function then the set A is called domain and the set B is called co-domain.

Definition (One One function) : Let f be a function from A to B then the function f is said to be one one function if every element of A is assigned to a single element of B.

Definition (Onto function) : Let f be a function from A to B then the function f is said to be onto function if every element of B is assigned.

Definition (Product function) : Let f be a function of A into B and let g be a function of B, the co-domain of f, into C. The new function is called a product function or composite function of f and g and it is denoted by $(go\,f)$ or $(g\,f)$

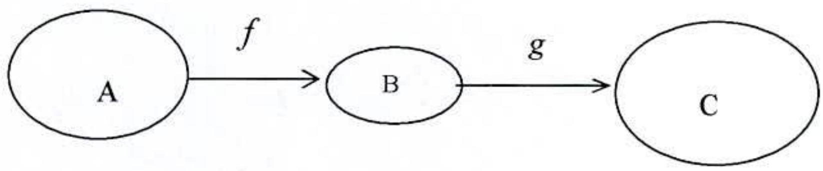


Fig. 1.6

1.2 Relation ,Lattice, Convex Sublattice .

Definition (Relation) : A relation R from A to B is a subset of $A \times B$.

Example 1.2.1 : Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then

$R = \{(1, a), (1, b), (3, c)\}$ is a relation from A to B.

Definition (Equivalance Relation) : A relation R in a set A is an equivalence relation if

(i) R is reflexive, that is for every $a \in A$, $(a,a) \in R$

(ii) R is symmetric, that is ,$(a,b) \in R$ implies $(b,a) \in R$

(iii) R is transitive, that is $(a,b) \in R$, and $(b,c) \in R$ implies $(a,c) \in R$

Example 1.2.2 : Let $A = \{1, 2, 3\}$ be a set and

$R = \{(1,1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3)\}$

be a relation of $A \times A$ then the relation is an equivalence relation, since

(i) R is reflexive, $(1,1), (2, 2), (3, 3) \in R$,

(ii) R is symmetric, $(1, 2), (2, 1), (1, 3), (3, 1) \in R$ and

(iii) R is transitive, $(2, 1), (1, 3), (2, 3) \in R$.

Definition (Partially ordered set): A nonempty set P, together with a binary relation is said to form a partially ordered set or a poset if the following conditions hold:

P1: Reflexivity: aRa for all $a \in P$

P2: Anti-symmetry: If aRb, bRa then $a = b$ $(a, b \in P)$

P3: Transitivity: If aRb, bRc then aRc $(a, b, c \in P)$

Example 1.2.3 : Let X be a non empty set. Then $P(X)$, the power set of X (ie, set of all subsets of X) under $\subseteq$ forms a poset. Here if $A, B \in P(X)$, then $A \leq B$ means $A \subseteq B$.

Definition (Totally ordered set) : If P is a poset in which every two members are comparable it is called a totally ordered set or a toset or a chain.

Thus if P is a chain and $x, y \in P$ then either $x \leq y$ or $y \leq x$.

Clearly also if x, y are distinct elements of a chain then either $x < y$ or $y < x$.

Definition (Greatest element of a poset) : Let P be a poset. If $\exists$ an element $a \in P$ s.t. $x \leq a$ for all $x \in P$ then a is called greatest or unit element of P. Greatest element if exists, will be unique.

Definition (Least element of a poset) : Let P be a poset. If $\exists$ an element $b \in P$ s.t. $b \leq x$ for all $x \in P$ then b is called least or zero element of P. Least element if exists, will be unique.

Example 1.2.4 : Let $X = \{1, 2, 3\}$. Then $(P(X), \subseteq)$ is a poset.

Let $A = \{\phi, \{1, 2\}, \{2\}, \{3\}\}$ then $(A, \subseteq)$ is a poset with ϕ as least element. A has no greatest element. Let $B = \{\{1, 2\}, \{2\}, \{3\}, \{1, 2, 3\}\}$ then B greatest element $\{1, 2, 3\}$ but no least element. If $C = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ then C has both least and greatest elements namely, ϕ and $\{1, 2\}$

Definition (Maximal element) : An element a in a poset P is called maximal element of P if $a < x$ for no $x \in P$.

Definition (Minimal element) : An element b in a poset P is called a minimal element of P if $x < b$ for no $x \in P$.

Definition (Upper bound of a set) : Le S be a non empty subset of a poset P. An element $a \in P$ is called an upper bound of S if $x \leq a \; \forall x \in S$

Definition (Least upper bound of a set) : If a is an upper bound of S s.t. $a \leq b$ for all upper bounds b of S then a is called least upper bound (l.u.b) or supremum of S. We write sup S for supremum S.

It is clear that there can be more upper bound of a set. But sup, if it exists, will be unique.

Definition (Lower bound of a set) : An element $a \in P$ will be called a lower bound of S if $a \leq x \ \forall x \in S$.

Definition (greatest lower bound of a set) : If a is a lower bound of a set S. Then a will be called greatest lower bound (g.l.b) or Infimum S (Inf S) if of a set $b \leq a$ for all lower bounds b of S.

Example 1.2.5: Let $(Z, \leq)$ be the poset of integers

Let $S = \{\cdots, -2, -1, 0, 1, 2\}$ then $2 = Sup\, S$

Again the poset $(R, \leq)$ of real numbers if $S = \{x \in R \mid x < 0, x \neq 0\}$ then $Sup\, S = 0$ (and it does not belong to S).

Definition (Lattice) : A poset $(L, \leq)$ is said to form a lattice if for every $a, b \in L$, $Sup\{a,b\}$ and $Inf\,\{a,b\}$ exist in L.

In that case, we write

$Sup\{a,b\} = a \vee b$ (read a join b)

$Inf\,\{a,b\} = a \wedge b$ (read a meet b)

Other notations like $a + b$ and $a \cdot b$ or $a \cup b$ and $a \cap b$ are also used for $Sup\{a,b\}$ and $Inf\,\{a,b\}$.

Example 1.2.6 : Let X be a non empty set, then the poset $(P(X), \subseteq)$ of all subset of is a lattice. Here for $A, B \in P(X)$

$$A \wedge B = A \cap B \text{ and } A \vee B = A \cup B$$

As particular case, when $X = \{1,2,3\}$

$$P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

It represented by the following figure

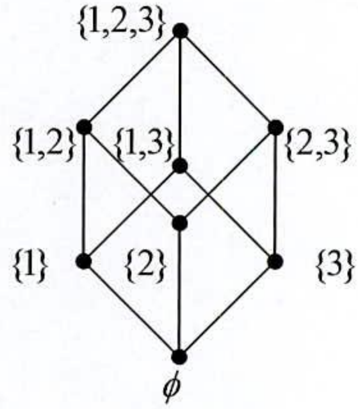


Fig. 1.7

Example 1.2.7 : Every chain is a lattice. Since any two elements a, b of a chain are comparable, say $a \leq b$, we find

$$a \wedge b = Inf\{a,b\} = a, a \vee b = Sup\{a,b\} = b$$

Example 1.2.8 : The set $L = \{1, 2, 3, 4, 6, 12\}$ of factors of 12 under divisibility forms a lattice. It is represented by the following diagram

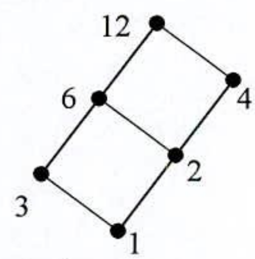


Fig. 1.8

Theorem 1.2.1: A poset $(L, \leq)$ is a lattice iff every non empty subset of L has Sup and Inf.

Proof: Let $(L, \leq)$ be a lattice. Let S be any non empty finite subset of L

Case (i) S has one element a, then $Inf\ S = Sup\ S = a$

Case (ii) S has two elements a, b; then by definition of lattice, *Sup S* and *Inf S* exist.

Case (iii) S has three elements. Let $S = \{a,b,c\}$

Since by definition of lattice any two elements of L have Sup and Inf.

We take $d = Inf\{a, b)$, $e = Inf\{c, d\}$.

We show $e = Inf\{a, b, c\}$

By definition of d and e, $d \le a, d \le b, e \le c, e \le d$

Thus $e \le a, e \le b, e \le c$

$\Rightarrow e$ is lower bound of $\{a, b, c\}$.

If f is any lower bound of $\{a, b, c\}$then

$f \le a, f \le b, f \le c$

$f \le a, f \le b,$ and $d = Inf\{a, b\}$ gives $f \le d$

$f \le c, f \le d$ and $e = Inf\{c, d\}$ gives $f \le e$

Hence $e = \inf\{a, b, c\} = \inf S$

Similarly Sup S exists.

The result can similarly be extended to any finite number of elements in S.

Indeed

$$\inf S = \inf\{.....\inf\{\inf\{a_1, a_2\}, a_3\},.....a_n\}$$

If $S = \{a_1, a_2,......,a_n\}$.

Conversely, the result holds trivially as when every non empty finite subset

Has Sup. and Inf. , a subset with two elements has Sup. and Inf. •

Theorem 1.2.2: If L is any lattice, then for any $a, b, c \in L,$ the following results hold

(1) $a \wedge a = a, a \vee a = a$ (Idempotency)

(2) $a \wedge b = b \wedge a, a \vee b = b \vee a$ (Commutativity)

(3) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (Associativity)

$a \vee (b \vee c) = (a \vee b) \vee c$

(4) $a \wedge b \leq a, \ b \leq a \vee b$

(5) $a \leq b \Leftrightarrow a \wedge b = a$ (Consistency)

$\Leftrightarrow a \vee b = b$

(6) If o, $u \in L,$ then

$o \wedge a = o, o \vee a = a$

$u \wedge a = a, \quad u \vee a = u$

(7) $a \wedge (a \vee b) = a$ (Absorption)

$a \vee (a \wedge b) = a$

(8) $a \leq b, c \leq d \Rightarrow a \wedge c \leq b \wedge d$

$a \vee c \leq b \vee d$

In particular, $a \leq b \Rightarrow a \wedge x \leq b \wedge x$

$a \vee x \leq b \vee x \quad \forall x \in L$

Proof: We prove the results for the meet operation and urge the reader to Prove similarly the results for join operation.

(1) $a \wedge a = \inf\{a, a\} = \inf\{a\} = a.$

$a \vee a = \sup\{a, a\} = \sup\{a\} = a.$

(2) $a \wedge b = \inf\{a, b\} = \inf\{b, a\} = b \wedge a.$

$a \vee b = \sup\{a, b\} = \sup\{b, a\} = b \vee a.$

(3) Let $b \wedge c = d$, then $d = Inf\{b, c\}$

$\Rightarrow \quad d \le b, d \le c$

Let $e = \text{Inf}\ \{a, d\}$ then $e \le a, \quad e \le d$

Thus $e \le a, \quad e \le b, \quad e \le c$ (using transitivity)

Now proceeding as in proof of theorem 1.2.1 we find

$e = a \wedge d = a \wedge (b \wedge c) = \inf\ \{a, b, c\}$

Similarly, we can show that $(a \wedge b) \wedge c = \inf\ \{a, b, c\}$.

Let $b \vee c = d$, then $d = \sup\{b, c\}$

$\Rightarrow \quad d \ge b, \quad d \ge c$

Let $e = \sup\{a, d\}$ then $e \ge a, \quad e \ge d$

Thus $e \ge a, \quad e \ge b, \quad e \ge c$ (using transitivity)

Now proceeding as in proof of theorem 1.2.1 we find

$e = a \vee d = a \vee (b \vee c) = \sup\ \{a, b, c\}$

Similarly, we can show that $(a \vee b) \vee c = \sup\ \{a, b, c\}$.

Hence $a \vee (b \vee c) = (a \vee b) \vee c$

(4) Follows by definitions of meet and join.

(5) $a \le b, a \le a$ (by reflexivity)

$\Rightarrow a$ is lower bound of $\{a, b\}$ and therefore $a = a \wedge b.$

$a \le b, b \le b$ (by reflexivity)

$\Rightarrow b$ is upper bound of $\{a, b\}$ and therefore $a = a \vee b.$

(6) Since $o \le x \le u$, for all $x \in L$, the results are trivial for meet and join.

(7) $a \le a \vee b$ by (4)

$\therefore \quad a \wedge (a \vee b) = a$ by (5).

$a \wedge b \leq a$ by (4)

$\therefore \quad (a \wedge b) \vee a = a$ by (5)

$a \vee (a \wedge b) = a$ by (2)

(8) $a \wedge c \leq a \leq b$

$a \wedge c \leq c \leq d$

Thus $a \wedge c$ is lower bound of $\{b, d\}$

Hence $a \wedge c \leq b \wedge d$, the g.1.b. $\{b, d\}$

Also then $\quad a \leq b, x \leq x \Rightarrow a \wedge x \leq b \wedge x.$

We also proof the result for the join operation.

Proposition 1.2.3: Show that idempotent laws follow from the absorption laws.

Proof : We have $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

Take, $b = a \wedge b$ in first and we get $a \wedge (a \vee (a \wedge b)) = a$

or $\quad a \wedge a = a$. Similarly we can show $a \vee a = a$. •

Theorem 1.2.4: In any lattice the distributive inequalities

$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

hold for any a, b, c.

Proof: $\quad a \wedge b \leq a$

$a \wedge b \leq b \leq b \vee c$

$\Rightarrow a \wedge b$ is lower bound of $\{a, b \vee c\}$

$$\Rightarrow a \wedge b \leq a \wedge (b \vee c) \tag{1}$$

Again $\quad a \wedge c \leq a$

$a \wedge c \leq c \leq b \vee c$

$$\Rightarrow \quad a \wedge c \leq a \wedge (b \vee c) \tag{2}$$

(1) and (2) show that $a \wedge (b \vee c)$ is an upper bound of $\{a \wedge b, a \wedge c\}$

$\Rightarrow$ $$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$$

Similarly we can prove the other inequality.

The above are also called semi distributive laws. •

Theorem 1.2.5: In only lattice L, the modular inequality

$$a \wedge (b \vee c) \geq b \vee (a \wedge c)$$

holds for all $a, b, c \in L, a \geq b.$

Proof: Follows from previous theorem as $a \geq b \Rightarrow a \wedge b = b.$

The dual of the modular inequality reads as:

$$a \vee (b \wedge c) \leq b \wedge (a \vee c) \quad \forall\, a, \; b, \; c \text{ with } a \leq b$$ •

Theorem 1.2.6: In any lattice L,

$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$, for all $a, b, c \in L$

Proof: Since $a \wedge b \leq a \vee b$

$a \wedge b \leq b \leq b \vee c$

$a \wedge b \leq a \leq c \vee a$

We find $(a \wedge b) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Similarly, $(b \wedge c) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

and $(c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Hence $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$ •

Definition(Algebric lattice) : A non empty set L together with two binary compositions (operations) $\wedge$ and $\vee$ is said to form a algebric lattice if for all $a, b, c \in L,$ the following conditions hold:

(i) Idempotency: $a \wedge a = a, \quad a \vee a = a$

(ii) Commutativity: $a \wedge b = b \wedge a, \quad a \vee b = b \vee a$

(iii) Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c, a \vee (b \vee c) = (a \vee b) \vee c$

(iv) Absorption: $a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a.$

Theorem 1.2.7 : Show that a poset is a lattice iff it is algebrically a lattice.

Proof : Clearly L is a non empty set.

So set $a \wedge b = \inf\{a,b\}$ and $a \vee b = \sup\{a,b\}$

Then $a \wedge a = \inf\{a,a\} = a ; a \vee a = \sup\{a,a\} = a$

So $\wedge$ and $\vee$ are idempotent

$a \wedge b = \inf\{a,b\} = \inf\{b,a\} = b \wedge a$

$a \vee b = \sup\{a,b\} = \sup\{b,a\} = b \vee a$

$\because$ $\wedge$ and $\vee$ are commutative.

Next, $a \wedge (b \wedge c) = \inf\{a, b \wedge c\} = \inf\{a, \inf\{b, c\}\}$

$= \inf\{\inf\{a, b\}, c\} = \inf\{a \wedge b, c\}$

$= (a \wedge b) \wedge c$

$a \vee (b \vee c) = \sup\{a, b \vee c\} = \sup\{a, \sup\{b, c\}\}$

$= \sup\{a, b \vee c\} = \sup\{a, \sup\{b, c\}\}$

$= \sup\{\sup\{a, b\}, c\} = \sup\{a \vee b, c\}$

$= (a \vee b) \vee c$

so $\wedge$ and $\vee$ are associative.

Finally, $a \wedge (a \vee b) = a \wedge \sup\{a, b\} = \inf\{a, \sup\{a, b\}\} = a$

$a \vee (a \wedge b) = a \vee \inf\{a, b\} = \sup\{a, \inf\{a, b\}\} = a$

Hence $\wedge$ and $\vee$ satisfy two Absorption identity

So $L^a = (L; \wedge, \vee)$ is a lattice.

(ii) Since $\wedge$ is idompotent ie $a \wedge a = a \qquad \forall a \in L$

So $\qquad a \leq a$

$\therefore \quad \leq$ is reflixive

Since $\wedge$ is commutative

$\therefore a \wedge b = b \wedge a$

$\Rightarrow a = b$ [$\therefore a \wedge b = a$ and $a \vee b = b$]

So, $\leq$ is anti symmetric.

Let $a \leq b$ and $b \leq c$

Than $a = a \wedge b$, $b = b \wedge c$

$= a \wedge (b \wedge c)$

$= (a \wedge b) \wedge c$

$= a \wedge c$

$\Rightarrow a = a \wedge c$

$\Rightarrow a \geq c$

So, $\geq$ is transitive

$\therefore \quad (L, \geq)$ is a poset. •

Example 1.2.8 : Every non empty subset of chain is a sublattice.

If S be a non empty subset of a chain L, then

$a, b \in S \Rightarrow a, b \in L \Rightarrow a, b$ are comparable

Let $a \leq b$. Then $a \wedge b = a \in S$

$a \vee b = b \in S$

Definition (Meet-semilattices) : A non empty set P together with a binary operation (meet) $\wedge$ is called a meet-semilattice if for all $a, b, c \in P$,

(i) $a \wedge a = a$

(ii) $a \wedge b = b \wedge a$

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Definition (Joint-semilattices) : A non empty set P together with a binary operation (joint) $\vee$ is called a joint-semilattice if for all $a, b, c \in P$,

(i) $a \vee a = a$

(ii) $a \vee b = b \vee a$

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$

Theorem 1.2.8 : If A and B be two lattice, that the product of A and B is a lattice.

Proof : It is given that A and B are two lattice then

$A \times B = \{(x, y) | x \in A, y \in B\}$ is a poset under the relation $\leq$ defined by $(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2$ in A, $y_1 \leq y_2$ in B

We show that $A \times B$ forms a lattice.

Let $(x_1, y_1), (x_2, y_2) \in A \times B$ be any element

Then $x_1, x_2 \in A$ and $y_1, y_2 \in B$

Since A and B are lattices, $\{x_1, x_2\}$ and $\{y_1, y_2\}$ have Sup and Inf in A and B respectively.

Let $x_1 \wedge x_2 = \inf\{x_1, x_2\}$ and $y_1 \wedge y_2 = \inf\{y_1, y_2\}$

then $x_1 \wedge x_2 \leq x_1$, $x_1 \wedge x_2 =\leq x_2$ $y_1 \wedge y_2 \leq y_1$, $y_1 \wedge y_2 =\leq y_2$

$\Rightarrow (x_1 \wedge x_2, y_1 \wedge y_2) \leq (x_1, y_1)$

$(x_1 \wedge x_2, y_1 \wedge y_2) \leq (x_2, y_2)$

$\Rightarrow (x_1 \wedge x_2, y_1 \wedge y_2)$ is a lower bound of $\{(x_1, y_1), (x_2, y_2)\}$

Suppose (z, w) is any lower bound of $\{(x_1, y_1), (x_2, y_2)\}$

Then $(z, w) \leq (x_1, y_1)$

$(z, w) \leq (x_2, y_2)$

$\Rightarrow z \leq x_1, z \leq x_2$ $w \leq y_1, w \leq y_2$

$\Rightarrow$ z is a lower bound of $\{x_1, x_2\}$ in A.

w is a lower bound of $\{y_1, y_2\}$ in B.

$\Rightarrow \quad z \leq x_1 \wedge x_2 = \inf\{x_1, x_2\}$

$w \leq y_1 \wedge y_2 = \inf\{y_1, y_2\}$

$\Rightarrow \quad (z, w) \leq (x_1 \wedge x_2, y_1 \wedge y_2)$

or that $(x_1 \wedge x_2, y_1 \wedge y_2)$ is g.l.b. $\{(x_1, y_1), (x_2, y_2)\}$

Similarity (by duality) we can say that

$(x_1 \vee x_2, y_1 \vee y_2)$ is l.u.b. $\{(x_1, y_1), (x_2, y_2)\}$

Hence $A \times B$ is a lattice.

Also $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2), (y_1 \wedge y_2)$

$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2), (y_1 \vee y_2)$. •

Definition (Convex sublattices) : A sublattice K of a lattice L is called a convex sublattice if for all $x, y \in K$ $[x \wedge y, x \vee y] \subseteq K$.

Example 1.2.9 : In the lattice $\{1, 2, 3, 4, 6, 12\}$ under divisibility $\{1, 6\}$ is a sublattice which is not convex as $2, 3 \in [1, 6]$, but $2, 3 \notin \{1, 6\}$

Diagrammatically the lattice $\{1, 2, 3, 4, 6, 12\}$ can be represented by the following figure

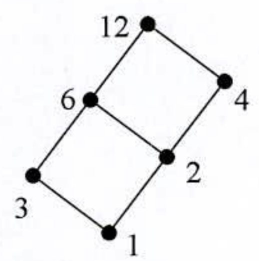


Fig. 1.9

Theorem 1.2.9 : A sublattice of a lattice L is a convex sublattice if and only if for all $\forall x, y \in K (x \leq y)$, $[x, y] \subseteq K$.

Proof : Let K be a convex sublattice of L.

Let $x, y \in K(x \leq y)$ be any elements, then by definition $[x \wedge y, x \vee y] \subseteq K$

$[x, y] \subseteq K$ as $x \leq y \Rightarrow x \wedge y = x$ $x \leq y \Rightarrow x \wedge y = x$ $x \vee y = y$.

Conversely, let $[x, y] \subseteq K$ $\forall$ $x, y, (x \leq y)$

Let $x, y \in K$ be a sublattice

Also these are comparable. Thus by contrition.

$[x \wedge y, x \vee y] \subseteq K$. •

1.3 Bounded Lattice, Complete Lattice and Ideal of a Lattice.

Definition (Bounded Lattice) : A lattice with a largest and a smallest element is called a bounded lattice. Smallest element is denoted by zero and the largest element is denoted by 1.

Definition (Complete Lattice) : A lattice L is called complete if for its every subset K, both Sup K and Inf. K exists in L.

Definition (Finite Lattice) : A Lattice L is called finite if it contain a finite number of elements.

Example 1.3.1: Let $L = \{1, 2, 5, 10\}$ be a lattice under divisibility. Here in the lattice the finite number of element in L. So, L is finite lattice.

Definition (Ideal of a Lattice) : A non empty subset I of a lattice L is called an ideal of L if

(i) $a, b \in I \Rightarrow a \vee b \in I$

(ii) $a \in I, i \in L \Rightarrow a \wedge i \in I$

Example 1.3.2: Let $L = \{1, 2, 5, 10\}$ be lattice of factors of 10 under

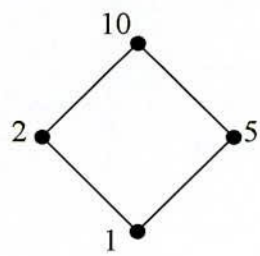


Fig. 1.10

divisibility. Then $\{1\}, \{1, 2\}, \{1, 5\}, \{1, 2, 5, 10\}$ are all the ideals of L.

Definition (Prime Ideal) : An ideal P of L is called a prime ideal if for any $x, y \in L$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

Definition (Principal ideal) : An ideal which is generated by a single element is called principal ideal.

Example 1.3.3 : Let $(a] = \{x | x \leq a\}$ then the ideal $(a]$ is generated by the element a. Hence $(a]$ is principal ideal.

Definition (Filter or Dual ideal) : A subset D of a lattice L is called a dual ideal if

(i) $d_1, d_2 \in D \Rightarrow d_1 \wedge d_2 \in D$

(ii) $d \in D$ and $x \in L . \Rightarrow x \vee d \in D$

Proposition 1.3.1. : Let $\Phi : L \to K$ be an onto homomorphism. Let I be an ideal of L, and let J be an ideal of K. Show that $\Phi(I)$ is an ideal of K, and $\Phi^{-1}(J) = \{a \mid a \in L, \Phi(a) \in J\}$ is an ideal of L.

Proof : To prove that $\Phi(I)$ is an ideal of K, let $x, y \in \Phi(I)$.

Then $\exists\ a, b \in I$ such that $\Phi(a) = x$ and $\Phi(b) = y$.

Now every $x \vee y = \Phi(a) \vee \Phi(b) = \Phi(a \vee b)$ [$\because \Phi$ is a homomorphism]

Since I is an ideal, $a \vee b \in I$ and so $\Phi(a \vee b) \in \Phi(I)$

ie $x \vee y \in \Phi(I)$.

Now $z \leq x \Rightarrow \Phi(a) \leq \Phi(b)$

$\Rightarrow \Phi(a) = \Phi(a) \wedge \Phi(b) = \Phi(a \wedge b)$. But $a \wedge b \in I$

Since $a \in L, b \in I$ & I is an ideal of L

Hence $z = \Phi(a) = \Phi(a \wedge b) \in \Phi(I)$

Therefore $\Phi(I)$ is an ideal of K.

Again, let $x, y \in \Phi^{-1}(J)$. Then $\Phi(x), \Phi(y) \in J$

$\Rightarrow \Phi(x) \vee \Phi(y) \in J$ [$\because$ J is an ideal]

$\Rightarrow \Phi(x \vee y) \in J$

$\Rightarrow x \vee y \in \Phi^{-1}(J)$.

Now let $x \in \Phi^{-1}(J)$ and $z \in L$ with $z \leq x$; then $\Phi(x) \in J$

$\therefore \Phi(z) \wedge \Phi(x) = \Phi(z \wedge x) = \Phi(z)$

$\Rightarrow \Phi(z) \leq \Phi(x) \in J$

$\Rightarrow \Phi(z)) \in J$

$\Rightarrow z \in \Phi^{-1}(J)$

Hence $\Phi^{-1}(J)$ is an ideal of L. •

Proposition 1.3.2 : Prove that every ideal of a Lattice L is prime if and only if L is a chain.

Proof : First suppose that every ideal of L is prime. Now we are to show that L is a chain.

Let $a, b \in L$. Then $a \wedge b \in L$. Now consider the ideal $I = (a \wedge b]$ generated by $a \wedge b$. By hypothesis I is prime.

Now $a \wedge b \in I$ $\Rightarrow$ either $a \in I$ or $b \in I$

$\Rightarrow$ either $a \leq a \wedge b$ or $b \leq a \wedge b$

$\Rightarrow$ either $a = a \wedge b$ or $b = a \wedge b$

$\Rightarrow$ either $a \leq b$ or $b \leq a$

$\Rightarrow$ L is a chain.

Conversely, let L be a chain and P be an ideal of L, we are to show that P is prime.

Let $x, y \in L$ with $x \wedge y \in P$. Since L is a chain

Then either $x \leq y$ or $y \leq x$ $\Rightarrow$ either $x \wedge y = x$ or $x \wedge y = y$

$\Rightarrow$ either $x \in P$ or $y \in P$

$\Rightarrow$ P is a prime ideal of L

Hence proved. •

Theorem 1.3.3: Let L be a lattice and K and I be non-empty subset of L

(i) I is an ideal iff for all $x, y \in I$, $x \vee y \in I$ and for all $x \in I$, $t \leq x$ implies $t \in I$

(ii) $(K] = \{x \in L | x \leq k_1 \vee k_2 \vee \cdots \vee k_n$ for some $k_1, k_2, \cdots, k_n \in H\}$.

(iii) $(a] = \{x \in L | x \leq a\}$

Proof : (i) Suppose I is an ideal. So I is a sublattice and so for all $x, y \in I$, $x \vee y \in I$. Now let $x \in I$, $t \leq x$ implies $t \in L$.

Then $t = t \wedge x \in I$

Conversely, suppose I has the stated properties .let $x, y \in I$ then $x \wedge y \leq x$ implies $x \wedge y \in I$. ie I is a sublattice.

Now suppose $i \in I, x \in L$. Then $i \wedge x \leq i$ implies $i \wedge x \in I$.

Therefore I is an ideal

(ii) Let $x, y \in R.H.S$.

Then $x \leq k_1 \vee k_2 \vee \cdots \vee k_n$ for some $k_1, k_2, \cdots, k_n \in K$

$y \leq k_1 \vee k_2 \vee \cdots \vee k_n$ for some $k_1, k_2, \cdots, k_m \in K$

So $x \vee y \leq k_1 \vee k_2 \vee \cdots \vee k_n \vee k_1 \vee k_2 \vee \cdots \vee k_m$

Which implies $x \vee y \in R.H.S.$

If $x \in R.H.S$ and $t \in L$ with $t \leq x$, then $x \leq k_1 \vee k_2 \vee \cdots \vee k_n$ for some $k_1, k_2, \cdots, k_n \in K$. and $t \leq x \leq k_1 \vee k_2 \vee \cdots \vee k_n$ implies $t \in R.H.S.$

Hence R.H.S. is an ideal.

Obviously R.H.S. contain K.

Let I_1 be an ideal then $x \leq k_1 \vee k_2 \vee \cdots \vee k_n$ for some $k_1, k_2, \cdots, k_n \in K$. Since k is an ideal containing K, $x \in K$

Therefore R.H.S. is the smallest ideal containing K.

(iii) Obvious from (ii) •

Theorem 1.3.4: Set of all ideals $I(L)$ of a lattice L again a lattice.

Proof: Let $I\ J \in I(L)$. Then clearly $I \wedge J = I \cap J$. Now we claim that

$I \vee J = \{x \in L | x \leq i \vee j\}$ for some $i \in I,\ j \in J$.

To prove this, let $x, y \in R.H.S$. Then $x \leq i \vee j$ for some $i \in I,\ j \in J$ and $y \leq i_1 \vee j_1$ for some $i_1 \in I,\ j_1 \in J$.

So, $x \vee y \leq (i \vee j) \vee (i_1 \vee j_1) = (i \vee i_1) \vee (j \vee j_1)$ $[i \vee i_1 \in I,\ j \vee j_1 \in J]$

which implies $x \vee y \in R.H.S$.

If $x \in R.H.S$ and $t \in L$ with $t \leq x$ then $x \leq i \vee j$ for some $i_1 \in I,\ j_1 \in J$.

So $t \leq i \vee j$ implies $t \in R.H.S$

Therefore R.H.S. is an ideal.

Obviously this contain both I and J.

Let $x \in R.H.S$ then $x \leq i \vee j$ for some $i_1 \in I,\ j_1 \in J$. Since I_1 is an ideal containing both I and J So $i \vee j \in I_1$ Hence $i \vee j \in I_1$ and hence $x \in I_1$. ie $R.H.S \leq K$ ie R.H.S. is the smallest ideal.

Therefore $R.H.S = I \vee J$ and so $I(L)$ is a lattice •

Theorem 1.3.5 : Prove that if D and F are dual ideals of L. Then

(i) $D \wedge F = D \cap F$

(ii) $D \vee F = \{x \in L |\ x \geq d \wedge f \quad for \quad some \quad d \in D, f \in F\}$

(iii) $[a) = \{x \in L |\ x \geq a\}$

complement element. •

1.4 Complimented and Relatively complimented lattice

Definition (Complimented lattice) : Let $[a, b]$ be an interval in a lattice L,

Let $x \in [a, b]$ be any element. If $\exists\, y \in L$ s.t., $x \wedge y = a, x \vee y = b$. We say y is a complement of x relative to $[a, b]$, or y is a relative complement of x in $[a, b]$.

Definition (Relatively complimented lattices) : If every element x of an interval $[a, b]$ has at least one complement relative to $[a, b]$, the interval $[a, b]$ is said to be complemented.

Further, if every interval in a lattice is complemented, the lattice is said to be relatively complemented.

Theorem 1.4.1 : Let A be a non-empty finite set. Show that $(\rho(A), \subseteq)$ is uniquely complemented lattices.

Proof : Let $A = \Phi$ finite set and $\rho(A)$ be the power set of A. We know $(\rho(A), \subseteq)$ form a lattice with least element Φ and greatest element A.

Any $X, Y \in \rho(A)$ $X \wedge Y = X \cap Y$ and $X \vee Y = X \cup Y$

since $A \wedge (A - X) = A \cap (A - X) = \Phi$

$A \vee (A - X) = A \cup (A - X) = A$

We see $A - X$ is complemented of X relative to $[\phi, A]$

Then $\rho(A)$ is complemented lattice.

Suppose Y is any complemented of X then

$X \wedge Y = X \cap Y = \phi$

$X \vee Y = X \cup Y = A$

ie, $X \cap Y = A \cap (A - X)$.

$X \cup Y = A \cup (A - X)$

$Y = A - X$ ....(i)

or that A-X is uniquely complemented of X.

So $(\rho(A), \subseteq)$ is an uniquely complemented lattice.

Now we prove $\rho(A)$ is also relative complemented.

Consider any interval $[X, Y]$ in $\rho(A)$.

Let $Z \in [X, Y]$ be any number, Then

$Z \cap (X \cup (Y - Z)) = (Z \cap X) \cup (Z \cap (Y - Z)) = X \cup \phi = X$

$Z \cup (X \cup (Y - Z)) = (Z \cup X) \cup (Y - Z) = Z \cup (Y - Z) = Y$

Showing that $X \cup (Y - Z)$ is the complemented of Z relative [X, Y]

Z is any element of any interval of $\rho(A)$.

Hence $\rho(A)$ is relative complemented . •

Theorem 1.4.2 : Two bounded lattices A and B are complemented if and only if $A \times B$ is complemented.

Proof : Let A and B be complemented and suppose o, u and o', u' are the universal bounds of A and B respectively.

Then (o, o') and (u, u') will be least and greatest elements of $A \times B$.

Let $(a, b) \in A \times B$ be any element.

Then $a \in A$, $b \in B$ and as A, B are complemented, $\exists\, a' \in A, b' \in B$ s.t.,

$a \wedge a' = o,\ a \vee a' = u\ b \wedge b' = o',\ b \vee b' = u'$.

Now $\quad (a, b) \wedge (a', b') = (a \wedge a', b \wedge b') = (o, o')$

$\quad (a, b) \vee (a', b') = (a \vee a', b \vee b') = (u, u')$

Shows that (a', b') is complement of (a, b) in $A \times B$.

Hence $A \times B$ is complemented.

Conversely, let $A \times B$ be complemented.

Let $a \in A$, $b \in B$ be any elements.

Then $(a, b) \in A \times B$ and thus has a complement, say (a', b')

Then $(a, b) \wedge (a', b') = (o, o')$, $(a, b) \vee (a', b') = (u, u')$

$\Rightarrow (a \wedge a', b \wedge b') = (o, o')$, $(a \vee a', b \vee b') = (u, u')$

$\Rightarrow a \wedge a' = o \quad a \vee a' = u$

$b \wedge b' = o' \quad b \vee b' = u'$

ie., a' and b' are complements a & b respectively. Hence A and B are complemented. •

Theorem 1.4.3 : Two lattice A and B are relatively complemented if and only if $A \times B$ is relatively complemented.

Proof : Let A ,B be relatively complemented .

Let $[(a_1, b_1), (a_2, b_2)$ be any interval of $A \times B$ and suppose (x, y) is any element of this interval.

Then $(a_1, b_1) \leq (x, y) \leq (a_2, b_2) \qquad a_1, a_2, x \in A \quad b_1, b_2, y \in B$

$\Rightarrow a_1 \leq x \leq a_2 \quad b_1 \leq y \leq b_2$

$\Rightarrow x \in [a_1, a_2]$ an interval in A, $y \in [b_1, b_2]$ an interval in B.

Since A, B are relatively complemented, x, y have complements relative to $[a_1, a_2]$ and $[b_1, b_2]$ respectively.

Let x' and y' be these complements. Then

$$x \wedge x' = a_1 \quad y \wedge y' = b_1$$

$$x \vee x' = a_2 \quad y \vee y' = b_2$$

Now $(x, y) \wedge (x', y') = (x \wedge x', y \wedge y') = (a_1, b_1)$

$(x, y) \vee (x', y') = (x \vee x', y \vee y') = (a_2, b_2)$

$\Rightarrow (x', y')$ is complement of (x, y) related to $[(a_1, b_1), (a_2, b_2)]$.

Thus any interval in $A \times B$ is complemented.

Hence $A \times B$ is relative complemented.

Conversely, let $A \times B$ be relatively complemented.

Let $[a_1, a_2]$ and $[b_1, b_2]$ be any intervals in A & B.

Let $x \in [a_1, a_2]$, $y \in [b_1, b_2]$ be any elements.

Then $a_1 \le x \le a_2$, $b_1 \le y \le b_2$

$\Rightarrow$ $(a_1, b_1) \le (x, y) \le (a_2, b_2)$

$\Rightarrow$ $(x, y) \in [(a_1, b_1), (a_2, b_2)]$, an interval in $A \times B$

$\Rightarrow$ (x, y) has a complement, say (x', y') relative to this interval.

Thus $$(x, y) \wedge (x', y') = (a_1, b_1)$$

$$(x, y) \vee (x', y') = (a_2, b_2)$$

$$\Rightarrow \quad (x \wedge x', y \wedge y') = (a_1, b_1)$$

$$(x \vee x', y \vee y') = (a_2, b_2)$$

$$\Rightarrow \quad x \wedge x' = a_1, x \wedge x' = a_2$$

$$y \wedge y' = b_1, y \vee y' = b_2$$

$\Rightarrow$ x' is complement of x relative to $[a_1, a_2]$

y' is complement of y relative to $[b_1, b_2]$

Hence A, B are relatively complemented. •

Theorem 1.4.4 : Dual of a complemented lattice is complemented.

Proof.: let (L, ρ) be a complemented lattice with o, u as least and greatest elements. Let $(\overline{L}, \overline{\rho})$ be the dual of (L, ρ). Then u, o are least and greatest elements of $\overline{L}$.

Let $a \in \overline{L} = L$ be any element .

Since $a \in L$, L is complemented ,$\exists$ $a' \in L$ s.t.,

$$a \wedge a' = o, a \wedge a' = u \text{ in L}$$

i.e., $o = \inf\{a, a'\}$ in L

$\Rightarrow o\rho a, \quad o\rho a'$

$\Rightarrow a\bar{\rho} o, \quad a'\bar{\rho} o$ in $\bar{L}$

$\Rightarrow o$ is an upper bound of $\{a, a'\}$ in $\bar{L}$

If k is an upper bound of $\{a, a'\}$ in $\bar{L}$ then $a\bar{\rho}k, a'\bar{\rho}k$

$\Rightarrow \quad k\rho a, k\rho a' \Rightarrow \quad k\rho o$ as o is Inf.

$\Rightarrow \quad o\bar{\rho}k$

i.e., o is l.u.b. $\{a, a'\}$ in $\bar{L}$

i.e., $a \vee a' = o$ in $\bar{L}$

Similarly, $\quad a \wedge a' = u$ in $\bar{L}$

or that a' is complement of a in $\bar{L}$

Hence $\bar{L}$ is complemented. ●

1.5 Atom and Dual atom.

Definition (Atom) : An element a in a lattice L is called an atom if it is covers o. In other words a is an atom iff $a \neq o$ and $x \wedge a = a$ or $x \wedge a = o \ \vee x \in L$.

Definition (Dual atom) : An element b is called dual atom if u, the greatest element of the lattice covers b.

Definition (Length) : A finite chain with n elements is said to have length $n-1$, (i.e. , length is the number of 'links' that the chain has.)

Definition (Cover) : If a and b two elements in a chain $b < a$ if there exist no element c s.t. $b < c < a$ then we say a cover b.

Definition (Height or dimension) : Let L be a lattice of finite length with least element o. An element $x \in L$ is said to have height or dimension n if $l[o, x] = n$ and in that case we write $h(x) = n$.

Proposition 1.5.1 : Show that no ideal of a complemented lattice which is a proper sublattice can contain both an element and its complement.

Proof : Let L be complemented lattice. Then $o, u \in L$. Let I be an ideal of L such that I is a proper sublattice of L. Suppose $\exists$ an element x in I such that its complement x' is also in I.

Then $x \wedge x' = o, \quad x \vee x' = u$

since I is a sublattice $x \wedge x', \quad x \vee x'$ are in I i.e., $o, u \in I$

Now if $l \in L$ be any element then as $u \in I$.

$$l \wedge u \in I$$

$\Rightarrow l \in L \Rightarrow L \subseteq I \Rightarrow I = L$, a contradiction. •

Proposition 1.5.2 : Let L be a uniquely complemented lattice and let a be an atom in l. Show that a' the complement of a is a dual atom of L.

Proof : Since L is uniquely complemented lattice, every element has a unique complement.

Suppose a' is not a dual atom, then $\exists$ at least one x s.t., $a' < x < u$

$\Rightarrow \quad a' \vee a \leq x \vee a$

$\Rightarrow \quad u \leq x \vee a \leq u$

$\Rightarrow \quad u = x \vee a$

Now if $a \leq x$ then $x \vee a = x \Rightarrow x = u$, not true. Again if $a \nleq x$, then $a \wedge x = o$ (note a is an atom)

Thus $a \wedge x = o, a \vee x = u \Rightarrow x = a'$, again a contradiction.

Hence a' is a dual atom. •

Proposition 1.5.3: Let L be a lattice, let P be a prime ideal of L, and let $a, b, c \in L$. Prove that if $a \vee (b \wedge c) \in P$, then $(a \vee b) \wedge (a \vee c) \in P$.

Proof. : Since $a \vee (b \wedge c) \in P$ then $a \in P$ and $b \wedge c \in P$ [P is an ideal and $a, b \wedge c \in P$]

$\Rightarrow a \in P$ and $b \in P$ or $c \in P$ [as P is prime ideal]

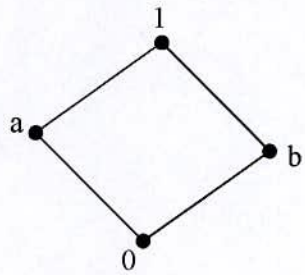
$\Rightarrow$ either $a \vee b \in P$ or $a \vee c \in P$

$\Rightarrow (a \vee b) \wedge (a \vee c) \in P$ [as P is prime ideal] •

"Modular and Distributive Lattice"

2.1 Introduction.

In this chapter we discuss the definition of homomorphism, isomorphism, join-reducible element, hereditary. An element $a \in L$ is call a join-reducible element if $b, c \in L, a = b \vee c$ implies that either $b = a$ or $c = a$



$a = a \vee 0, \quad b = b \vee 0$

Fig. 2.1

Here a, b are all join irreducible elements. We denote $J(L)$ as the set of all join irreducible elements.

In this chapter we also prove the following theorem, "A Lattice L is distributive if and only if for all $x, y \in L\ x < y$. There exist a prime ideal P with $x \in P, y \notin P$"

2.2 Modular and Distributive Lattice

Definition(Modular Lattice): A lattice L is called modular lattice if all

$a, b, c \in L$ with $a \geq b$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = [b \vee (a \wedge c)]$

Definition (Distributive Lattice): A lattice L is called distributive lattice if

all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Example 2.2.1 : The lattice $(\rho(x), \subseteq)$ is a distributive lattice as

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

For a distributive Lattice L $J(L)$ denotes the set of all nonzero join irreducible elements, regarded as a poset under the partial ordering of L. $H(J(L))$ denotes the set of all hereditary subsets partially ordered by set inclusion. $H(J(L))$ is a Lattice in which meet & join are intersection & union respectively. Hence $H(J(L))$ is a distributive Lattice for $a \in L$, set $r(a) = \{x \in J(L) \mid x \leq a\}$

Theorem 2.2.1: Every maximal chain C of a finite distributive Lattice L is a length $|J(L)|$ (order of $J(L)$)

Proof : For $a \in J(L)$, Let $m(a)$ be the smallest element of C containing a.

Define a map $\Phi : J(L) \to C - \{0\}$ by $\Phi(a) = m(a)$

Let $\Phi(a) = \Phi(b)$. Then $m(a) = m(b)$. let $m(a) > x$ and $x \in C$

Then, $x \vee a = x \vee b$. Therefore,

$$\begin{aligned} a &= a \wedge (x \vee a) \\ &= a \wedge (x \vee b) \\ &= (a \wedge x) \vee (a \wedge b) \end{aligned}$$

either $a = (a \wedge x)$ or $a = (a \wedge b)$ $[\because a \in J(L)]$

But $a = (a \wedge x) \Rightarrow a \leq x \Rightarrow m(a) \leq x < m(a)$; a contradiction.

Therefore, $a = (a \wedge b)$ and so $a \leq b$. Similarly $b \leq a$.

$\therefore a = b$ which proves that Φ is one –one.

To show the onto-ness, let $y \in C - \{0\}$ and $y > z,\ z \in C$.

Then $r(y) \supset r(z)$, and so $y = m(a)$ for any $a \in r(y) - r(z)$

$$= \Phi(a)$$

Therefore, Φ is onto. $\therefore J(L) \cong C - \{0\}$. Which proves that C of length $|J(L)|$. Hence proved. •

Proposition 2.2.2. : Prove that L is distributive if and only if the identity

$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ hold in L

Proof : Suppose L is distributive then

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad (1)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad (2)$$

for all $x, y, z \in L$

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = [(y \wedge x) \vee (y \wedge z)] \vee (z \wedge x)$$

$$= [y \wedge (x \vee z)] \vee (z \wedge x) \text{ by (1)}$$

$$= [y \vee (z \wedge x)] \wedge [(x \vee z) \vee (z \wedge x)] \text{ by (2)}$$

$$= (y \vee z) \wedge (y \vee x) \wedge (x \vee z) \text{ by (2)}$$

$$= (y \vee z) \wedge (x \vee y) \wedge (z \vee x)$$

$$= (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$$

Conversely,

suppose $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ hold in L

Let $x, y, z \in L$ with $z \leq x$.

Then $x \wedge (y \vee z) = [x \wedge (x \vee y)] \wedge (y \vee z)$

$$= (x \vee z) \wedge (x \vee y) \wedge (y \vee z) \quad [\because z \leq x \text{ then } x \vee z = x]$$

$= (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$

$= (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

$= (x \wedge y) \vee (y \wedge z) \vee z \; [\because z \leq x \text{ then } x \wedge z = z]$

$= (x \wedge y) \vee z$

$\Rightarrow$ L is modular.

Now for any $a, b, c \in L$,

$a \wedge (b \vee c) = [a \wedge (a \vee b)] \wedge (b \vee c)$

$= [a \wedge (a \vee c)] \wedge (a \vee b) \wedge (b \vee c)$

$= a \wedge [(a \vee b) \wedge (b \vee c) \wedge (c \vee a)]$

$= a \wedge [(b \wedge c) \vee \{(a \wedge b) \vee (c \wedge a)\}]$

$= (a \wedge b \wedge c) \vee (a \wedge b) \vee (c \wedge a)$

$= (a \wedge b) \vee (c \wedge a)$

ie $\quad a \wedge (b \vee c) = (a \wedge b) \vee (c \wedge a)$

Thus L is distributive. •

Theorem 2.2.3: A lattice L is modular if and only if no sublattice isomorphic to N_5.

Proof. : Suppose L is a modular lattice. Then its every sublattice is also modular. Since $N_5 = \{0, a, b, c, 1\}$. Where $c \leq a$, $a \wedge b = b \wedge c = 0$.

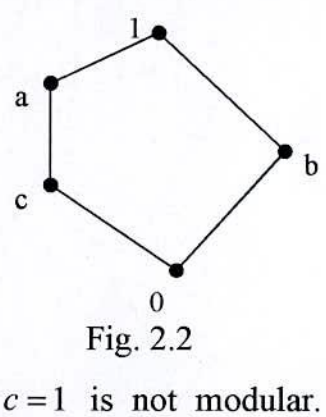


Fig. 2.2

And $a \vee b = b \vee c = 1$ is not modular. So L does not containing sublattice isomorphic to N_5.

To prove the converse, Let L be not modular. Then there exists elements $x, y, z \in L$ with $z \leq x$ such that $x \wedge (y \vee z) \neq (x \wedge y) \vee z$.

But $x \wedge (y \vee z) > (x \wedge y) \vee z$. Then the elements $x \wedge y$,y, $(x \wedge y) \vee z$, $x \wedge (y \vee z)$ $y \vee z$ form a lattice . Diagram as follows,

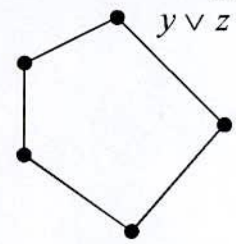


Fig. 2.3

Observe that, $(x \wedge (y \vee z)) \wedge y = x \wedge [(y \vee z) \wedge y]$

$$= x \wedge y$$

and $\qquad y \wedge (x \wedge (y \vee z)) = x \wedge y$

Again, $\qquad y \vee [(x \wedge y) \vee z] = [y \vee (x \wedge y)] \vee z$

$$= y \vee z$$

and $\qquad y \vee [x \wedge (y \vee z)] = y \vee z$

If $y = x \wedge y$ then $y \leq x$ and so $y \vee z = (x \wedge y) \vee z$

$$\Rightarrow (x \wedge y) \vee z = y \vee z \qquad (i)$$

Also, $y \leq x$ and $z \leq x$,$\Rightarrow y \vee z \leq x$ and

so $\qquad \Rightarrow x \wedge (y \vee z) = y \vee z \qquad (ii)$

Hence we have, $x \wedge (y \vee z) = (x \wedge y) \vee z$

Which is a contradiction, Since L is not modular. So $y \neq x \wedge y$. Similarly we can show that, $(x \wedge y) \vee z \neq x \wedge y$, $y \neq y \vee z$, $x \wedge (y \vee z) \neq y \vee z$.

Hence the five elements are distinct and they form a sublattice of L which is isomorphic to N_5.

Therefore L is modular. •

Theorem 2.2.4 : A modular lattice is distributive if and only if it has no sublattice isomorphic M_5.

Proof. : 1st suppose a modular lattice L is distributive. Then its every sublattice is also distributive.

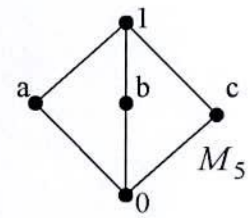


Fig. 2.4

Since M_5 is not distributive (For $a \wedge (b \vee c) = a \wedge 1 = a$ but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$) So, L cannot contain any sublattice isomorphic to M_5.

Conversely, suppose that L is not distributive. Then there exist elements $x, y, z \in L$ such that $x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z)$ but

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$$
$$\Rightarrow (x \wedge y) \vee (x \wedge z) < x \wedge (y \vee z)$$

Thus every modular lattice which is not distributive contains a sublattice isomorphic to M_5.

Hence L is a distributive. •

2.3 Sectionally complemented Lattice

Definition (Sectionally complemented) : A lattice L with 0 is called sectionally complemented if for each $x \in L$, $[0, x]$ is complemented.

Definition (Generalized Boolean lattice) : A sectionally complemented distributive lattice L is called a generalized Boolean lattice.

Theorem 2.3.1: A lattice L is distributive if and only if every element has at most one relative complement in any interval.

Proof.: 1st suppose a modular lattice L is distributive. Let $a, b, c \in L$ with $b \leq a \leq c$.

Suppose a has two relative complements d and e in $[b, c]$. Then we have

$$a \wedge d = b \qquad a \vee d = c$$

and $a \wedge e = b \qquad a \vee e = c$

Now,
$$\begin{aligned} d &= d \wedge c \\ &= d \wedge (a \vee e) \\ &= (d \wedge a) \vee (d \wedge e) \\ &= b \vee (d \wedge e) \\ &= (a \wedge e) \vee (d \wedge e) \\ &= e \wedge (a \vee d) \\ &= e \wedge c \\ &= e \end{aligned}$$
$$\Rightarrow d = e$$

Hence a has one relative complement in any interval.

Conversely, suppose L is not distributive. Therefore it contains a sublattice isomorphic to either M_5 or N_5 given below:

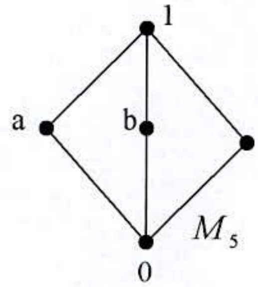


Fig. 2.5

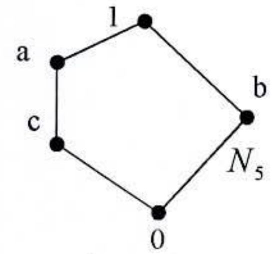


Fig. 2.6

In both case c has two relative complements. Which gives a contradiction.

Hence L is distributive. •

Theorem 2.3.2: A lattice L is distributive if and only if for any two ideal I and J of L

$$I \vee J = \{i \vee j \mid i \in I, j \in J\}$$

Proof. : 1st suppose a modular lattice L is distributive. Then clearly $R.H.S \subseteq I \vee J$.

Now, let $t \in I \vee J$ .Then we have $t \leq i \vee j$ for some $i \in I$ and $j \in J$.

$\therefore \quad t = t \wedge (i \vee j)$

$= (t \wedge i) \vee (t \wedge j)$

$= i' \vee j'$ where, $i' = t \wedge i \in I$ and $j' = t \wedge j \in J$

Hence $t \in R.H.S$. $I \vee J \subseteq R.H.S$

Therefore , $I \vee J = \{i \vee j \mid i \in I, j \in J\}$

Conversely, suppose L is not distributive. Therefore it contains elements a, b, c in M_5 or N_5

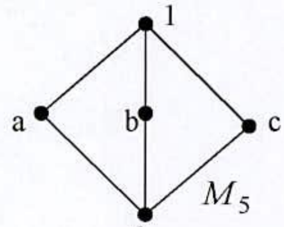


Fig. 2.7

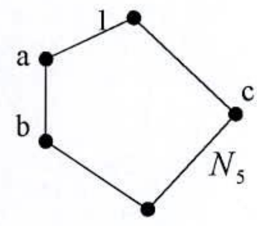


Fig. 2.8

Let $I=(b]$ and $J=(c]$, since $a \leq b \vee c$, then we have $a \in I \vee J$.

However a has no representation as in given theorem.

For if $a=i \vee j$, $i \in I$, $j \in J$. Then $j \leq a$. Also $j \leq c$. Therefore $j \leq a \wedge c < b$. Thus $j \in I$.

Which gives a contradiction.

Hence L is distributive. •

Theorem 2.3.3: For any two ideals I and J of a distributive lattice L if $I \wedge J$ and $I \vee J$ are principal then both I and J are principal.

Proof.: Let $I \wedge J = (x]$ and $I \vee J = (y]$

Then $y = i \vee j$ for some $i \in I$ and $j \in J$. Set $c = x \vee i$ and $b = x \vee j$.

Then clearly $c \in I$ and $b \in J$.

We have to show that $I=(c]$ and $J=(b]$.

If $I \neq (c]$, then there exists an element $a > c$ such that $a \in I$.

Moreover, the set $\{x, a, b, c, y\}$ form a lattice isomorphic to N_5

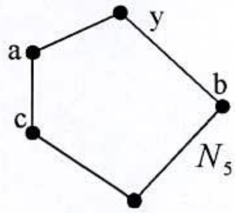


Fig. 2.9

ie, L is not distributive. Which is a contradiction

Hence $I = (c]$. Therefore I is a principal ideal.

Similarly, we can show that, $J = (b]$. ie J is also a principal ideal.

Hence proved. •

2.4 Homomorphism and Isomorphism.

Definition (Homomorphism) : Let L and M be lattices. A mapping $\theta : L \to M$ is called a meet homomorphism if $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$. It is called join homomorphism if $\theta(a \vee b) = \theta(a) \vee \theta(b)$. If θ is both meet as well as join homomorphism, it is called a homomorphism. A homomorphism is sometimes called a morphism.

Definition (Isomorphism) : Let L and M be lattices. A mapping $\theta : L \to M$ is called an isomorphism if for all $a, b \in L$, then $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$, $\theta(a \vee b) = \theta(a) \vee \theta(b)$ and θ is one-one and onto.

Theorem 2.4.1 : Let L be a distributive and $a \in L$, the map $\varphi : x \to \langle x \wedge a, x \vee a \rangle$ is an embedding of L into $(a] \times [a)$. It is an isomorphism if a has a complement.

Proof.: For $x, y \in L$

we have, $\varphi(x) = \langle x \wedge a, x \vee a \rangle$ and $\varphi(y) = \langle y \wedge a, y \vee a \rangle$.

Then
$$\begin{aligned} \varphi(x \wedge y) &= \langle (x \wedge y) \wedge a, (x \wedge y) \vee a \rangle \\ &= \langle x \wedge y \wedge a, (x \vee a) \wedge (y \vee a) \rangle \\ &= \langle (x \wedge a) \wedge (y \wedge a), (x \vee a) \wedge (y \vee a) \rangle \\ &= \langle x \wedge a, x \vee a \rangle \wedge \langle y \wedge a, y \vee a \rangle \\ &= \varphi(x) \wedge \varphi(y) \end{aligned}$$

and
$$\begin{aligned} \varphi(x \vee y) &= \langle (x \vee y) \wedge a, (x \vee y) \vee a \rangle \\ &= \langle (x \wedge a) \vee (y \wedge a), (x \vee a) \vee (y \vee a) \rangle \\ &= \langle x \wedge a, x \vee a \rangle \vee \langle y \wedge a, y \vee a \rangle \\ &= \varphi(x) \vee \varphi(y) \end{aligned}$$

Hence φ is a homomorphism.

Now let $\varphi(x)=\varphi(y), x, y \in L$.

Then $\langle x \wedge a, x \vee a\rangle=\langle y \wedge a, y \vee a\rangle$ and

So, $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$.

Now,
$$\begin{aligned} x &= x \wedge (x \vee a) \\ &= x \wedge (y \vee a) \\ &= (x \wedge y) \vee (x \wedge a) \\ &= y \wedge (x \vee a) \\ &= y \wedge (y \vee a) \\ &= y \end{aligned}$$

$\Rightarrow x = y$ and so φ is one-one.

Hence φ is an embedding.

2nd part: Let $a \in L$ has a complement. Choose an element $\langle x, y\rangle \in (a]\times[a)$, then $x \le a \le y$. Since a has a complement in L so it has a relative complement b in the interval $[x, y]$.

Then we have , $a \wedge b = x$ and $a \vee b = y$

$$\begin{aligned} \therefore \langle x, y\rangle &= \langle a \wedge b, a \vee b\rangle \\ &= \varphi(b) \end{aligned}$$

Hence φ is onto. Therefore φ is an isomorphism. •

Zorn'a Lemma 2.4.2: Let A be a subset and let χ be a non empty subset of $P(A)$. Let us assume that χ has the following property :

If $C \subseteq \chi$ and C is a chain , then $\cup\{X \mid X \in C\} \in \chi$. Then χ has a maximal number. •

Corollary 2.4.3: In a distributive lattice L every ideal is the intersection of al prime ideals containing it.

Proof.: Let I be any ideal of L. Let $I_1 = \cap\{P \mid P \supseteq I\}$, P is a prime ideal of L. We have to show that $I = I_1$.

If $I \neq I_1$, then there exists an element $x \in I_1$ but $x \notin I$, Then by Stone theorem there exists a prime ideal $P_1 \supseteq I$ but $x \in P_1$. This implies that $x \notin I_1$. Which is a contradiction.

Hence $I = I_1$. •

Theorem 2.4.4: Let L be a distributive lattice with 0 and 1. Then L is a Boolean lattice if and only if $P(L)$, the set of all prime ideals of L is unordered.

Proof.: First suppose L is a Boolean lattice.

Suppose $P(L)$ is not unordered. Then there exist $P, Q \in P(L)$. Then there exists an $a \in Q - P$.

Now $a \wedge a' = 0 \in P$. Since P is prime and $a \notin P$ implies $a' \in P \subset Q$.

$\Rightarrow a' \in Q$.

Thus $a \vee a' = 1 \in Q$. Which is a contradiction as Q is prime.

Hence $P(L)$ is unordered.

Conversely, Suppose that $P(L)$ is unordered. We have to show that L is a Boolean lattice.

If L is not Boolean, then there exist an element $a \in L$ which has no complement.

Set $D = \{x \mid a \vee x = 1\}$. Then is a dual ideal.

Consider $D_1 = D \vee [a) = \{x \mid x \geq d \wedge a\}$ for some $d \in D$.

$[D = \{x \mid a \vee x = 1\}$, $[a) = \{x \mid a \leq x\}$, $D \vee a = \{x \mid x \geq a \geq a \wedge d\}$ for some $d \in D]$

Now we have to show that D_1 does not contain 0.

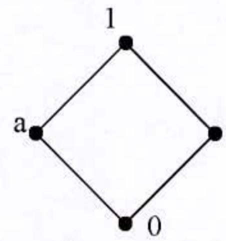
If D_1 contain 0, then $0 = d \wedge a$ for some $d \in D$. Then we have $d \vee a = 1$. Which gives a contradiction as L is not Boolean.

Hence $0 \notin D_1$. Then there exists a prime P such that $P \cap D_1 = \Phi$.

Now $1 \notin [a) \vee P$ for otherwise $1 = a \vee p$ for some $p \in P$.

Which is a contradiction. •

Definition (Join irreducible element): An element $a \in L$ is called a join irreducible element if for $b, c \in L$, $a = b \vee c$ implies that either $b = a$ or $c = a$.



$a = a \vee 0, \;\; b = b \vee 0$

Fig. 2.10

Here a, b are all join irreducible. We denote $J(L)$ as the set of all join irreducible element.

Definition (Hereditary): A subset A of a poset P is called hereditary if for any $x \in A$ and $y \leq x; (y \in P)$ implies $y \in A$.

$H(P)$ denote the set of all hereditary subset of P.

Theorem 2.4.5: Let L be a finite distributive lattice. Then the map $\varphi: a \to r(a)$ is a isomorphism between L and $H(J(L))$.

Proof.: Define $\varphi: L \to H(J(L))$ by $\varphi(a) = r(a)$, $a \in L$.

Since L is finite, so every element is the join of join irreducible elements. Thus $a \in L \Rightarrow a = \vee\, r(a)$.

Obviously $\varphi(a \wedge b) = \varphi(a) \cap \varphi(b)$. So φ is a meet homomorphism.

To show that φ is a join homomorphism. We are to show that

$r(a \vee b) = r(a) \cup r(b)$.

Now $r(a) \cup r(b) \subseteq r(a \vee b)$ is obvious.

Let $x \in r(a \vee b)$

$$\Rightarrow x \leq a \vee b$$

$$\Rightarrow x = x \wedge (a \vee b)$$

$$= (x \wedge a) \vee (x \wedge b)$$

Since $x \in J(L)$, so we have either $x = x \wedge a$ or $x = x \wedge b$

$\Rightarrow$ either $x \leq a$ or $x \leq b$

$\Rightarrow$ either $x \in r(a)$ or $x \in r(b)$

$\Rightarrow$ $x \in r(a) \cup r(b)$

Hence, $r(a \vee b) \subseteq r(a) \cup r(b)$.

Therefore, $r(a \vee b) = r(a) \cup r(b)$. So φ is a join homomorphism.

Therefore, φ is a homomorphism.

Suppose $\varphi(a) = \varphi(b)$, $a, b \in L$

$$\Rightarrow r(a) = r(b)$$

$$\Rightarrow \vee r(a) = \vee r(b)$$

$$\Rightarrow a = b$$

Hence φ is one-one.

To show φ is onto. Let $A \in H(J(L))$ and $a \in L$. Set $a = \vee A$. We are to show that $r(a) = A$.

Clearly, $A \subseteq r(a)$.

Let $x \in r(a) \Rightarrow x \leq a$

$$\Rightarrow x = x \wedge a$$

$= x \wedge (\vee A)$

$= \vee(x \wedge t \mid t \in A)$ (since L is distributive)

Since $x \in J(L)$ so $x = x \wedge t$ for some $t \in A$.

$\Rightarrow x \leq t$

$\Rightarrow x \in A$ as $A \in H(J(L))$

$\Rightarrow r(a) \subseteq A$

$\therefore r(a) = A$

$\Rightarrow \varphi(a) = A$

Hence φ is onto.

Therefore, $L \cong H(J(L))$. •

Proposition 2.4.6 : Let L be a lattice, let P be a prime ideal of L and let $a, b \in L$. Prove that if $a \vee (b \wedge c) \in P$ then $(a \vee b) \wedge (a \vee c) \in P$.

Proof.: Suppose $a \vee (b \wedge c) \in P$, then we have $a \in P$ and $b \wedge c \in P$, since $a \leq a \vee (b \wedge c)$, $b \wedge c \leq a \vee (b \wedge c)$ and P is ideal.

$\Rightarrow a \in P$ and $b \in P$ or $c \in P$

$\Rightarrow$ either $a \vee b \in P$ or $a \vee c \in P$

$\Rightarrow$ $(a \vee b) \wedge (a \vee c) \in P$ [as P is prime ideal.] •

Proposition 2.4.7: Show that the lattice L is distributive if and only if for all $x, y \in L$, $x < y$.There exists a prime ideal P with $x \in P$, $y \notin P$.

Proof. : Suppose L is distributive, let $x, y \in L$ with $x < y$. Consider $I = (x]$ and $D = [y)$, then $I \cap D = \Phi$ and so there exist a prime ideal P such that $P \supseteq I$ and $P \cap D = \Phi$, then $x \in P$, $y \notin P$.

Conversely, let us assume that for all $x, y \in L$ with $x < y$ there exists a prime ideal P such that $x \in P$, $y \notin P$.

We have to prove that L is distributive.

If possible , let L is not distributive. Then there exists $a, b, c \in L$ such that $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$ as $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

$\Rightarrow a \vee (b \wedge c) < (a \vee b) \wedge (a \vee c)$

Hence there exist a prime ideal P such that $a \vee (b \wedge c) \in P$,

$(a \vee b) \wedge (a \vee c) \notin P$

Now, $a \vee (b \wedge c) \in P$

$\Rightarrow$ $a \in P$ and $b \wedge c \in P$ [$\because$ P is prime ideal]

$\Rightarrow$ $a \in P$ and either $b \in P$ or $c \in P$

$\Rightarrow$ either $a, b \in P$ or $a, c \in P$

$\Rightarrow$ either $a \vee b \in P$ or $a \vee c \in P$

$\Rightarrow$ $(a \vee b) \wedge (a \vee c) \in P$ which is a contradiction.

Hence L must be distributive.

Since P is prime, it is follows that $a \vee (b \wedge c) \in P$ &

$(a \vee b) \wedge (a \vee c) \notin P$ gives in a contradiction.

Hence L is distributive. •

Proposition 2.4.8: A lattice L is distributive if and only if $I(L)$ is distributive ; $I(L)$ is the set of all ideals.

Proof. : Suppose L is distributive. Let $I, J, K \in I(L)$. We need to show that $I \wedge (J \vee K) = (I \wedge J) \vee (I \wedge K)$.

The relation $(I \wedge J) \vee (I \wedge K) \subseteq I \wedge (J \vee K)$ is obviously true. Let $x \in I \wedge (J \vee K)$, then $x \in I$ and $x \in J \vee K$. Since L is distributive. So $x = x \wedge (j \vee k) = (x \wedge j) \vee (x \wedge k) \in (I \wedge J) \vee (I \wedge K)$ for some $j \in J$, $k \in K$.

Then, $I \wedge (J \vee K) \subseteq (I \wedge J) \vee (I \wedge K)$

$\therefore I \wedge (J \vee K) = (I \wedge J) \vee (I \wedge K)$

$\therefore\ I(L)$ is distributive.

Conversely, suppose, $I(L)$ is distributive. let $x, y, z \in L$. Then

$$\begin{aligned}(x \wedge (y \vee z)] &= (x] \wedge (y \vee z] \\ &= (x] \wedge [(y] \vee (z]) \text{ as } I(L) \text{ is distributive.} \\ &= (x \wedge y] \vee (x \wedge z] \\ &= ((x \wedge y) \vee (x \wedge z)]\end{aligned}$$

$$\Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

So L is distributive. •

"Pseudocomplemented Lattice"

3.1 Introduction.

In this chapter we discuss presodocomplemented lattice, stone and algebric lattice. presodocomplemented lattice have been studied by several authors (17), (22), (25), (26), (27), (29)

Recall that let L be a lattice with 0 and 1 and $a \in L$. An elements $a^* \in L$ is called pseudocomplement of a if $a \wedge a^* = 0$ and $a \wedge x = 0$ $(x \in L)$ implies $x \leq a^*$.

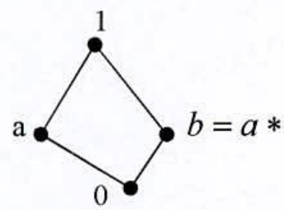


Fig. 3.1

b pseudocomplement of a ie $b = a*$

We denotes pseudocomplement of a by $a*$.

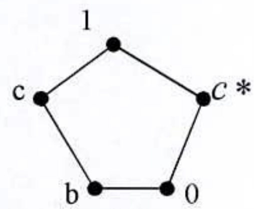


Fig. 3.2

A lattice L with 0 and 1 is called pseudocomplement if its every element has a pseudocomplement.

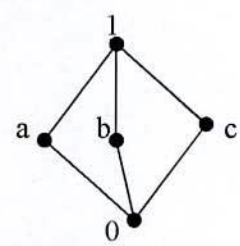


Fig. 3.3

Every finite distributive lattice is called pseudocomplemented.

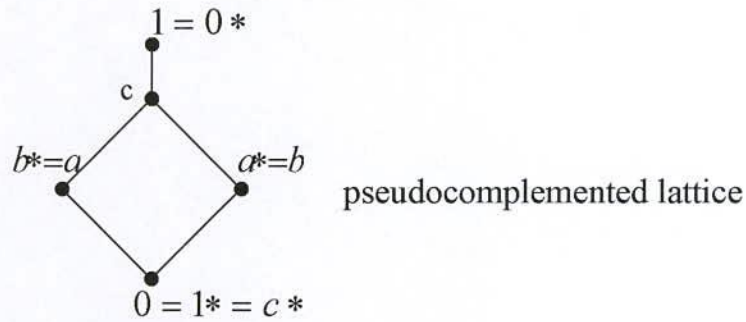
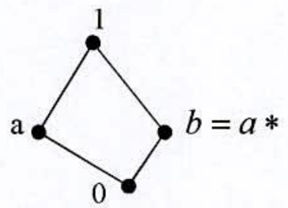


Fig. 3.4

3.2 Pseudocomplemented Lattice.

Definition (Pseudocomplemented) :

Let L be a lattice with 0 and 1 and $a \in L$. An elements $a* \in L$ is called pseudocomplement of a if $a \wedge a* = 0$ and $a \wedge x = 0$ $(x \in L)$ implies $x < a *$



b pseudocomplement of a ie $b = a *$

Fig. 3.1

We denotes pseudocomplement of a by $a *$.

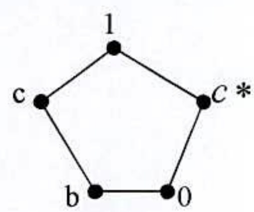


Fig 3.6

A lattice L with 0 and 1 is called pseudocomplement if its every element has a pseudocomplement.

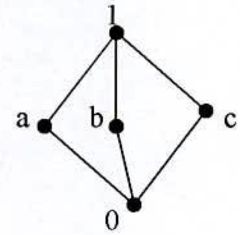
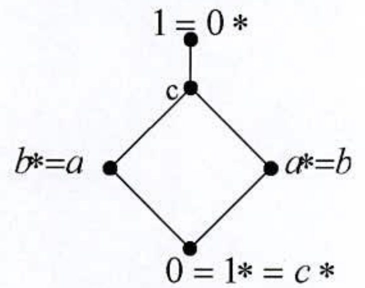


Fig 3.7

Every finite distributive lattice is called pseudocomplemented.

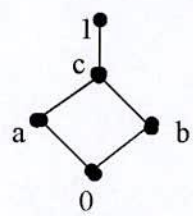


pseudocomplemented lattice

Fig. 3.8

Definition (Dense element) : If the pseudocomplemented zero of an element is called dense element and denoted by D(L).

$\therefore \quad D(L) = \{a \in L \mid a* = 0\}$

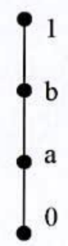


$D(L) = \{1, c\}$

Fig. 3.9

Definition (Dense Lattice) : A pseudocomplemented lattice is called dense lattice if $S(L) = \{0, 1\}$.

$S(L)$ is called the skeletal of L. The elements of $S(L)$ are called skeletal elements.



$S(L) = \{0, 1\}$

Fig 3.10

Proposition 3.2.1 : Let L be a pseudocomplemented meet semilattice and let $a, b \in L$ verify that formulas

$(a \wedge b)* = (a** \wedge b)* = (a** \wedge b**)*$.

Proof. : We know that, $(a \wedge b)* = (a \wedge b)*** * = ((a \wedge b)**)*$

ie, $(a \wedge b)* = (a ** \wedge b **)*$ (i)

Again, $(a ** \wedge b)* = (a ** \wedge b)*** * = ((a ** \wedge b)**)*$

$= (a **** \wedge b **)*$

ie, $(a ** \wedge b)* = (a ** \wedge b **)*$ (ii)

Now from (i) and (ii), we get,

$(a \wedge b)* = (a ** \wedge b)* = (a ** \wedge b **)*$ •

Proposition 3.2.2 : Let L be a pseudocomplemented distributive lattice. Prove that for each $a \in L$, $(a]$ is a pseudocomplemented distributive lattice, In fact, the pseudocomplement of $x \in (a]$ in $(a]$ is $x * \wedge a$.

Proof. : Since L is distributive lattice, then for each $a \in L$, $(a]$ is also distributive lattice. We shall now show that $(a]$ is pseudocomplemented.

let $x \in (a]$ then $x \wedge (x * \wedge a) = (x \wedge x *) \wedge a = 0 \wedge a = 0$.

Furthermore, if $x \wedge t = 0$ then $t \leq x * \Rightarrow t \wedge a \leq x * \wedge a$

$\Rightarrow t \leq x * \wedge a$ $[\because t \in (a] \Rightarrow t \leq a \Rightarrow t \wedge a = t.]$

From the above it follows that $x * \wedge a$ is the pseudocomplement of x. Therefore, $(a]$ is a pseudocomplemented distributive lattice.

The proof. is thus complete. •

3.3 Minimal prime ideal.

Definition (Minimal prime ideal) :

A prime ideal P of a lattice L is called a minimal prime ideal if there is no prime ideal Q such that $Q \subset P$.

Theorem 3.3.1.: Let L is a lattice with 0. Then every prime ideals contains a minimal prime ideal.

Proof.: Let P be a prime ideal of L and χ denotes the set of all prime ideals Q contained in P. Then χ is nonempty, since $P \in \chi$.

Let C is a chain in χ and let $M = \cap\{\chi \mid \chi \in C\}$.

Then M is nonempty and $0 \in M$.

Clearly M is an ideal. Let $a \wedge b \in M$ for some $a, b \in L$, then $a \wedge b \in \chi$ for $a \vee \chi \in C$.

Since χ is prime, so either $a \in \chi$ or $b \in \chi$.

ie either $M = \cap\{\chi \mid a \in \chi\}$ or $M = \cap\{\chi \mid b \in \chi\}$

ie either $a \in M$ or $b \in M$.

Hence M is a prime ideal. Therefore every chain in χ has a smallest element.

Therefore by Zorn's Lemma χ has a minimal ideal R.

In other words P contains a minimal prime ideal R. ●

Theorem 3.3.2.: Let L be a pseudocomplementd distributive lattice and P be a prime ideal of L. Then the following conditions are equivalent.

(i) P is minimal.

(ii) $x \in P$ implies $x* \notin P$.

(iii) $x \in P$ implies $x** \in P$.

(iv) $P \cap D(L) = \Phi$

Proof.: $(i) \Rightarrow (ii)$

Suppose (i) holds. ie P is a minimal prime ideal.

Let $x \in P$. I f (ii) fails, then $x^* \in P$.

Let $D = (L - P) \vee [x)$ we claim that $0 \notin D$,

for if $0 \in D$, then $0 = q \wedge x$ for some $q \in L - P$.

$\Rightarrow q \leq x^* \in P \Rightarrow q \in P$ which is a contradiction.

Hence $0 \notin D$.

Then by Stone representation theorem there exist a prime ideal Q such that $Q \cap D = \Phi$.

$\Rightarrow (L - P) \cap Q = \Phi$ and so $Q \subseteq P$.

Moreover $x \in P$ but $x \notin Q$ and so $Q \subset P$.

Which is a contradiction.

Hence $x^* \notin P$. ie (ii) holds.

$(ii) \Rightarrow (iii)$

Suppose (ii) holds. ie $x \in P$ implies $x^* \notin P$.

Now $x^* \wedge x^{**} = 0 \in P$. Since P is prime and $x^* \notin P$, so $x^{**} \in P$.

ie (iii) holds.

$(iii) \Rightarrow (iv)$

Suppose (iii) holds. ie $x \in P$ implies $x^{**} \in P$.

Let $x \in P \cap D(L)$. then $x \in P$ and $x \in D(L)$.

Then $x^* = 0 \Rightarrow x^{**} = 1$. But $x^{**} \in P \Rightarrow 1 \in P$.

Which is a contradiction.

Therefore $P \cap D(L) = \Phi$. Hence (iv) holds.

$(iv) \Rightarrow (i)$

Suppose (iv) holds. ie $P \cap D(L) = \Phi$

If (i) does not hold, then there exists a prime ideal Q such that $Q \subset P$.

Let $x \in P - Q$. Then $x \notin Q$. Now $x \wedge x* = 0 \in Q$.

Since Q is prime and $x \notin Q$ then $x* \in Q \subset P$.

$$\Rightarrow x* \in P$$

Therefore $x \vee x* \in P$. Moreover $(x \vee x*)* = x* \wedge x** = 0$.

$\Rightarrow x \vee x* \in D(L)$

$\Rightarrow x \vee x* \in P \cap D(L)$. Which contradict (iv)

Hence P is minimal. ie (i) holds. •

3.4 Stone lattice, Algebraic Lattice and Compact element

Definition (Stone lattice) :

A distributive pseudocomplemented lattice L is called Stone lattice if for all $a \in L$ $a * \vee a ** = 1$

Example 3.4.1 (Every Boolean lattice is Stone lattice Converse is not true).

Fig. 3.11

Stone lattice but not Boolean lattice

Definition (Stone Algebra) :

A pseudocomplemented distributive lattice is called a Stone algebra if for each $a * \vee a ** = 1$.

Definition (Generalized Stone Lattice):

A lattice L with 0 is called generalized Stone lattice if $(x]* \vee (x]** = L$ for each $x \in L$.

The generalize pseudocomplemented lattices (ie. a lattice with 0 such that $(x]*$ is a principal for each x.)

Katrinak [5, Lemma 8, p.134] proved the following result.

Lemma 3.4.1 : A lattice with 0 is a generalized Stone lattice if and only if each interval $[0, x]$, $0 < x \in L$, is a Stone lattice.

We remark that a Stone lattice can be considered as either a generalized Stone lattice with 1 or a pseudocomplemented lattice in which $x * \vee x ** = 1$ for each x where $(x]* = (x]$. •

Theorem 3.4.2 : For a distributive lattice L with pseudocomplementation the following condition are equivalent :

(i) L is a stone algebra.

(ii) For $a, b \in L$ $(a \wedge b)^* = a^* \vee b^*$.

(iii) $a, b \in S(L)$ implies that $a \vee b \in S(L)$.

(iv) $S(L)$ is a subalgebra.

Proof : $(i) \Rightarrow (ii)$

Suppose (i) holds. ie L is a Stone algebra.

We shall that, $(a \wedge b)^* = a^* \vee b^*$

Let $a, b \in L$. Then,

$$(a \wedge b) \wedge (a^* \vee b^*) = (a \wedge b \wedge a^*) \vee (a \wedge b \wedge b^*)$$

[$\because L$ is distributive lattice]

$$= (a \wedge a^* \wedge b) \vee (a \wedge b \wedge b^*)$$

$$= (0 \wedge b) \vee (a \wedge 0)$$

$$= 0 \vee 0$$

$$= 0$$

Now suppose $x \in L$ such that $(a \wedge b) \wedge x = 0$.

$\Rightarrow (b \wedge x) \wedge a = 0 \Rightarrow b \wedge x \leq a^*$. Meeting both sides with a^{**} we get,

$$a^{**} \wedge (b \wedge x) \leq a^{**} \wedge a^* = 0$$

$$\Rightarrow (x \wedge a^{**}) \wedge b = 0$$

$$\Rightarrow x \wedge a^{**} = b^*.$$

since L is a Stone algebra, then we have

$$a^* \vee a^{**} = 1$$

Now $x = x \wedge 1 = x \wedge (a^* \vee a^{**})$

$$= (x \wedge a^*) \vee (x \wedge a^{**})$$

$$\leq a^* \vee b^*$$

Hence $a*\vee b*$ is the pseudocomplement of $a\wedge b$. ie (ii) holds.

$(ii)\Rightarrow(iii)$

Suppose (ii) holds. Let $a, b\in S(L)$ we have $a=a**$ and $b=b**$.

$$\therefore\quad a\vee b=(a**\vee b**)$$
$$=(a*\wedge b*)*$$
$$=(a\vee b)**$$
$$\Rightarrow a\vee b\in S(L)$$

$(iii)\Rightarrow(iv)$

Suppose (iii) holds. ie $a, b\in S(L)$ implies that $a\vee b\in S(L)$.

As $a, b\in S(L)$, so we have

$a\vee b\in S(L)$.

Hence (iv) holds, ie $S(L)$ is a subalgebra.

$(iv)\Rightarrow(i)$

Suppose (iv) holds, ie $S(L)$ is a subalgebra of L.

Now, for any $a\in L$, $a*\in S(L)$, $a**\in S(L)$.

Hence $a*\vee a**=(a**\wedge a***)*$ [From $a\vee b=(a*\wedge b*)*$]

$$=0*$$
$$=1$$

Hence L is stone algebra. ie (i) holds. •

Theorem 3.4.3.: Let L be a distributive lattice with pseudocomplemented. Then L is a Stone algebra if and only if $P\vee Q=L$ for any two distinct minimal prime ideal.

Proof.: 1st Suppose L is a Stone algebra. Suppose P & Q are two distinct minimal prime ideals.

Let $a\in Q-P$. Then $a\notin P$. Now $a\wedge a*=0\in P$. Since P is prime and $a\notin P$ so $a*\in P$.

Now L - Q is a minimal dual prime ideal.

Thus $(L-Q)\vee[a)=L$. So $a=x\wedge a$ for some $x\in L-Q$.

$$\Rightarrow\ a^*\geq x\in L-Q$$

$$\Rightarrow\ a^*\in L-Q$$

$$\Rightarrow\ a^*\notin Q$$

$$\Rightarrow\ a^*\in P-Q$$

Similarly we have, $a^{**}\in P-Q$.

Hence $a^*\vee a^{**}\in P\vee Q$. But since L is a Stone algebra then $a^*\vee a^{**}=1$.

$\Rightarrow\ 1\in P\vee Q\ \Rightarrow\ L=P\vee Q$.

Conversely,

Suppose $P\vee Q=L$ for any two distinct minimal prime ideals.

We have to show that L is a Stone algebra.

If L is not Stone algebra, then there exists $a\in L$ such that $a^*\vee a^{**}\neq 1$. Then there exists a prime ideal R such that $a^*\vee a^{**}\in R$.

We claim that, $(L-R)\vee[a^*)\neq L$

For if $(L-R)\vee[a^*)=L$ then $x\wedge a^*=0$, for some $x\in(L-R)$

$\Rightarrow a^{**}\geq x\in(L-R)\Rightarrow a^{**}\in(L-R)\Rightarrow a^{**}\notin R$

Which is a contradiction.

Hence $(L-R)\vee[a^*)\neq L$

Let F be a maximal dual prime ideal containing $(L-R)\vee[a^*)$ and G be a maximal dual prime ideal containing $(L-R)\vee[a^{**})$.

Put $P=L-F$ and $Q=L-G$.

Then P and Q are minimal prime ideal and $P \neq Q$. as $a* \in Q$ but $a* \notin P$ and $a** \in P$ but $a** \notin Q$.

ie P and Q are distinct.

Also $P, Q \subseteq R$ and thus $P \vee Q \subseteq R \neq L$.

Which is a contradiction.

Hence L is a Stone algebra. •

Proposition 3.4.4 : Show that a distributive pseudocomplemented lattice is a Stone lattice if and only if $(a \vee b)** = a** \vee b**$ for $a, b \in L$.

Proof.: Let L be a distributive pseudocomplemented lattice. If L is a Stone lattice, then for $a, b \in L$.

we have ,

$(a \wedge b)* = a* \vee b*$, and for any pseudocomplemented lattice,

$(a \vee b)* = a* \wedge b*$

Hence $(a \vee b)** = (a* \wedge b*)* = a** \vee b**$

Conversely, let $(a \vee b)** = a** \vee b**$ for all $a, b \in L$.

Now for $x \in L$. Let $x* \vee x** = y$, then

$(x* \vee x**)** = y**$

or $x*** \vee x**** = y**$

or $x* \vee x** = y**$

or $y = y**$

Now $y* = (x* \vee x**)*$

$= x** \wedge x***$

$= x** \wedge x*$

$= 0$

$\therefore y** = 0* = 1$

$\Rightarrow y = 1$

Hence $x*\vee x**=1$. Therefore L is Stone lattice. •

Proposition 3.4.5 : Show that in a Stone algebra every prime ideal contain exactly one minimal prime ideal.

Proof: Let p be a prime ideal and let Q_1 & Q_2 be two minimal prime ideals contains in p with $Q_1 \neq Q_2$. Let $x \in Q_1 - Q_2$, then $x \in Q_1$ but $x \notin Q_2$

Now $x \wedge x^* = 0 \in Q_2$. $\Rightarrow x^* \in Q_2 \Rightarrow x^* \in p$.

Again since a_1 is minimal, then $x \in Q_1 \Rightarrow x^{**} \in Q_1 \Rightarrow x^{**} \in P$.

Hence $1 = x^* \vee x^{**} \in P$ which contradict the fact that P is prime.

Hence $Q_1 = Q_2$.

Hence in a Stone algebra every prime ideal contains exactly one minimal prime ideal . •

Proposition 3.4.6 : If P is a prime ideal of a lattice L, then $\dfrac{L}{R(P)}$ is a two element chain. The elements are P, $L-P$.

Proof: Let $x, y \in L-P$.

If for some $l \in L, x \wedge l \in P$,then $l \in P$ ($\because x \notin P$ and P is prime).

Hence $y \wedge l \in P$

ie $\forall\, l \in L$, $x \wedge l \in P$

$\Leftrightarrow y \wedge l \in P$

$\Rightarrow x \equiv y\ R(P)$. •

Definition (Compact element) :

Let L be a lattice. An element $a \in L$ is called compact if for any $X \subseteq L$ with $a \leq \vee X$ implies the existence of a finite subset $X_1 \subseteq X$ such that $a \leq \vee X_1$.

Definition (Algebraic Lattice) :

A complete lattice L is called algebraic if its every element is the supremum of compact element.

Theorem 3.4.7: Every distributive algebraic lattice is pseudocomplemented.

Proof.: Let L be a distributive algebraic lattice. Then $L \cong I(S)$, the lattice of ideals where S is a join semilattice with 0, let $I, I_k \in I(S)$ for $k \in K$ (index set). Then $I \wedge I_r \subseteq I \wedge \vee(I_k \mid k \in K)$ for any $r \in K$.

Clearly $\vee(I_k \mid k \in K) \subseteq I \wedge \vee(I_k \mid k \in K)$. To prove the reverse inequality. Let $a \in I \wedge \vee(I_k \mid k \in K)$. Then $a \in I$ and $a \in \vee(I_k \mid k \in K)$. Then there exist indices $\lambda_{1,} \lambda_2, \cdots, \lambda_n$ such that $a \leq i_{\lambda_1} \vee i_{\lambda_2} \vee \cdots \vee i_{\lambda_n}$ for some $i_{\lambda_k} \in I_{\lambda_k}$ for some $k \in 1, 2, \cdots, n$.

Thus $a \in I_{\lambda_1} \vee I_{\lambda_2} \vee \cdots \vee I_{\lambda_n}$ and so

$a = I \wedge (I_{\lambda_1} \vee I_{\lambda_2} \vee \cdots \vee I_{\lambda_n})$.

$= (I \wedge I_{\lambda_1}) \vee (I \wedge I_{\lambda_2}) \vee \cdots \vee (I \wedge I_{\lambda_n})$ as $I(S)$ is distributive

$\subseteq \vee(I_k \mid k \in K)$

ie $I \wedge \vee(I_k \mid k \in K) \subseteq \vee(I_k \mid k \in K)$

Therefore $I \wedge \vee(I_k \mid k \in K) = \vee(I_k \mid k \in K)$.

This shows that $I(S)$ has the join infinite distributive property. Moreover as $0 \in S$, $I(S)$ is complete . Therefore $I(S)$ is pseudocomplemented and so L is pseudocomplemented. •

Theorem 3.4.8: Let L be a pseudocomplemented meet semilattice $S(L) = \{a* \mid a \in L\}$. Then the partial ordering of L partially orders $S(L)$ and make $S(L)$ into a Boolean lattice . For $a, b \in S(L)$. We have $a \wedge b \in S(L)$ and the join in $S(L)$ we described by $a \vee b = (a * \wedge b*)*$.

Proof.: We start with the following observations.

(i) $\forall a \in L, a \leq a**$

(ii) $a \leq b \Rightarrow a* \geq b*$

(iii) $a* = a***$

(iv) $a \in S(L)$ if and only if $a = a**$

(v) For $a, b \in S(L)$, $a \wedge b \in S(L)$

(vi) For $a, b \in S(L)$, $a \vee b = (a* \wedge b*)*$

(vii) Since $a* \wedge a = a \wedge a* = 0$. Also $a* \wedge a** = 0$.

So $a \leq a**$ from the definition of pseudocomplement.

(viii) $a \leq b$, so $.a \wedge b* \leq b \wedge b* = 0$

ie $a \wedge b* = 0 \Rightarrow b* \leq a*$

from the definition of pseudocomplement.

(ix) By (i) $a* \leq (a*)** = a***$,

again $a \leq a**$ by (i)

so by (ii) $a*** \leq a*$.

Hence $a* = a***$.

(x) Let $a \in S(L)$ then $a = b*$ for some $b \in L$.

Hence $a** = b*** = b* = a$. If $a = a**$ then $a = (a*)*$ and so, $a \in S(L)$.

(xi) Let $a, b \in S(L)$ then $a = a**, b = b**$ so $a \geq (a \wedge b)**$ $b \geq (a \wedge b)**$. So $(a \wedge b)** \leq a \wedge b$. Again by (i) & (ii) $a \wedge b \leq (a \wedge b)**$.

Hence $a \wedge b = (a \wedge b)**$.

So, $a \wedge b \in S(L)$, $a \geq a \wedge b$

$$\Rightarrow a ** \geq (a \wedge b) ** \text{ by (ii)}$$

$$\Rightarrow a \geq (a \wedge b) **$$

For $a, b \in S(L)$

we have $a* \geq a * \wedge b *$. So by (ii) and (iv) $a \leq (a * \wedge b*)*$. Similarly $b \leq (a * \wedge b*) *$.

Now if $a \leq x$, $b \leq x$ $(x \in S(L))$, then $a* \geq x *$, $b* \geq x *$. So $a * \wedge b* \geq x *$.

Hence, $x ** \geq (a * \wedge b*) *$ ie $x \geq (a * \wedge b*) *$ as $x \in S(L)$, Hence, $(a * \wedge b*)* = \sup\{a, b\} = a \vee b \in S(L)$.

Thus $S(L)$ is a lattice. Moreover $0, 1 \in S(L)$. Therefore $S(L)$ is a bounded lattice

Now for any $a \in S(L)$, $a \wedge a* = 0$ and $a \vee a* = (a * \wedge a **)* = 0* = 1$. ie $a *$ is the complement of a in $S(L)$. Hence $(S(L); \wedge, \vee)$ is a complemented lattice. Then we only to show that $S(L)$ is distributive. Let $x, y, z \in S(L)$ $x \wedge z \leq x \vee (y \wedge z)$ and $y \wedge z \leq x \vee (y \wedge z)$. Hence $x \wedge z \wedge (x \vee (y \wedge z))* = 0$ and $y \wedge z \wedge (x \vee (y \wedge z))* = 0$. Thus, $z \wedge (x \vee (y \wedge z))* \leq x *$ and $y *$, and so $z \wedge (x \vee (y \wedge z))* \leq x * \wedge y *$

Consequently, $z \wedge (x \vee (y \wedge z)) * \wedge (x * \wedge y*) = 0$, which implies $z \wedge (x * \wedge y*)* = (x \vee (y \wedge z)) ** = x \vee (y \wedge z)$ So by (vi) and (iv) $z \wedge (x \vee y) = x \vee (y \wedge z)$.

Therefore $S(L)$ is distributive

Definition (Dense set) :

$D(L) = \{a \in L \mid a* = 0\}$, $D(L)$ is called the dense set.

$D(L)$ is a filter or Dual ideal, $1 \in D(L)$. If L is a pseudocomplemented lattice L then some properties hold in L

(i) $a \wedge a* = 0$

(ii) $a \leq b \Rightarrow a* \geq b*$

(iii) $a \leq a**$

(iv) $a* = a***$

(v) $(a \vee b)* = a* \wedge b*$

(vi) $(a \wedge b)** = a** \wedge b**$

(vii) $a \wedge b = 0 \Leftrightarrow a** \wedge b** = 0$

(viii) $a \wedge (a \wedge b)* = a \wedge b*$

(ix) $0* = 1, \quad 1* = 0$

(x) $a \in S(L) \Leftrightarrow a = a**$

(xi) $a, b \in S(L) \Rightarrow a \wedge b \in S(L)$

(xii) $Sup\{a, b\} = (a* \wedge b*)*$

(xiii) $0, 1 \in S(L), 1 \in D(L), S(L) \wedge D(L) = \{1\}$

(xiv) $a, b \in D(L) \Rightarrow a \wedge b \in D(L)$

(xv) $a \in D(L), b \geq a \Rightarrow b \in D(L)$

(xvi) $a \vee a* \in D(L)$

(xvii) $x \rightarrow x**$ is a meet homomorphism of L onto $S(L)$.

Proof.:

(v) $(a \vee b)* = a* \wedge b*$

$$(a \vee b) \wedge a* \wedge b* = (a \wedge a* \wedge b*) \vee (b \wedge a* \wedge b*)$$
$$= 0 \vee 0 = 0$$

Now, Let $(a \vee b) \wedge x = 0, \quad x \in L$

Then $(a \wedge x) \vee (b \wedge x) = 0$

$\Rightarrow a \wedge x = 0, \quad b \wedge x = 0$

$\Rightarrow x \leq a*, \quad \Rightarrow x \leq b*$

$\Rightarrow x \leq a* \wedge b*$

i.e. $a* \wedge b*$ is the pseudocomplemented of $a \vee b$

Hence $(a \vee b)* = a* \wedge b*$

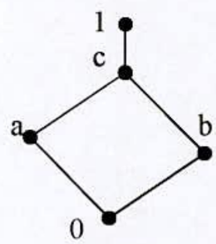


Fig 3.12

$a \wedge b = 0 \quad \Rightarrow (a \wedge b)* = 0* = 1$

$a* = b, \quad b* = a$

$a* \vee b* = a \vee b = c$

$(a \wedge b)* \neq a* \vee b*$

(vi) $(a \wedge b)** = a** \wedge b**$

Proof:- we know that $x**$ is the smallest element in $S(L)$ Continuing x.

Also we know that for any $p, q \in S(L), \ p \wedge q \in S(L)$.

Then for any $a, b \in L$,$a** \wedge b** \in S(L)$

and it is obviously the smallest element of $S(L)$ Containing $a \wedge b$.

Therefore $(a \wedge b)** = a** \wedge b**$. •

(viii) Since $a \wedge b \leq b$, so $(a \wedge b)* \geq b*$

So $a \wedge (a \wedge b)* \geq a \wedge b*$

(xii) Let $a \wedge b = 0$, Then $(a \wedge b)** = 0** = 0$

So by (vi) $a** \wedge b** = 0$

Conversely, Suppose $a** \wedge b** = 0$

So $a \wedge b \wedge a** \wedge b** = a \wedge b \wedge 0$

or, $(a \wedge a**) \wedge (b \wedge b**) = 0$

or, $a \wedge b = 0$

(xiii) Let $x \in S(L) \cap D(L)$. Then $x \in S(L)$ and $x \in D(L)$.

Now $x \in S(L) \Rightarrow x = x**$ & $x \in D(L) \Rightarrow x* = 0$.

So, $x = x** = (x*)* = (0)* = 0* = 1$

Hence $S(L) \wedge D(L) = \{1\}$

(xiv) $a, b \in D(L)$

$\Rightarrow a* = b* = 0$

$\Rightarrow a** = b** = 1$

$\therefore$ by (vi), $(a \wedge b)** = a** \wedge b** = 1 \wedge 1 = 1$

$\therefore (a \wedge b)* = (a \wedge b)*** = 1* = 0$

So, $a \wedge b \in D(L)$.

(xv) $a \in D(L), b \geq a,\ b* \leq a* = 0 \Rightarrow b* = 0, \Rightarrow b \in D(L)$

(xvi) $(a \wedge a*)* = a* \wedge a**$ by (v)

$= 0$

$\Rightarrow a \vee a* \in D(L)$

(xvii) follows from (vi)

$\varphi(x \wedge y) = (x \wedge y)** = x** \wedge y**$

$= \varphi(x) \wedge \varphi(y)$

Theorem 3.4.9: For a distributive lattice with pseudocomplemention L, the following condition are equivalent

(i) L is a Stone algebra

(ii) $\forall\ a, b \in L$, $(a \wedge b)* = a* \vee b*$

(iii) $a, b \in S(L)$ implies $a \vee b \in S(L)$

(iv) $S(L)$ is a subalgebra of L.

Proof. : (i) $\Rightarrow$ (ii)

Suppose L is a Stone algebra.

We shall show that, $a * \vee b *$ is the pseudocomplement of $a \wedge b$

We have, $a \wedge b \wedge (a * \vee b *) = (a \wedge b \wedge a *) \vee (a \wedge b \wedge b *)$

$= 0 \vee 0 = 0$

Now, Suppose $a \wedge b \wedge x = 0$ for some $x \in L$

Then $(b \wedge x) \wedge a = 0$ which implies $b \wedge x \leq a *$

Multiplying both sides by $a **$.

We have $b \wedge x \wedge a ** \leq a * \wedge a ** = 0$

i.e. $(x \wedge a **) \wedge b = 0$

Which imply $x \wedge a ** \leq b **$.

Now by, Stone identity $a * \vee a ** = 1$.

So, $x = x \wedge 1 = (x \wedge a *) \vee (x \wedge a **) \leq a * \vee b *$

Therefore, $(a \wedge b)* = a * \vee b *$

(ii) $\Rightarrow$ (iii)

Suppose (ii) holds

Let $a, b \in S(L)$. Then $a = a **, b = b **$

So by (ii) $a \vee b = a ** \vee b **$

$= (a * \wedge b *)*$ by (iii)

$= a \vee b$

i.e. $a \vee b \in S(L)$

(iii) $\Rightarrow$ (iv) is trivial

(v) $\Rightarrow$ (i)

Suppose (iv) holds

Let $a \in L$. Then $a*, a** \in S(L)$

Since (iv) holds , $a* \vee a** = a* \vee a**$

$$= (a** \wedge a***)*$$

$$= 0* = 1$$

Hence L is Stone . •

Theorem 3.4.10: Let L be a pseudocomplemented distributive lattice and P be a Prime ideal of L. Then the following conditions are equivalent

(i) P is minimal

(ii) $x \in P, \Rightarrow x* \notin P$

(iii) $x \in P, \Rightarrow x** \in P$

(iv) $P \wedge D(L) = \varphi$

Proof:- (i) $\Rightarrow$ (ii)

Let P be minimal.

Suppose, It (ii) fails there exists $x \in P$ such that $x* \in P$.

Let $D = (L-P) \vee [x)$ Then $0 \in D$.

For otherwise $0 = q \wedge x$ for some $q \in L - P$, which implies $q \leq x* \in P$.

Therefore, $q \in P$, which is a contradiction.

Hence $0 \notin P \cap D$. Then by Stones representation theorem there exists a prime ideal Q such that $Q \cap D = \varphi$. This implies $Q \cap (L-P) = \varphi$ and So $Q \subseteq P$. But $Q \neq P$ as $x \in Q$. This contradict the minimally of P.

Hence (ii) follows.

(ii) $\Rightarrow$ (iii)

Suppose (ii) holds and $x \in P$.

Now $x* \wedge x** = 0 \in P$. But $x* \notin P$ and P is prime, So $x** \in P$

i.e. (iii) holds.

(iii) $\Rightarrow$ (iv)

Suppose (iii) hold

Let $x \in P \cap D(L)$.

Then $x \in P$ and $x \in D(L)$.

$\therefore x \in D(L)$ implies $x* = 0$

By (iii) $x** \in P$

$\therefore x** = (x*)* = 0* = 1 \in P$ which is impossible as P is prime .

So (iv) holds.

(v) $\Rightarrow$ (i)

Suppose P is not minimal, Then there exist a prime ideal $Q \subset P$. Let $x \in P - Q$

Now, $x \wedge x* = 0 \in Q$. Since $x \notin Q$ and Q is prime,

So, $x* \in Q \subset P$. Then $x, x* \in P$

So , $x \vee x* \in P$

Now $(x \vee x*)* = x* \wedge x** = 0$ implies $x \vee x* \in D(L)$

i.e. $P \cap D(L) \neq \varphi$

and So (iv) does not hold. •

"Boolean Algebras"

4.1 Introduction.

In this chapter we introduce and study on Boolean algebra, Imbedding mapping and obtain their several features.

A complimented distributive lattice is called Boolean algebra. If Boolean lattices so considered are called Boolean Algebra.

In this chapter we have also proved the following theorem

In a Boolean algebra, the following result hold

(i) $(a')' = a$

(ii) $(a \wedge b)' = a' \vee b'$ [De Morgan's Law]

(iii) $(a \vee b)' = a' \wedge b'$ [De Morgan's Law]

(iv) $a \leq b \Leftrightarrow a' \geq b'$

(v) $a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = u$

4.2 Boolean Algebra, Dual Meet and Join Homomorphism.

Definition (Boolean algebra): A complimented distributive lattice is called Boolean algebra. If Boolean lattices so considered are called Boolean Algebra.

The main results of this paper are

(i) Let M be a bounded distributive lattice and $c \in M$. Thus M can be imbedded in $[0,c]\times[c,u)$

(ii) A Boolean Algebra is self-dual.

Definition (Boolean lattice): A complemented distributive lattice is called a Boolean lattice.

Since complements are unique in a Boolean lattice we can regard a Boolean lattice as an algebra with two binary operations $\wedge$ and $\vee$ and one unary operation $'$. Boolean lattices so considered are called Boolean algebras In other words, by a Boolean Algebra, we mean a system consisting of a nonempty set L together with two binary operation $\wedge$ and $\vee$ and one unary operation (1), satisfying $(\forall a,b,c \in L)$

(i) $a \wedge a = a, \quad a \vee a = a$

(ii) $a \wedge b = b \wedge a, \quad a \vee b = b \vee a$

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \quad a \vee (b \vee c) = (a \vee b) \vee c$

(iv) $a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a$

(v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(vi) $\forall a \in L, \exists\, a' \in L,$ s.t., $a \wedge a' = 0, a \vee a' = u$ where o,u are elements of L satisfying $0 \le x \le u \;\forall\, x \in L$

(a' will be unique and is the complement of a)

Example 4.2.1: Let $A = \{0, a, b, u\}$. Define $\wedge$, $\vee$ and complementation $'$ by

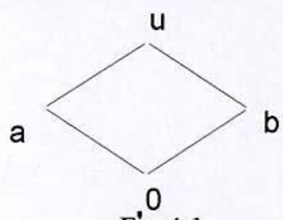


Fig.4.1

∧	0	a	b	u
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
u	0	a	b	u

∨	0	a	b	u
0	0	a	b	u
a	a	a	u	u
b	b	u	b	U
u	u	u	u	u

	′
0	u
a	b
b	a
u	0

Then A form a Boolean algebra under these operations $\wedge, \vee, '$.

Theorem 4.2.1: L and M are Boolean algebras iff L×M is a Boolean algebra.●

Theorem 4.2.2: A Boolean lattice is relatively complemented and relative complements are unique. ●

4.3 Boolean Subalgebra.

Definition (Boolean subalgebra) : A subalgebra (or Boolean subalgebra) is a non empty subset S of a Boolean algebra L s.t. $a, b \in S \Rightarrow a \wedge b, a \vee b, a' \in S$.

We thus realize that a subalgebra differs from a sublattice in as such as it is closed under complementation also. Notice that if $[a,b]$be an interval in a Boolean algebra L, where $a > 0$, then $[a,b]$ is a sublattice of L, but is not a subalgebra as

$$a \in [a,b] \Rightarrow a' \in [a,b]$$
$$\Rightarrow a \wedge a' \in [a,b]$$
$$\Rightarrow 0 \in [a,b]$$

which is not possible as $a > 0$.

Hence a Boolean sublattice may not be a Boolean subalgebra. (The converse being, of course, true).

Theorem 4.3.1: Every interval of a Boolean algebra is itself a Boolean algebra.

Proof : Let $[a, b]$ be any interval of a Boolean algebra L, then $[a, b]$ being a sublattice will be distributive.

Since L is distributive complemented lattice, it is relatively complemented.

i.e., each interval in L is complemented

i.e. $[a,b]$ is complemented distributive lattice and hence is a Boolean algebra. ●

Proposition 4.3.2: Show that a non empty subset S of a Boolean algebra is a subalgebra if it is closed under $\vee$ and complementation.

Proof : We need prove that for any $a, b \in S$, $a \vee b \in S$

Now $$(a \wedge b)' = a' \vee b' \in S$$

$$(a \wedge b) = ((a \wedge b)')' \in S$$

similarly, one can show that S would be a subalgebra if it is closed under $\vee$ and complementation. •

Theorem 4.3.3 : In a Boolean algebra, the following result hold

(i) $(a')' = a$

(ii) $(a \wedge b)' = a' \vee b'$ [De Morgan's Law]

(iii) $(a \vee b)' = a' \wedge b'$ [De Morgan's Law]

(iv) $a \leq b \Leftrightarrow a' \geq b'$

(v) $a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = u$

Proof : (i) Let $(a')' = a''$, then

$$a \wedge a' = 0 \Leftrightarrow a \vee a' = u$$

$$a' \wedge a'' = 0 \Leftrightarrow a' \vee a'' = u$$

$$\Rightarrow \quad a \wedge a' = a'' \wedge a', a \vee a' = a'' \vee a'$$

$$\Rightarrow \quad a'' = a$$

(ii) We have $(a \wedge b) \wedge (a' \vee b') = [(a \wedge b) \wedge a'] \vee [(a \wedge b) \wedge b']$

$$= [(a \wedge a') \wedge b] \vee [a \wedge (b \wedge b')]$$

$$= [0 \wedge b] \vee [a \wedge (0)] = 0 \vee 0 = 0$$

$$(a \wedge b) \vee (a' \vee b') = (a' \vee b') \vee (a \wedge b)$$

$$= [(a' \vee b') \vee a] \wedge [(a' \vee b') \vee b]$$

$$= [(a' \vee a) \vee b'] \wedge [a' \vee (b' \vee b)]$$

$$= [u \vee b'] \wedge [a' \vee u] = u \wedge u = u$$

Hence $(a \wedge b)' = a' \vee b'$

(ii) Similar as (ii)

(iii) $a \leq b \Rightarrow a = a \wedge b$

$$\Rightarrow a' = (a \wedge b)' = a' \vee b'$$

$$\Rightarrow b' \leq a'$$

$b' \leq a' \Rightarrow b'' \leq a'' \Rightarrow b \leq a$

(iv) $a \leq b \Rightarrow a \wedge b' \leq b \wedge b' \Rightarrow 0 \leq a \wedge b' \leq 0 \Rightarrow a \wedge b' = 0$

Again, let $a \wedge b' = 0$

Then, $a = a \wedge u = a \wedge (b \vee b')$

$$= (a \wedge b) \vee (a \wedge b') = (a \wedge b) \vee 0 = (a \wedge b)$$

$\Rightarrow a \leq a \wedge b$. •

4.4 Dual Meet Homomorphism.

Definition (Dual meet homomorphism) : Let L, M be two lattice a map

$\theta : L \to M$ is called a dual meet homomorphism

if, $\theta(a \wedge b) = \theta(a) \vee \theta(b)$ (1)

where $a, b \in L$ and θ is called a dual join homomorphism

if $\theta(a \vee b) = \theta(a) \wedge \theta(b)$ (2)

It is called a dual homomorphism if it satisfied the above conditions.

Theorem 4.4.1 : A Boolean algebra is it self dual

Proof. Let L be a Boolean algebra.

Define a map $\theta : L \to L$, s.t.,

$$\theta(x) = x'$$

then θ is well defined as for each $x \in L, x'$ exists and unique.

Now $\theta(x) = \theta(y) \Rightarrow x' = y'$

$$\Rightarrow (x')' = (y')' \Rightarrow x = y$$

Thus θ is 1-1.

For any $y \in L$, y' is it required pre image under θ showing that θ is onto.

Also $\theta(x \wedge y) = (x \wedge y)' = x' \vee y' = \theta(x) \vee \theta(y)$

$$\theta(x \vee y) = (x \vee y)' = x' \wedge y' = \theta(x) \wedge \theta(y)$$

shows that θ is a dual homomorphism.

Thus θ is a dual isomorphism and hence L is self dual. •

4.5 Imbedding Mapping.

Definition (Imbedding) : Let L, M be two lattices a one-one homomorphism $\theta : L \to M$ is called an imbedding mapping. Also in that case we say L is imbedded in M.

Theorem 4.5.1 : Let L be a bounded distributive lattice and $a \in L$ then L can be imbedded into $[0,a]\times[a,u]$.

Proof. : Define a map $\theta : L \to [0,a]\times[a,u]$, s.t.,

$$\theta(x) = (x \wedge a, x \vee a)$$

Clearly then $x \wedge a \in [0,a]$and $x \vee a \in [a,u]$.

Let $\quad x = y \Rightarrow x \wedge a = y \wedge a$

$$x \vee a = y \vee a$$

$$\Rightarrow (x \wedge a, x \vee a) = (y \wedge a, y \vee a)$$

$$\Rightarrow \theta(x) = \theta(y)$$

Thus θ is well defined.

Again, if $\quad \theta(x) = \theta(y)$

then $\quad (x \wedge a, x \vee a) = (y \wedge a, y \vee a)$

$$\Rightarrow \quad x \wedge a = y \wedge a$$

$$x \vee a = y \vee a$$

$$\Rightarrow \quad x = y$$

Thus θ is 1-1

Now $\quad \theta(x \wedge y) = ((x \wedge y) \wedge a, (x \wedge y) \vee a)$

$$= ((x \wedge a) \wedge (y \wedge a), (x \vee a) \wedge (y \vee a))$$

and $\quad \theta(x) \wedge \theta(y) = ((x \wedge a, x \vee a) \wedge (y \wedge a, y \vee a))$

shows that $\quad \theta(x \wedge y) = \theta(x) \wedge \theta(y)$

similarly $\quad \theta(x \vee y) = \theta(x) \vee \theta(y)$

Hence θ is a 1-1 homomorphism, i.e.,θ is an imbedding map. •

Theorem 4.5.2: If L is a Boolean algebra and $a \in L$, then $L \cong [0,a] \times [a,u]$.

Proof.: By previous theorem

$$\theta : L \to [0,a] \times [a,u] \text{ s.t.,}$$
$$\theta(x) = (x \wedge a, x \vee a)$$

is a 1-1 homomorphism. (Note a Boolean algebra is distributive). We show θ is onto.

Let $(y,z) \in [0,a] \times [a,u]$ be any element

then $0 \le y \le a, \quad a \le z \le u$ (i)

Take $x = y \vee (z \wedge a')$, then

$$\begin{aligned}\theta(x) &= \theta(y \vee (z \wedge a')) \\ &= ((y \vee (z \wedge a')) \wedge a, y \vee (z \wedge a') \vee a) \\ &= ((y \wedge a) \vee ((z \wedge a') \wedge a), y \vee ((z \vee a) \wedge (a' \vee a))) \\ &= (y \vee (z \wedge 0), y \vee (z \wedge u)) \\ &= (y,z)\end{aligned}$$

Hence θ is an isomorphism. •

Proposition 4.5.3 : If A, B, C are lattices such that $B \cong C$, then

$A \times B \cong A \times C$

Proof : Let $f : B \to C$ be the given isomorphism

Define $\theta : A \times B \to A \times C$, s.t.,

$$\theta((a,b)) = (a, f(b))$$

then since $\theta((a,b)) = \theta((c,d))$

$\Leftrightarrow \quad (a, f(b)) = (c, f(d))$

$\Leftrightarrow \quad a = c, \quad f(b) = f(d)$

$\Leftrightarrow \quad a = c, \quad b = d$ (f being well defined 1-1 map)

$\Leftrightarrow \quad (a,b) = (c,d)$

We find θ is a well defined 1-1 map.

Again, for any $(x,y)\in A\times C$, as $y\in C, f:B\to C$ is onto, $\exists\, b\in B$ s.t., $f(b)=y$

Now $\theta((x,b))=(x,f(b))=(x,y)$ and thus θ is onto.

Finally,

$$\theta((a,b)\wedge(c,d))=\theta(a\wedge c,b\wedge d)=(a\wedge c,f(b\wedge d))$$
$$=(a\wedge c,f(b)\wedge f(d))=(a,f(b))\wedge(c,f(d))$$
$$=\theta(a,b)\wedge\theta(c,d)$$

Similarly, $\theta((a,b)\vee(c,d))=\theta(a,b)\vee\theta(c,d)$

Hence θ is an isomorphism. •

Proposition 4.5.4 : If L is a Boolean algebra and $a\in L$, then show that $L\cong[0,a]\times[0,a']$.

Proof : By theorem 5.2.8

$$L\cong[0,a]\times[a,u]$$

Define a map $f:[a,u]\to[0,a']$ s. t.

$$f(x)=x\wedge a'$$

Now $x\in[a,u] \;\Rightarrow\; a\le x\le u$

$$\Rightarrow\; a\wedge a'\le x\wedge a'\le u\wedge a'$$
$$\Rightarrow\; 0\le x\wedge a'\le a'$$
$$\Rightarrow\; x\wedge a'\in[0,a]$$

and $x=y \;\Rightarrow x\wedge a'=y\wedge a' \;\Rightarrow f(x)=f(y)$

we find f is well defined

Again, $f(x)=f(y)$

$\Rightarrow$ $x\wedge a'=y\wedge a'$

$\Rightarrow$ $(x\wedge a')\vee a=(y\wedge a')\vee a$

$$\Rightarrow \quad (x \vee a) \wedge (a' \vee a) = (y \vee a) \wedge (a' \vee a)$$

$$\Rightarrow \quad x \wedge u = y \wedge u \quad \Rightarrow x = y$$

Thus f is 1-1.

Now $f(x \wedge y) = (x \wedge y) \wedge a' = (x \wedge a') \wedge (y \wedge a') = f(x) \wedge f(y)$

$f(x \vee y) = (x \vee y) \wedge a' = (x \wedge a') \vee (y \wedge a') = f(x) \vee f(y)$

Hence f is a homomorphism.

Finally, let $y \in [o, a']$ be any element.

Then $\quad 0 \leq y \leq a'$

$$\Rightarrow \quad a \vee 0 \leq a \vee y \leq a \vee a' \text{ or } a \leq a \vee y \leq u$$

$$\Rightarrow \quad a \vee y \in [a, u]$$

and as $f(a \vee y) = (a \vee y) \wedge a'$

$$= (a \wedge a') \vee (y \wedge a') = 0 \vee y = y$$

we find f is onto and hence an isomorphism using Theorem 5.2.8 we get,

$$L \cong [0, a] \times [a, u] \cong [0, a] \times [0, a'] \bullet$$

"Boolean Ring"

5.1 Introduction.

In this chapter, we introduced and study on Boolean ring , Disjunctive normal form, Conjunctive normal form and obtain their several features.

Recall that a Boolean function is said to be a Disjunctive normal form (DN form) in n variables $x_1, x_2, x_{3,} ----x_n$ if it can be written as join of terms of the type $f_1(x_1) \wedge f_2(x_2) \wedge f_3(x_3) \wedge ---- \wedge f_n(x_n)$. Where $f_i(x_i) = x_i$ or x_i' for all $i = 1, 2, 3, -----, n$ and no two terms are same. Also 1 and 0 are said to be in Disjunctive normal form.

5.2 Ring, Ring with zero devisor ,Boolean Ring

Definition (Ring) : A non-empty set R together with two binary operations addition (denoted by "+") and multiplication (denoted by ".") is called a ring if it is satisfied the following laws:

1. Associative law of addition:
   $\forall \quad a,b,c \in R \qquad \Rightarrow (a+b)+c = a+(b+c)$
2. Existence of additive identity zero:
   $\exists \quad 0 \in R \quad \Rightarrow a+0 = 0+a, \quad \forall\, a \in R$
3. Existence of additive inverse:
   $a \in R \Rightarrow \exists \quad -a \in R \Rightarrow a+(-a) = (-a)+a = 0, \quad \forall\, a \in R$
4. Commutative law of addition :
   $\forall \quad a,b \in R \Rightarrow a+b = b+a$
5. Associative law of Multiplication:
   $\forall \quad a,b,c \in R \Rightarrow (a.b).c = a.(b.c)$
6. Distributive laws:
   (i) Left : $\forall \quad a,b,c \in R \Rightarrow a.(b+c) = a.b + a.c$
   (ii) Right $\forall \quad a,b,c \in R \Rightarrow (a+b).c = a.c + b.c$

Definition (Ring with unity): A ring R is called a ring with unity if there exists an element $1 \neq 0 \in R$ such that $a\,.1 = 1.\,a = a$, $\forall\, a \in R$ where 1 is called the multiplicative identity or multiplicative unity.

Definition (Commutative Ring): A ring R is called **Commutative Ring** if under the binary operation of multiplication $a\,.b = b.\,a \;\; \forall \quad a,b \in R$.

Definition (Ring with zero divisor): A ring R is called with zero divisors if there exist at least two elements a and b of R such that $a\,b = 0$ where $a \neq 0$ and $b \neq 0$

Example 5.2.1: The rings Z Q R and C are integral domains.

Definition (Subring): Let R and S be two rings with respect to the two binary operations addition and multiplication. If S is a subset of R, then S is called a subring of R.

Theorem 5.2.1 : Let S be a subring of a ring $(R,+,\cdot)$. Then show that S is an additive subgroup of R.

Proof.: Let $a,b \in S$.

Since S is a subribg of R then $b \in S \Rightarrow -b \in S$.

Now $a-b \in S$

$\Rightarrow a+(-b) \in S$, by the closure property of addition

$\Rightarrow a-b \in S$

Thus S is a subgroup of R. •

Definition (Boolean Ring): A ring R is called Boolean Ring if

$a^2 = a \ \forall \ \ a \in R$.

Example 5.2.2 : Show that a ring R with $x^2 = x \ \forall \ \ x \in R$ must be commutative.

Solution : We have $x^2 = x \ \forall \ \ x \in R$

Now $(x+x)^2 = x+x$

$\Rightarrow (x+x)(x+x) = (x+x)$

$\Rightarrow (x+x)x + (x+x)x = x+x$ [by distributive law]

$\Rightarrow (x^2+x^2)+(x^2+x^2) = x+x$

$\Rightarrow (x+x)+(x+x) = x+x \qquad [\because x^2 = x]$

$\Rightarrow (x+x)+(x+x) = (x+x)+0$

$\Rightarrow x+x = 0$ [by left cancellation law for addition]

$\Rightarrow x+x = 0, \forall\, x \in R$.

Let $a,b \in R \Rightarrow a^2 = a,\ b^2 = b$ and $(a+b)^2 = a+b$.

Now $(a+b)^2 = a+b$

$\Rightarrow (a+b)(a+b) = a+b$

$\Rightarrow (a+b)a+(a+b)b = a+b$ [by distributive law]

$\Rightarrow (a^2+ba)+(ab+b^2) = a+b$

$\Rightarrow (a+ba)+(ab+b) = a+b$ [$\because a^2 = a,\ b^2 = b$]

$\Rightarrow (a+b)+(ba+ab) = (a+b)+0$

$\Rightarrow ba+ab = 0$ [by left cancellation law for addition]

$\Rightarrow ba+ab = ba+ba$ [$\because x+x=0$]

$\Rightarrow ab = ba$ [by left cancellation law for addition]

$\Rightarrow ab = ba, \forall\, a,b \in R$

$\Rightarrow$ R is commutative.

Example 5.2.3: If R is a Boolean ring. Then show that

(i) $a+a=0, \forall\, a \in R$

(ii) $a+b=0 \Rightarrow a=b$

Solution : (i)we already proof in example 8.

(iii) $a+b=0$

$\Rightarrow a+b = a+a$ [$\because a+a=0$]

$\Rightarrow b=a$ [by left cancellation law of addition in R]

Theorem 5.2.2: Every Boolean algebra is a Boolean ring with unity.

Proof.: A Boolean ring is a ring in which $x^2 = x\ \forall x$.

Let $(A, \wedge, \vee, ')$ be a Boolean algebra.

Define two operation (+) and $(\cdot)$ on A by

$$a \cdot b = a \wedge b$$

$$a+b = (a \wedge b') \vee (a' \wedge b) \qquad a,b \in A$$

Then (+) and $(\cdot)$ are clearly binary compositions on A.

To show that $< A, +, \cdot >$ forms a Boolean ring, we verify all the conditions in the definition.

Let $a, b, c \in A$ be any members.

$$a+b=(a\wedge b')\vee(a'\wedge b)=(b\wedge a')\vee(b'\wedge a)=b+a$$

$$(a+b)+c=[(a+b)\wedge c']\vee[(a+b)'\wedge c]$$

$$=[\{(a\wedge b')\vee(a'\wedge b)\}\wedge c']\vee[\{(a\wedge b')\vee(a'\wedge b)\}'\wedge c]$$

$$=[(a\wedge b'\wedge c')\vee(a'\wedge b\wedge c')]\vee[(a\wedge b')'\wedge(a'\wedge b)'\wedge c]$$

$$=[(a\wedge b'\wedge c')\vee(a'\wedge b\wedge c')]\vee[(a'\vee b)\wedge(a\vee b')\wedge c]$$

$$=[(a\wedge b'\wedge c')\vee(a'\wedge b\wedge c')]\vee[\{(a'\vee b)\wedge a\} \vee\{(a'\vee b)\wedge b'\}\wedge c]$$

$$=[(a\wedge b'\wedge c')\vee(a'\wedge b\wedge c')]\vee [\{(a'\wedge a)\vee(b\wedge a)\vee(a'\wedge b')\vee(b\wedge b')\}\wedge c]$$

$$=(a\wedge b'\wedge c')\vee(a'\wedge b\wedge c')\vee[\{(b\wedge a)\vee(a'\wedge b')\}\wedge c]$$

$$=(a\wedge b'\wedge c')\vee(a'\wedge b\wedge c')\vee[(b\wedge a\wedge c)\vee(a'\wedge b'\wedge c)]$$

$$=(a\wedge b'\wedge c')\vee(a'\wedge b\wedge c')\vee(a\wedge b\wedge c)\vee(a'\wedge b'\wedge c)$$

Since the resulting value is symmetric in a, b, c it will also be equal to $(b+c)+a=a+(b+c)$ (by commutativity of +).

Hence + is associative.

Again, $a+0=(a\wedge u)\vee(a'\wedge 0)=a=0+a$

Also, $a+a=(a\wedge a')\vee(a'\wedge a)=0$

Thus $(A,+)$ forms an abelian group.

Since $a\cdot b=a\wedge b$ and $\wedge$ is commutative and associative.

We find also $(\cdot)$ is also commutative and associative.

Again, $a(b+c)=a\wedge(b+c)=a\wedge[(b\wedge c')\vee(b'\wedge c)]$

$$=(a\wedge b\wedge c')\vee(a\wedge b'\wedge c)$$

$$ab+ac=(a\wedge b)+(a\wedge c)$$

$$=[(a\wedge b)\wedge(a\wedge c)']\vee[(a\wedge b)'\wedge(a\wedge c)]$$

$$=[(a\wedge b)\wedge(a'\vee c')]\vee[(a'\vee b')\wedge(a\wedge c)]$$

$$= (a \wedge b \wedge a') \vee (a \wedge b \wedge c')$$
$$\vee (a \wedge c \wedge a') \vee (a \wedge c \wedge b')$$
$$= (a \wedge b \wedge c') \vee (a \wedge b' \wedge c)$$

Hence $a(b+c) = ab + ac$

Similarly, $(b+c)a = ba + ca$

Finally, since $a \cdot u = a \wedge u = a = u \wedge a = u \cdot a.$

We find $(A, +, \cdot)$ forms a commutative ring with unity u

Also as $a \cdot a = a \wedge a = a \quad \forall a$

We gather that A forms a Boolean ring. •

Theorem 5.2.3 : Every Boolean ring with unity is a Boolean algebra.

Proof.: Let $< A, +, \cdot >$ be a Boolean ring with unity.

We define two operations $\wedge$ and $\vee$ on A by

$$a \wedge b = a \cdot b$$
$$a \vee b = a + b + ab$$

Then since $(\cdot)$ is commutative (a Boolean ring is commutative) and associative, we find $\wedge$ is commutative and associative.

Again, $a \vee a = a + a + aa = (a+a) + a = 0 + a$

(In Boolean ring $a + a = 0 \quad \forall a$, where 0 is zero of the ring)

Also $a \vee b = a + b + ab = b + a + ba = b \vee a$

$$(a \vee b) \vee c = (a \vee b) + c + (a \vee b) \cdot c = (a + b + ab) + c + (a + b + ab) \cdot c$$
$$= a + b + ab + c + ac + bc + abc$$

Since, $a \vee (b \vee c) = (b \vee c) \vee a$ (by commutativity of $\vee$)

By symmetry,

$$(b \vee c) \vee a = b + c + bc + a + ba + ca + abc$$

Hence $\vee$ is associative.

Finally to check absorption, we find

$$a \wedge (a \vee b) = a(a + b + ab) = a^2 + ab + a^2b = a + ab + ab$$
$$= a + 2ab$$
$$= a$$

(as $x + x = 0 \quad \forall x$)

$a \vee (a \wedge b) = a \vee ab == a + ab + aab = a + 2ab = a$

Thus A is a lattice.

We verify distributively for A. Let now $a \in A$ be any element. We show it has a complement, namely, $a + 1$ (where 1 is unity of ring A)

Now $a \wedge (a+1) = a(a+1) = a^2 + a = a + a = 0$

$a \vee (a+1) = a + a + 1 + a(a+1) = 2a + 1 + a + a = 1 + 2a = 1$

Showing that $a' = a + 1$

Notice, in the ring A $0 \cdot a = 0 \quad \forall a \in A$ (0 being zero of ring)

$\Rightarrow \quad 0 \wedge a = 0 \quad \forall a \in A$.

Again $1 \cdot a = a \quad \forall a$

ie. $1 \wedge a = a \quad \forall a \in A$.

Thus 0 and 1 are least and greatest elements of the lattice A. •

5.3 Disjunctive Normal form, Minterms, Boolean Expression.

Definition (Disjunctive normal form) : A Boolean function (Expression) is said to be in **Disjunctive normal form** (DN form) in n variables $x_1, x_2, x_3, \cdots\cdots x_n$ if it can be written as join of terms of the type $f_1(x_1) \wedge f_2(x_2) \wedge f_3(x_3) \wedge \cdots\cdots \wedge f_n(x_n)$ where $f_i(x_i) = x_i$ or x_i' for all i=1,2,3 . . . . . n and no two terms are same, Also 1 and 0 are said to be in disjunctive normal form.

Definition (Minterms or Minimal polynomials) : Again, in that case, terms of the type $f_1(x_1) \wedge f_2(x_2) \wedge f_3(x_3) \wedge \cdots\cdots \wedge f_n(x_n)$ are called minterms or minimal polynomials,(A normal form is also called a canonical form)

For instance, $(x \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z)$ is in disjunctive normal form (in 3 variables) and each of the brackets is a minterm.

Definition (Boolean expressions or Boolean polynomials) : Let $(A, \wedge, \vee, ')$ be a Boolean algebra. Expressions involving members of A and the operations $\wedge, \vee$ and complementation are called Boolean expressions or Boolean polynomials. For example, $x \vee y', x, x \wedge 0$ etc are all Boolean expressions. Any function specifying these Boolean expressions is called a Boolean function. Thus if $f(x, y) = x \wedge y$ then f is the Boolean function and $x \wedge y$ is the Boolean expressions (or value of the function f). Since it is normally the function value (and not the function) that we are interested in, we call these expressions the Boolean functions.

Theorem 5.3.1: Every Boolean function can be put in disjunctive normal form.

Proof. : We prove the result by taking the following steps.

(1) If primes occur outside brackets, then open brackets by using De Morgan's law

$(a \wedge b)' = a' \vee b', (a \vee b)' = a' \wedge b'$

(2) Open all brackets by using distributivity and simplify using any of the definition conditions like idempotency, absorption etc.

(3) If any of the terms does not contain a certain variable $x_i (\text{or } x_i')$ then take meet of that term with $x_i \vee x_i'$. Do this with each such term. (It will not affect the function as $x_i \vee x_i' = 1$ and $1 \wedge a = a$)

Now, open brackets and drop all terms of the type $a \wedge a' (= 0)$. Again , if any of the terms occur more than once, these can be omitted because of idempotency. The resulting expression will be in DN form.

Hence every function in a Boolean algebra is equal ton a function in DN form. •

Proposition 5.3.2: Put the function $f = [(x \wedge y')' \vee z'] \wedge (x' \vee z)'$ in the DN form.

Proof : We have,

$$f = [(x' \vee y'') \vee z'] \wedge (z' \wedge x'') = (x' \vee y \vee z') \wedge (z' \wedge x)$$

$$= (x' \wedge z' \wedge x) \vee (y \wedge z' \wedge x) \vee (z' \wedge z' \wedge x)$$

$$= 0 \vee (x \wedge y \wedge z') \vee (x \wedge z')$$

$$= (x \wedge y \wedge z') \vee [(x \wedge z') \wedge (y \vee y')] \qquad \text{(Note this step)}$$

$$= (x \wedge y \wedge z') \vee [(x \wedge z' \wedge y) \vee (x \wedge z' \wedge y')]$$

$$= (x \wedge y \wedge z') \vee (x \wedge z' \wedge y'). \quad \bullet$$

Proposition 5.3.3: Put the function

$f = [(x' \wedge y) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x' \wedge y' \wedge z' \wedge t) \vee t']'$ in the DN form .

Proof : We have,

$$f = [(x' \wedge y) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x' \wedge y' \wedge z' \wedge t) \vee t']'$$

$$= (x' \wedge y)' \wedge (x \wedge y \wedge z')' \wedge (x \wedge y' \wedge z)' \wedge (x' \wedge y' \wedge z' \wedge t)' \wedge t$$

$$= (x \vee y') \wedge (x' \vee y' \vee z) \wedge (x' \vee y \vee z') \wedge (x \vee y \vee z \vee t') \wedge t$$

$$= [(x \vee y') \wedge (x' \vee y' \vee z)] \wedge (x' \vee y \vee z') \wedge [(x \vee y \vee z \vee t') \wedge t]$$

$$= [(x \wedge x') \vee (x \wedge y') \vee (x \wedge z) \vee (y' \wedge x') \vee (y' \wedge y') \vee (y' \vee z)] \wedge (x' \vee y \vee z') \wedge [(x \wedge t) \vee (y \wedge t) \vee (z \wedge t) \vee (t \wedge t')]$$

$$= [(x \wedge y') \vee (x \wedge z) \vee (y' \wedge x') \vee y' \vee (y' \vee z)] \wedge [(x' \wedge y \wedge t) \vee (x' \wedge z \wedge t) \vee (y \wedge x \wedge t) \vee (y \wedge t) \vee (y \wedge z \wedge t) \vee (z' \wedge x \wedge t) \vee (z' \wedge y \wedge t)]$$

$$= (x \wedge y' \wedge z' \wedge t) \vee (x \wedge z \wedge y \wedge t) \vee (x' \wedge y' \wedge z \wedge t) \vee (y' \wedge z \wedge t \wedge x) \vee (y' \wedge z' \wedge x \wedge t) \vee (y' \wedge z \wedge x' \wedge t)$$

$$= (x \wedge y' \wedge z' \wedge t) \vee (x \wedge z \wedge y \wedge t) \vee (x' \wedge y' \wedge z \wedge t).$$

Note: Some times it is easy to use the notation (+) for $\vee$ and (.) for $\wedge$ while simplifying. Thus, for instance, the above solution would read

$$f = (x'y + xyz' + xy'z + x'y'z't + t')'$$

$$= (x'y)'(xyz')'(xy'z)'(x'y'z't)'t$$

$$= (x + y')(x' + y' + z)(x' + y + z')(x + y + z + t')t$$

$$= (xx' + xy' + xz + y'x' + y'y' + y'z)(x' + y + z')(xt + yt + zt + t't)$$

$$= (xy' + xz + y'x' + y' + y'z)(x'yt + x'zt + yxt + yt + yzt + z'xt + z'yt)$$

$$= xy'z't + xyzt + y'x'zt + y'ztx' + y'zx't$$

$$= xy'z't + xyzt + y'x'zt$$

We have shown above that every function can be expressed in DN form. •

Proposition 5.3.4: Write the function $x \vee y'$ in the disjunctive normal form in three variables x, y, z.

Proof. : We have

$$x \vee y' = [x \wedge (y \vee y') \wedge (z \vee z')] \vee [y' \wedge (x \vee x') \wedge (z \vee z')]$$

$$= [\{(x \wedge y) \vee (x \wedge y')\} \wedge (z \vee z')] \vee [\{(y' \wedge x) \vee (y' \wedge x')\} \wedge (z \vee z')]$$

$$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (y' \wedge x \wedge z) \vee (y' \wedge x \wedge z') \vee (y' \wedge x' \wedge z) \vee (y' \wedge x' \wedge z')$$

$$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (y' \wedge x' \wedge z) \vee (y' \wedge x' \wedge z')$$ •

Proposition 5.3.5: Find the Boolean expression for the function f given by

$$f(x,y,z) = \begin{cases} 1 & When \quad x = z = 1, y = 0 \\ & x = 1, y = z = 0 \\ 0 & Otherwise \end{cases}$$

Proof : The function is specified by the minterms $(x \wedge y' \wedge z)$ and $(x \wedge y' \wedge z')$

i.e. the function in the DN form is

$(x \wedge y' \wedge z) \vee (x \wedge y' \wedge z')$ •

Example 5.3.1 : Let A={0,1} and $f : A^2 \to A$, be defined by

$f(x,y) = (x \wedge y) \vee (x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y')$

i.e. f is complete DN form. We calculate all values of $f(x,y), \quad x, y \in A$.

$$f(0,0) = (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) = 1$$

$$f(1,0) = (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) = 1$$

$$f(0,1) = (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) = 1$$

$$f(1,1) = (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) = 1$$

(Note $x = 0 \Leftrightarrow x' = 1$)

We thus notice that in each case, one minterm is $1 \wedge 1 = 1$ and all others are zero. Also the resulting value of $f(x, y)$ is always 1.

If we go through similar process, with a function f which is in complete DN form in 3 variables x, y, z we'll get the same result. We can generalize this result.

Example 5.3.2 : Let A={0,1} and $f : A^3 \to A$, be the function defined by $f(x, y, z) = x \wedge (y \vee z)$, then the functional values of f are given by

$$f(0,0,0) = 0 \wedge (0 \vee 0) = 0 \qquad f(1,1,0) = 1 \wedge (\vee 0) = 1$$

$$f(1,0,0) = 1 \wedge (0 \vee 0) = 0 \qquad f(1,0,1) = 1 \wedge (0 \vee 1) = 1$$

$$f(0,1,0) = 0 \wedge (1 \vee 0) = 0 \qquad f(0,1,1) = 0 \wedge (1 \vee 1) = 0$$

$$f(0,0,1) = 0 \wedge (0 \vee 1) = 0 \qquad f(1,1,1) = 1 \wedge (1 \vee 1) = 1$$

which we sometimes write in the tabular form as

x	y	z	$f(x,y,z)$
0	0	0	0
1	0	0	0
0	1	0	0
0	0	1	0
1	1	0	1
1	0	1	1
0	1	1	0
1	1	1	1

Example 5.3.3 : Complete DN form in 2 variables is

$$(x \wedge y) \vee (x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y')$$

Let $f = (x \wedge y)$ [any one DN form]

$$f' = (x \wedge y)' = x' \vee y' = [x' \wedge (y \vee y')] \vee [y' \wedge (x \wedge x')]$$

$$= (x' \wedge y) \vee (x' \vee y') \vee (y' \wedge x) \vee (y' \wedge x')$$

$$= (x' \wedge y) \vee (x \vee y') \vee (y' \wedge x').$$

Thus what we gather from here is that if we pick up any DN form the complete DN form then complement of that DN form will contain the 'left out' term in the complete DN form.

Take for instance, $p = (x \wedge y) \vee (x' \wedge y)$

$$p' = [(x \wedge y) \vee (x' \wedge y)]' = (x \wedge y)' \wedge (x' \wedge y)'$$

$$= (x' \vee y') \wedge (x \vee y')$$

$$= (x' \wedge x) \vee y' = y' = y' \wedge (x \vee x')$$

$$= (y' \wedge x) \vee (y' \wedge x')$$

the 'left out' terms in the complete DN form.

Proposition 5.3.6: In a Boolean algebra, show that

$$f(x,y) = [x \wedge f(1,y)] \vee [x' \wedge f(0,y)]$$

Proof : We know that any function f (in 2 variables) in complete DN form

is $$f(x,y) = (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$$

$$= [x \wedge (y \vee y')] \vee [(x' \wedge (y \wedge y')] \qquad \ldots(1)$$

Put $x = 1, x' = 0$ and we get

$$f(1,y) = [1 \wedge (y \vee y')] \vee [(0 \wedge (y \wedge y')] = y \vee y'$$

Again by putting $x = 0, x' = 1$ we get

$$f(0,y) = [0 \wedge (y \vee y')] \vee [(1 \wedge (y \wedge y')] = y \vee y'$$

Thus (1) gives

$$f(x,y) = [x \wedge f(1,y)] \vee [x' \wedge f(0,y)] \bullet$$

Remarks 5.3.7: One can extended the above result to n variables and prove that,

$$f(x_1, x_2, \cdots, x_n) = \{x_1 \wedge f(1, x_2, x_3, \cdots, x_n)\} \vee \{x_1' \wedge f(0, x_2, x_3, \cdots, x_n)\}. \bullet$$

5.4 Conjunctive Normal form.

Definition (Conjunctive Normal form) : A Boolean function f is said to be conjunctive normal form (CN form) in n variable $x_1, x_2, \cdots, x_n$ if f is meet of terms of the type $f_1(x_1) \vee f_2(x_2) \vee \cdots \vee f_n(x_n)$ where $f_i(x_i) = x_i$ or x_i' for all $i = 1,2,\cdots,n$ and no two terms are same. Also 0 and 1 are said to be in CN form.

Proposition 5.4.1: Put the function $f = [(x \wedge y')' \vee z'] \wedge (x' \vee z)'$ in the CN form.

Proof : We have,

$$
\begin{aligned}
f &= [(x' \vee y) \vee z'] \wedge (x \wedge z') \\
&= (x' \vee y \vee z') \wedge [(x \wedge z') \vee (y \wedge y')] \\
&= (x' \vee y \vee z') \wedge \{[(x \wedge z') \vee y] \wedge [(x \wedge z') \vee y']\} \\
&= (x' \vee y \vee z') \wedge [(x \vee y) \wedge (z' \vee y) \wedge (x \wedge z') \vee y']\} \\
&= (x' \vee y \vee z') \wedge [\{x \vee y \vee (z \wedge z')\} \wedge \{(z' \vee y) \vee (x \wedge x')\} \\
&\quad \wedge \{(x \vee y') \vee (z \wedge z')\} \wedge \{(z' \vee y') \vee (x \wedge x')\}] \\
&= (x' \vee y \vee z') \wedge [\{x \vee y \vee (z \wedge z')\} \wedge \{(z' \vee y) \vee (x \wedge x')\} \\
&\quad \wedge \{(x \vee y') \vee (z \wedge z')\} \wedge \{(z' \vee y') \vee (x \wedge x')\}] \\
&= (x' \vee y \vee z') \wedge (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (z' \vee y \vee x) \\
&\quad \wedge (z' \vee y \vee x') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \\
&\quad \wedge (z' \vee y' \vee x) \wedge (z' \vee y' \vee x') \\
&= (x \vee y \vee z) \wedge (x' \vee y \vee z') \wedge (x \vee y \vee z') \\
&\quad \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z')
\end{aligned}
$$

Proposition 5.4.2: Put the function $x \wedge (y \vee z)$ in the CN form.

Proof : $x \wedge (y \vee z) = [x \vee (y \wedge y')] \wedge [(y \vee z) \vee (x \wedge x')]$

$$
\begin{aligned}
&= (x \vee y) \wedge (x \vee y') \wedge (y \vee z \vee x) \wedge (y \vee z \vee x') \\
&= (x \vee y) \vee (z \wedge z') \wedge (x \vee y') \\
&\quad \vee (z \wedge z') \wedge (x \vee y \vee z) \wedge (x' \vee y \vee z)
\end{aligned}
$$

$$= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z')$$
$$\wedge (x \vee y \vee z) \wedge (x' \vee y \vee z)$$
$$= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z)$$
$$\wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z) \quad \bullet$$

Proposition 5.4.3: Find the DN form of the function whose CN form is

$$f = (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z)$$

Proof : We know, $f = (f')'$. Thus,

$$f = [\{(x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z)$$
$$\wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z)\}']'$$
$$= [(x \vee y \vee z)' \vee (x \vee y \vee z')' \vee (x \vee y' \vee z)'$$
$$\vee (x \vee y' \vee z')' \vee (x' \vee y \vee z)']'$$

(by De Morgan's law)

$$= [(x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z')$$
$$\vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z')]'$$

(by De Morgan's law)

$$= (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \quad \text{(by previous problem)} \quad \bullet$$

Note: By similar steps we can find the CN form of a function from its DN form.

Proposition 5.4.4: Prove that in a Boolean Lattice ; $x \neq 0$ is join irreducible if and only if x is an atom.

Proof. : Suppose x is a join irreducible element. Consider the interval $[0, x]$. Let $a \in [0, x]$, we claim that either $a = 0$ or $a = x$.

Since L is Boolean then there exists $[0, x]$ such that

$a \wedge b = 0$ and $a \vee b = x$

But $a \vee b = x \Rightarrow$ either $a = x$ or $b = x$

If $a = x$ then nothing to prove.

If $b = x$ then $a \wedge b = 0 \Rightarrow a \wedge x = 0$

$$\Rightarrow a = 0 \ [\because a \le x]$$

Hence x is an atom.

Conversely, let x is an atom and $x = b \vee c$

$\therefore b \vee c > 0$ as $\therefore b \vee c \ge b \ge 0$

Then either $b = b \vee c = x$ or $b = 0$

Also, as above $\therefore b \vee c \ge c \ge 0 \ \Rightarrow$ either $c = b \vee c = x$ or $c = 0$.

As $b = 0$ and $c = 0 \Rightarrow x = 0$ which is impossible.

Hence $x = b$ or $x = c$.

ie x is join irreducible. •

Proposition 5.4.5 : Let L be a distributive lattice, $a, b, c \in L$, $a \le b$. Show that $[a, b]$ is Boolean if and only if $[a \vee c, b \vee c]$ are Boolean.

Proof. : Suppose $[a, b]$ is Boolean and let $t \in [a \vee c, b \vee c]$ then

$(t \vee a) \wedge b \in [a, b] \therefore$ there exists $z \in [a, b]$ such that,

$[(t \vee a) \wedge (b \vee z)] = a$ and $((t \vee a) \wedge b) \vee z = b$

$\Rightarrow ((t \wedge b) \vee (a \wedge b)) \wedge z = a$

$\Rightarrow (t \wedge b \wedge z) \vee (a \wedge b \wedge z) = a$

or $(t \wedge z) \vee (a \wedge z) = a$

or $(t \wedge z) \vee a = a \Rightarrow t \wedge z \le a$.

$\Rightarrow (t \wedge z) \wedge c \le a \wedge c \Rightarrow t \wedge (z \wedge c) \le a \wedge c$

Again, $a \wedge c \le t$ and $a \wedge c \le z \wedge c$

$\Rightarrow a \wedge c \le t \wedge (z \wedge c)$

Hence $t \wedge (z \wedge c) = a \wedge c$ (i)

Also, $(t \vee z) \wedge b = (t \wedge b) \vee (z \wedge b) = (t \wedge b) \vee b = b$

$\therefore b \le t \vee z$. But $t \le b \wedge c$ and $z \le b \Rightarrow t \vee z \le b$

Then $t \vee z = b$ (ii)

Again $a \wedge c \le t \le b \wedge c$

$\Rightarrow c \vee (a \wedge c) \leq c \vee t \leq c \vee (b \wedge c)$

$\therefore c \leq c \vee t \leq c \Rightarrow c \vee t = c$ (iii)

Now, $(z \wedge c) \vee t = (z \vee t) \wedge (c \vee t)$

$= b \wedge c$ [from (ii) and (iii)]

$\therefore t \vee (z \wedge c) = b \wedge c$ (iv)

Hence from (i) and (iv) we can conclude that $[a, b]$ is Boolean implies that $[a \vee c, b \vee c]$ is Boolean.

Again, let $t \in [a \vee c, b \vee c]$ then $(t \wedge b) \vee a \in [a, b]$ since $[a, b]$ is Boolean, there exist $z \in [a, b]$ such that

$(t \wedge b) \vee a \vee z = b$ (v)

and $((t \wedge b) \vee a) \wedge z = a$ (vi)

[$(t \wedge b) \vee a \geq a$, $t \wedge b \leq b$, $a \leq b \Rightarrow (t \wedge b) \vee a \leq b$.]

From (v), $(t \wedge b) \vee a \vee z = b$

or, $(t \wedge b) \vee (z \vee a) = b$

or, $(t \vee z \vee a) \wedge (b \vee z \vee a) = b$

or, $(t \vee z) \wedge b = b$

$\Rightarrow b \leq t \vee z$

Now, $t \wedge (z \vee c) = (t \wedge z) \vee (t \wedge c) \geq a \vee c$

or, $t \wedge (z \vee c) \geq a$ (vii)

But, $a \vee c \leq t$ and $a \leq z$ $\Rightarrow a \vee c \leq z \vee c$

$\Rightarrow a \vee c \leq t \wedge (z \vee c)$ (viii)

From (vii) and (viii), $t \wedge (z \vee c) = a \vee c$

Also, $t \vee (z \vee c) = (t \vee c) \vee z \leq (b \vee c) \vee z = (b \vee z) \vee c = b \vee c$

ie, $t \vee (z \vee c) \leq b \vee c$ (ix)

From $b \leq t \vee z$, we have $b \vee c \leq (t \vee z) \vee c$ (x)

From ((ix) and (x), $t \vee (z \vee c) = b \vee c$ (xi)

From (viii) and (xi), we conclude that $[a, b]$ Boolean implies that $[a \vee c, b \vee c]$ is Boolean. •

Theorem 5.4.6 : Let L be a distributive lattice with 0 and 1. Then L is Boolean if and only if $P(L)$, the set of all prime ideals of L is unordered.

Proof.: Let L is Boolean. If $P(L)$ is not unordered. Then there exists $P, Q \in P(L)$ such that $P \subset Q$. Choose $a \in Q - P$. Now, $a \wedge a' = 0$ $a \wedge a' \in P \subset Q$. Since P is prime and $a \notin P$. So $a' \in P \subset Q$. Thus $a, a' \in Q$ and so $1 = a \vee a' \in Q$ which is a contradiction.

Therefore $P(L)$ must be unordered.

Conversely, let $P(L)$ be unordered. Suppose L is not Boolean, then there exists $a \in L$ which has no complement.

Set $D = \{x \mid x \vee a = 1\}$, then D is a filter. Take $D_1 = D \vee [a)$. Filter D_1 does not contain D. For otherwise $0 = d \wedge a$ for some $d \in D$. Then $d \vee a = 1$ would imply that d is complement of a which is a contradiction. Thus $D \notin D_1$. Then by Stone representation theorem there exists a prime ideal P disjoint to D_1. Also note that $1 \notin (a] \vee P$ otherwise $1 = a \vee p$ for some $p \in P$. Contradicting $P \cap D = \Phi$. Then by Stone representation theorem there exists a prime ideal $Q = (a] \vee P$ and so $P \subset Q$ which is impossible, since $P(L)$ is unordered. Therefore L must be Boolean. •

5.5 Switching Circuits

One of the major applications of Boolean algebra is to the switching systems (an electrical network consisting of switches) that involve two state devices. The simplest example of such a device being an ordinary ON-OFF switch. By a switch we mean a contact or a device in an electric circuit which lets(or does not let) the current to flow through the circuit. The can assume two states 'closed' or 'open' (ON or OFF). In the first case the current flows and in the second the current does not flow. We will use $a, b, c, \cdots, x, y, z, \cdots$ etc. to denote switches in a current. There are two basic way in which switches are generally interconnected. These are referred to as 'in series' and ' in parallel'.

Two switches a, b are said to be connected 'in series' if the current can be pass only when both are in closed state and current doesn't flow if any one or both are open. We represent it as in the following diagram.

Fig. 5.1

Two switch a, b are said to be connected 'in parallel' if current flows when any one or both are closed does not pass when both are open. We represent this by the diagram

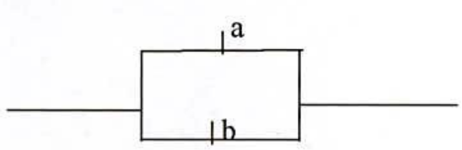


Fig.5.2

If two switches in a circuit be such that both are open(closed) simultaneously, we'll represent them by the same letter. Again if two switches be such that one is open iff the other is closed, we represent them by a and a'.

We show that the system discussed above forms a Boolean algebra.

Let 0 denote open circuit (current does not pass)

1 denote closed circuit (current passes).

Let 'in series' connection be represented by $\wedge$ (i.e. $a \wedge b$ denotes 'switches a and b are connected in series'). Also let $a \vee b$ denote 'switches a and b are connected in parallel'.

Consider the system $(B = \{0,1\}, \wedge, \vee)$.

Then B is a non empty set $\wedge$ and $\vee$ are two binary compositions (operations) on B as is evident the following tables

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

The conditions of idempotency, commutativity, associativity, and absorption are clearly seen to be satisfied.

e.g.. $1 \wedge (1 \vee 0) = 1 \wedge 1 = 1$

$1 \wedge (1 \wedge 0) = 1 \wedge 0 = (1 \wedge 1) \wedge 0$

In fact, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ also holds when a, b, c take values 0 or 1. Also since $0 \wedge 1 = 0,\ 0 \vee 1 = 1$ we find 0 and 1 are each others complements.

Hence B is a distributive lattice in which each element has a complement, i.e. it is a Boolean algebra.

the system $(\{0,1\}, \wedge, \vee, ')$ discussed above is usually called switching algebra which we have shown is a two valued Boolean algebra.

Proposition 5.5.1: Draw the circuit represented by the Boolean function

$$f = a \wedge (b \vee c)$$

Proof : The circuit is given by the diagram

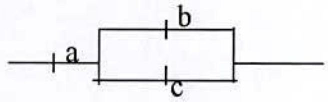


Fig. 5.3

Notice, the current would flow when a and b or a and c are closed i.e., when a and b or c is closed. •

Proposition 5.5.2 : Draw the circuit which realizes the function

$$a \wedge [(b \vee d') \vee (c' \wedge (a \vee d \vee c'))] \wedge b$$

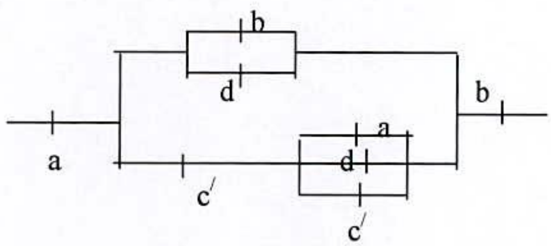


Fig. 5.4 •

Proposition 5.5.3 : Find the function that represents the circuit

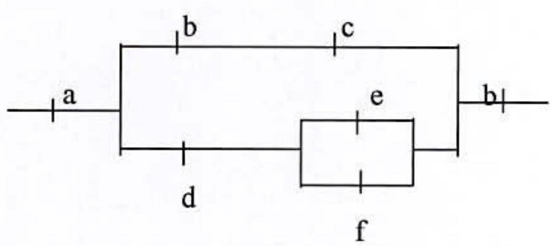


Fig. 5.5

Proof : The circuit given by the function

$$a \wedge [(b \wedge c) \vee (d \wedge (e \vee f))].$$

Let us consider the circuit given by the function

$$(a \wedge b) \vee (a \wedge c)$$

It is represented by

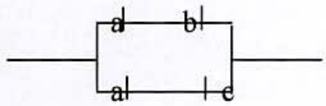


Fig.5.6

since $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$, the circuit could be simplified to

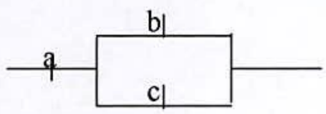


Fig. 5.7

Proposition 5.5.4: Simplify the circuit

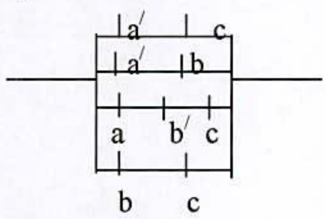


Fig.5.8

Proof: The circuit is represented by the function

$$(a' \wedge c) \vee (a' \wedge b) \vee (a \wedge b' \wedge c) \vee (b \wedge c)$$

which is equal to

$$(a' \wedge b) \vee (a' \vee (a \wedge b') \vee b) \wedge c$$

$$= (a' \wedge b) \vee [a' \vee (a \wedge b') \vee (a \vee a') \wedge b] \wedge c$$

$$= (a' \wedge b) \vee [a' \vee (a \wedge b') \vee (a \wedge b) \vee (a' \wedge b)] \wedge c$$

$$= (a' \wedge b) \vee [a' \vee \{a \wedge (b' \vee b)\} \vee (a' \wedge b)] \wedge c$$

$$= (a' \wedge b) \vee [a' \vee a \vee (a' \wedge b)] \wedge c$$

$= (a' \wedge b) \vee [1 \vee (a' \wedge b)] \wedge c$

$= (a' \wedge b) \vee c$

which is given by ,

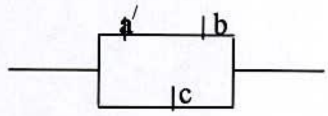


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Fig. 5.9

Proposition 5.5.5: Simplify the circuit represented by

$$f = (a \wedge c' \wedge d') \vee (a \wedge b' \wedge d) \vee (a \wedge c \wedge d')$$

Proof: We have,

$$f = (a \wedge c' \wedge d') \vee (a \wedge b' \wedge d) \vee (a \wedge c \wedge d')$$

$$= [(a \wedge d') \wedge (c' \vee c)] \vee (a \wedge b' \wedge d)$$

$$= [(a \wedge d') \wedge 1] \vee (a \wedge b' \wedge d)$$

$$= (a \wedge d') \vee (a \wedge b' \wedge d)$$

$$= a \wedge [d' \vee (b' \wedge d)]$$

$$= a \wedge [(d' \vee b') \wedge (d' \vee d)] \quad = a \wedge (b' \vee d')$$

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