A STUDY ON DISTRIBUTIVE NEARLATTICES

A Thesis

Presented for the degree of Doctor of Philosophy

by

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To my parents,
who have profoundly
influenced my life

TABLE OF CONTENTS.

SUMMARY

This thesis studies the nature of distributive nearlattices. By a nearlattice S we will always mean a (lower) semilattice which has the property that any two elements possessing a common upper bound, have a supremum, Cornish and Hickman in their paper [14], referred this property as the upper bound property, and a semilattice of this nature as a semilattice with the upper bound property. Cornish and Noor in [15] preferred to call these semilattices as nearlattices as the behaviour of such a semilattice is closer to that of a lattice than an ordiary semilattice. In this thesis we give several results on nearlattices which certainly extend and generalize many results in lattice theory.

In chapter 1 we discuss ideals, congruences and other results which are basic to this thesis. We include some characterizations of distributive and modular nearlattices. We generalize the separation properties given by M.H.Stone for distributive lattices. We also show that the set of prime ideals of a nearlattice S is unordered if and only if S is semiboolean.

Chapter 2 discusses the skeletal congruences of a distributive nearlattice. Skeletal congruences on distributive lattices have been studied extensively by Cornish in [11]. Here we extend several results of Cornish for nearlattices. We also introduce the notion of disjunctive nearlattices. A distributive nearlattice S with 0 is called disjunctive if for $0 \leq a < b$ there is an element $x \in S$ such that $x \wedge a = 0$ and $0 < x \leq b$. Then we give several characterizations of disjunctive nearlattices and semiboolean algebras using skeletal congruences. Finally we show that a distributive naerlattice is semiboolean if and only if $\theta \dashrightarrow \ker\theta$ is lattice isomorphism of Sc(S) onto KSc(S) whose inverse is the map $J \dashrightarrow \theta(J)$.

In chapter 3, we discuss on normal and n-normal nearlattices. Normal lattices have been studied by several authors including Cornish [8] and Monteiro [34]; while n-normal lattices have been studied by Cornish [9] and Davey [16]. In proving some of the results we have used Principle of Localization, which is an extension of lecture note of Dr. Noor on localization. This technique is very interesting and quite different from those of the previous authors.

Chapter 4 studies the multiplier extension (meet translation) of a distributive nearlattice. Previously multipliers on semilattices and lattices have been studied by several authors e.g, Szasz and Szendrie [54,55,56] Kolibiar [29], Cornish [10] and Niemenen [37] on a lattice. In a more recent paper, Noor and Cornish in [39] studied them on nearlattices. Here we extend some of their work. We also give a categorical result, where we see that the multiplier extension has a functorial character which is entirely different from that of the Lattice Theory, c.f. Cornish [10, theorem 2.4]. In section 2 of this chapter we discuss multipliers on sectionally pseudocomplemented distributive nearlattices which are sectionally in B_n^*, $-1 \leq n \leq \omega$ and generalize a number of results of [10]. We show that S is sectionally in B_n if and only if M(S), the lattice of multipliers is in B_n. Finally we show that for $1 \leq n < \omega$, above conditions are also equivalent to the condition that S is sectionally pseudocomplemented and for any n+1 minimal prime ideals

$$P_1, P_2, \ldots\ldots\ldots, P_{n+1},$$
$$P_1 \vee P_2 \vee \ldots\ldots\ldots \vee P_{n+1} = S.$$

STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

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Md.Bazlar Rahman

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CHAPTER — 1

IDEALS AND CONGRUENCES OF A DISTRIBUTIVE NEARLATTICE.

1. Preliminaries.

1.1. In this section it is intended only to outline and fix the notation for some of the concepts of nearlattices which are basic to this thesis. We also formulate some results on arbitrary nearlattices for latter use. For the background material in Lattice Theory we refer the reader to the texts of G. Gratzer [19], [18] and D.E. Rutherford [48].

By a nearlattice S we will always mean a (lower) semilattice which has the property that any two elements possessing a common upper bound, have a supremum. Cornish and Hickman, in their paper [14], referred this property as the upper bound property,and a semilattice of this nature as a semilattice with the upper bound property. We shall see latter, the behaviour of such a semilattice is closer to that of a lattice than an ordinary semilattice. For the sake of brevity, we prefer to use the term nearlattice in place of semilattice with the upper bound property.

Of course, a nearlattice with a largest element is a lattice. Since any semilattice satisfying the

descending chain condition has the upper bound property, all finite semilattices are nearlattices.

Now we give an example of a meet semilattice which is not a nearlattice .

Example. In $\mathbb{R}^2$ consider the set

$S = \{(0,0)\} \cup \{(1,0)\} \cup \{(0,1) \cup \{(1,y) : y > 1\}$

shown by the following figure 1.1

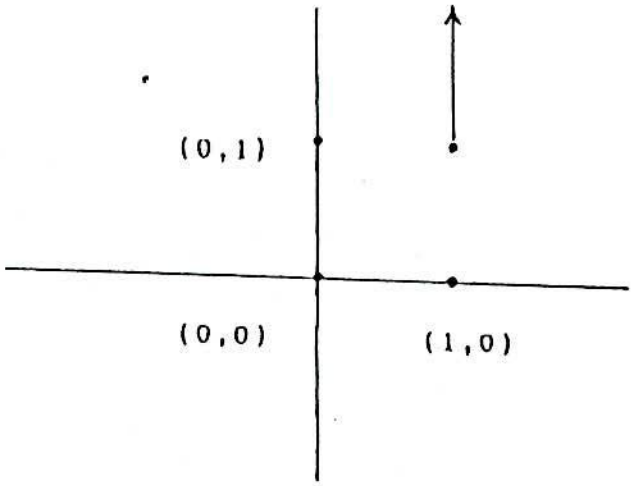


Figure 1.1

Define the partial ordering $\leq$ on S by $(x, y) \leq (x_1, y_1)$ iff $x \leq x_1$ and $y \leq y_1$. Observe that $(S ; \leq)$ is a meet

semilattice. Both (1,0) and (0,1) have common upper bounds. In fact $\{(1, y) \mid y > 1\}$ are common upper bounds of them. But the supremum of (1,0) and (0,1) does not exist. Therefore $(S ; \leq)$ is not a nearlattice.

The upper bound property, appears in Gratzer and Lasker [20],while Rozen [49,pp.17-20] shows that it is the result of placing certain associativity conditions on the partial join operation. Moreover, Evans in a more recent paper [17] referred nearlattices as conditional lattices. By a conditional lattice he means a (lower) semilattice S with the condition that for each $x \in S$, $\{ y \in S : y \leq x \}$ is a lattice; and it is very easy to check that this condition is equivalent to the upper bound property of S. Also, Nieminen refers to nearlattices as "partial lattices" in his paper [38].

Whenever a nearlattice has a least element we will denote it by 0. If $x_1, x_2, \ldots, x_n$ are elements of a nearlattice then by $x_1 \vee \ldots \ldots \vee x_n$, we mean that the supremum of $x_1, \ldots, x_n$ exists and $x_1 \vee \ldots \ldots \vee x_n$ is the symbol denoting this supremum.

A non empty subset K of a nearlattice S is called a subnearlattice of S if for any $a,b \in K$, both $a \wedge b$ and

a $\vee$ b (whenever it exists in S) belong to K ($\wedge$ and $\vee$ taken in S), and the $\wedge$ and $\vee$ of K are the restrictions of the $\wedge$ and $\vee$ of S to K. Moreover, a subnearlattice K of a nearlattice S is called a sublattice of S if a $\vee$ b $\in$ K for all a, b $\in$ K.

A nearlattice S is called modular if for any a, b, c $\in$ S with c $\leq$ a, a $\wedge$ (b $\vee$ c) = (a $\wedge$ b) $\vee$ c whenever b $\vee$ c exists. S is called distributive if for any x , x_1 ,....., x_n ,

x $\wedge$ (x_1 $\vee$....$\vee$ xn) = (x $\wedge$ x_1) $\vee$.....$\vee$ (x $\wedge$ x_n) whenever x_1 $\vee$...$\vee$ x_n exists.Notice that the right hand expression always exists by the upper bound property of S.

Lemma 1.2. A nearlattice S is distributive (modular) if and only if (x] = { y $\in$ S : y $\leq$ x } is a distributive (modular) lattice for each x $\in$ S. □

Consider the following lattices.

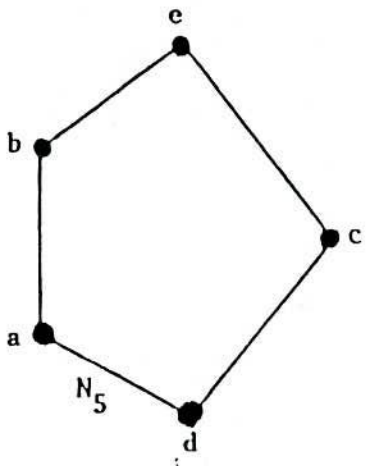


Figure 1.2

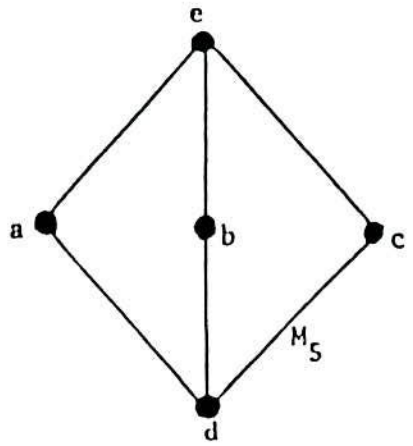


Figure 1.3

Hickman in [23] has given the following extension of a very fundamental result of Lattice Theory.

Theory 1.3. A nearlattice S is distributive if and only if S does not contain a sublattice isomorphic to N_5 or M_5. □

Now we give another extension of a fundamental result of Lattice Theory.

Theory 1.4. A nearlattice S is modular if and only if S does not contain a sublattice isomorphic to N_5.

Proof: Suppose S does not contain any sublattice isomorphic to N_5. Then, (x] does not contain any sublattice isomorphic to N_5 for each $x \in S$. Thus, a fundamental result of Lattice Theory says that (x] is modular for each $x \in S$ as (x] is a sublattice of S. Hence, S is modular by lemma 1.2.

Conversly, let S be modular. If S contains a sublattice isomorphic to N_5, then letting e as the largest element of the sublattice we see that (e] is not modular [by Lattice Theory]. Thus by lemma 1.2 above, S is not modular and this gives a contradiction.

This completes the proof. □

In this context it should be mentioned that many Lattice theorist e.g. R.Balbes [5], J.Verlet [58], R.C.Hickman [22] and K.P.Shum [53] have worked with a class of semilattices S which has the property that for each $x, a_1, \ldots, a_r \in S$, if $a_1 \vee \ldots \ldots \vee a_r$ exists then $(x \wedge a_1) \vee \ldots \ldots \vee (x \wedge a_r)$ exists and equals $x \wedge (a_1 \vee \ldots \ldots \vee a_r)$. [5] called them as prime semilattices while [53] referred them as weakly distributive semilattices.

Hickman in [23] has defined a ternary operation j by $j(x, y, z) = (x \wedge y) \vee (y \wedge z)$, on a nearlattice S (which exists by the upper bound property of S). In fact he has shown that (also see lyndon [30] Theorem 4) the resulting algebras of the type (S ; j) form a variety, which he referred to as the variety of join-algebras and following are its defining identities.

(i) $j(x, x, x) = x$

(ii) $j(x, y, x) = j(y, x, y)$.

(iii) $j(j(x, y, x), z, j(x, y, x)) = j(x, j(y, z, y), x)$

(iv) $j(x, y, z) = j(z, y, x)$.

(v) $j(j(x, y, z), j(x, y, x), j(x, y, z)) = j(x, y, x)$.

(vi) $j(j(x, y, x), y, z) = j(x, y, z)$.

(vii) $j(x, y, j(x, z, x)) = j(x, y, x)$.

(viii) $j(j(x, y, j(w, y, z)), j(x, y, z), j(x, y, j(x, y, z))) = j(x, y, z)$.

We do not want to elaborate it further as it is beyond the scope of this thesis.

We call a nearlattice S a medial nearlattice if for all $x, y, z \in S$, $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists. For a (lower) semilattice S, if $m(x, y, z)$ exists for all $x, y, z \in S$, then it is not hard to see that S has the upper bound property and hence is a medial nearlattice. Distributive medial nearlattices were first studied by Sholander in [51] and [52], and recently by Evans in [17]. Sholander preferred to call these as median semilattices. There he showed that every medial nearlattice S can be characterized by means of an algebra $(S ; m)$ of type $\langle 3 \rangle$, known as median algebra, satisfying the following two identities

(i) $m(a, a, b) = a$.

(ii) $m(m(a, b, c), m(a, b, d), e) = m(m(c, d, e), a, b)$.

Evans in [17] has studied nearlattices with the property that for any a , b , c $\in$ S, a $\vee$ b $\vee$ c exists whenever a $\vee$ b, b $\vee$ c and c $\vee$ a exists. He referred them as strong conditional lattices. It is not hard to see that these strong conditional lattices are precisely the medial nearlattices.

A family *A* of a subsets of a set A is called a closure system on A if

(i) A $\in$ *A* and

(ii) *A* is closed under arbitrary intersections.

Suppose *B* is a sub family of *A*. *B* is called a directed system if for any X, Y $\in$ *B* there exists Z in *B* such that X, Y $\sqsubset$ Z. If $\cup$ { X : X $\in$ *B* } $\in$ *A* for directed system *B* contained in the closure system *A*, then *A* is called algebraic. When ordered by set inclusion, an algebraic closure system forms an algebraic lattice.

A non empty subset H of a nearlattice S is called hereditary if, for any x $\in$ S and y $\in$ H, x $\leq$ y implies x $\in$ H. When S does not have a smallest element we also regard the empty set {Φ} as hereditary. Thus, the set H(S) of all hereditary subsets of S is a complete distributive lattice when partially ordered by

set-inclusion, where the meet and join in H(S) are given by set-theoretic intersection and union, respectively.

The largest element of H(S) is S, while the smallest element is {0}, if $0 \in S$, and the empty set, otherwise.

2. Ideals of Nearlattices.

A non empty subset I of a nearlattice S is called an ideal if it is hereditary and closed under existent finite suprema. We denote the set of all ideals of S by I(S). If S has a smallest element 0 then I(S) is an algebraic closure system on S, and is consequently an algebraic lattice. However, if S does not possess smallest element then we can only assert that $I(S) \cup \{\Phi\}$ is an algebraic closure system.

For any subset K of a nearlattice S, (K] denotes the ideal generated by K.

Infimum of two ideals of a nearlattice is their set theoretic intersection. Supremum of two ideals I and J in a lattice L is given by $I \vee J = \{x \in L : x \leq i \vee j$ for some $i \in I$, $j \in J\}$. Cornish and Hickman in [14] showed that in a distributive nearlattice S for two ideals I and J, $I \vee J = \{i \vee j : i \in I, j \in J$ where $i \vee j$ exists$\}$. But in a general nearlattice the formula for the supremum of two ideals is not very easy. We start this section with the following lemma which gives the formula for the supremum of two ideals. It is in fact exercise 22 of Gratzer [19, p-54] for partial lattice.

Lemma 2.1. Let I and J be ideals of a nearlattice S. Let $A_0 = I \cup J$, $A_n = \{ x \in S : x \leq y \vee z;\ y \vee z \text{ exists and } y, z \in A_{n-1} \}$, for $n = 1, 2, \ldots\ldots$, and $K = \bigcup_{n=0}^{\infty} A_n$. Then $K = I \vee J$.

Proof. Since $A_0 \sqsubset A_1 \sqsubset A_2 \sqsubset \ldots\ldots$ $\ldots \sqsubset A_n \sqsubset \ldots\ldots\ldots$, K is an ideal containing I and J. Suppose H is any ideal containing I and J. Of course, $A_0 \sqsubset H$. We proceed by induction. Suppose $A_{n-1} \sqsubset H$ for some $n \geq 1$ and let $x \in A_n$. Then $x \leq y \vee z$ with $y, z \in A_{n-1}$. Since $A_{n-1} \sqsubset H$ and H is an ideal, $y \vee z \in H$ and $x \in H$. That is $A_n \sqsubset H$ for every n. Thus, $K = I \vee J$. □

The following result is due to Cornish and Hickman in [14 ,Theorem 1.1].

Theorem 2.2. The following conditions on a nearlattice S are equivalent.

(i) S is distributive.

(ii) For any $H \in H(S)$, $(H] = \{ h_1 \vee \ldots \vee h_n : h_1, \ldots, h_n \in H \}$

(iii) For any $I, J \in I(S)$,
$I \vee J = \{ a_1 \vee \ldots\ldots \vee a_n : a_1, \ldots\ldots, a_n \in I \cup J \}$.

(iv) I(S) is a distributive lattice.

(v) The map H ---> (H] is a lattice homomorphism of H(S) onto I(S) (which preserves arbitrary supremum). □

Observe here that (iii) of above could easily be improved to (iii)′; for any I, J ∈ I(S),
I ∨ J = { i ∨ j : i ∈ I , j ∈ J}.

Let $I_f(S)$ from henceforth denote the set of all finitely generated ideals of a nearlattice S. Of course, $I_f(S)$ is an upper subsemilattice of I(S). Also for any $x_1,.....,x_n \in S$, $(x_1,.....,x_n]$ is clearly the supmremum $(x_1] \vee.....\vee (x_n]$. When S is distributive,

$(x_1,...,x_n] \cap (y_1,....,y_n]$

$= ((x_1] \vee....\vee (x_n]) \cap ((y_1] \vee...\vee (y_n])$

$= \bigvee_{i,j} (x_i \wedge y_i]$ for any $x_1,....,x_n , y_1,....,y_n \in S$

(by 1. 2.2) and so $I_f(S)$ is a distributive sublattice of I(S), c.f. Cornish and Hickman [14].

A nearlattice S is said to finitely smooth if the intersection of two finitely generated ideals is itself finitely generated. For example, (i) distributive nearlattices, (ii) finite nearlattices, (iii) lattices, are finitely smooth. Hickman in [23] exhibited a nearlattice which is not finitely smooth.

By Cornish and Hickman [14], we know that a nearlattice S is distributive if and only if I(S) is so. Our next result shows that the case is not the same with the modularity.

Theorem 2.3. Let S be a nearlattice. If I(S) is modular then S is also modular but the converse is not necessarily true.

Proof: Suppose I(S) is modular. Let a, b, c $\in$ S with c $\leq$ a and b $\vee$ c exists. Then (c] $\sqsubset$ (a].Since I(S) is modular. So (a $\wedge$ (b $\vee$ c)]=(a] $\wedge$ ((b] $\vee$ (c]=((a] $\wedge$ (b]) $\vee$ (c]=((a $\wedge$ b) $\vee$ c]. This implies a $\wedge$ (b $\vee$ c)=(a $\wedge$ b) $\vee$ c, and so S is modular.

Nearlattice S of figure 1.4 shows that the converse of this result is not true.

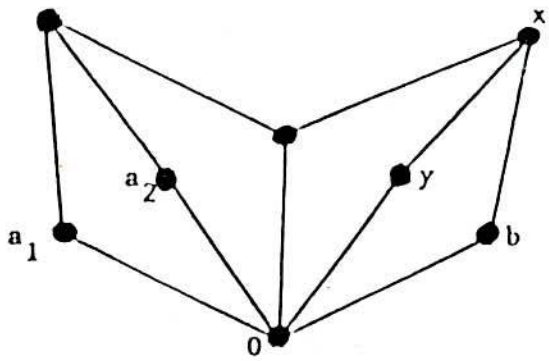


Figure 1.4 □

Notice that there (r] is modular for each $r \in S$. But in I(S), clearly {(0], (a], $(a_1, y]$, $(a_2, b]$, S} is a pentagonal sublattice.

We now give an extension of a well known result of Lattice Theory in presence of distributivity.

Theorem 2.4. Let I and J be two ideals in a distributive nearlattice S. If $I \wedge J$ and $I \vee J$ are principal, then both I and J are principal.

Proof: Suppose $I \wedge J = (x]$ and $I \vee J = (y]$.Then by [14],Theorem1.1] $y = i \vee j$ for some $i \in I$ and $j \in J$.Since $x \leq y$ and $i \leq y$, $x \vee i$ exists by the upper bound property of S. Moreover $x \vee j \in J$. Now $(y] = I \vee J \supseteq (x \vee i] \vee J \supseteq (i] \vee J \supseteq (y]$. This implies $I \vee J = (x \vee i] \vee J$. Again, $(x] = I \wedge J \supseteq (x \vee i] \wedge J \supseteq (x]$ implies $I \wedge J = (x \vee i] \wedge J$. Then from the distributivity of I(S) two equalities imply that $I = (x \vee i]$. That is, I is principal. Similarly, we can show that that J is also principal. □

A filter F in a nearlattice S is a non-empty subset of S such that if $f_1, f_2 \in F$ and $x \in S$ with $f_1 \leq x$,then both $f_1 \wedge f_2$ and x are in F. A filter G is called a prime

filter if $G \neq S$ and atleast one of $x_1, \ldots\ldots, x_n$ is in G whenever $x_1 \vee \ldots\ldots \vee x_n$ exists and is in G. An ideal P in a nearlattice S is called a prime ideal if $P \neq S$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$. It is not hard to see that a filter F of a nearlattice S is prime if and only if S-F is a prime ideal.

The set of filters of a nearlattice is an upper semilattice; yet it is not a lattice in general, as there is no guarantee that the intersection of two filters is non-empty. The join $F_1 \vee F_2$ of two filters is given by $F_1 \vee F_2 = \{ s \in S : s \geq f_1 \wedge f_2$ for some $f_1 \in F_1$ and $f_2 \in F_2 \}$. The smallest filter containing a subsemilattice H of S is $\{ s \in S : s \geq h$ for some $h \in H \}$ and is denoted by [H). Moreover, the description of the join of filters shows that for all $a, b \in S$, $[a) \vee [b) = [a \wedge b)$.

Now we will give an extension of a well known Theorem of Lattice Theory due to M.H.Stone; c.f [41].

Theorem 2.5. Let S be a nearlattice. The following conditions are equivalent:

(i) S is distributive.

(ii) For any ideal I and any filter F of S, such that $I \cap F = \Phi$, there exists a prime ideal $P \supseteq I$ and

disjoint from F.

Proof: (i) implies (ii). Let T be the set of all ideals containing I, but disjoint from F. T is non-empty, since $I \in T$.

Let C be a chain in T and let $M = \cup \{ X : X \in C \}$. Let $x, y \in M$, then $x \in X$, $y \in Y$ for some $x , y \in C$. Since C is a chain either $X \sqsubset Y$ or $Y \sqsubset X$.

Suppose $X \sqsubset Y$. Then both $x , y \in Y$. So if $x \vee y$ exists, then $x \vee y \in Y \sqsubset M$, as Y is an ideal. Now for $p \leq x$, $p \in X$ as X is an ideal and $p \in M$. Thus M is an ideal. Moreover M contains I and $F \cap M = \Phi$. So M is maximum element of C.

Hence by Zorn's Lemma T has a maximum element P. We claim that P is prime. If not, there exist $a , b \notin P$, but $a \wedge b \in P$. Because of maximality of P, $(P \vee (a]) \cap F \neq \Phi$. $(P \vee (b]) \cap F \neq \Phi$. Then by [14, theorem 1.1], there exist elements $p \vee a_1 \in F$ and $q \vee b_1 \in F$ for some $a_1 \leq a$ and $b_1 \leq b$. Then by $x = (p \vee a_1) \wedge (q \vee b_1) \in F$ and $p , q \in P$. Also $x = (p \wedge q) \vee (p \wedge b_1) \vee (a_1 \wedge q) \vee (a_1 \wedge b_1)$ implies $F \cap P \neq \Phi$, which is a contradiction. Hence P is a prime ideal.

(ii) implies (i). Let $x , y , z \in S$, such that $y \vee z$ exists. Then $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$. If $(x \wedge y) \vee (x \wedge z) < x \wedge (y \vee z)$.
Consider $I = ((x \wedge y) \vee (x \wedge z)]$ and $F = [x \wedge (y \vee z))$. Then $I \cap F = \Phi$, so by (ii) there exists a prime ideal $P \supseteq I$ such that $P \cap F = \Phi$.

Now $(x \wedge y) \vee (x \wedge z) \in P$ implies $x \wedge y \in P$ and $x \wedge z \in P$. Since P is prime, this implies either $x \in P$ or $y \vee z \in P$ and so $x \wedge (y \vee z) \in P$, which is a contradiction to $P \cap F = \Phi$. Therefore $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$ and so S is distributive. □

The following corollaries follow immediately from above theorem.

Corollary 2.6. A nearlattice S is distributive if and only if for any ideal I and $a \in S$ such that $a \notin I$, there exists a prime ideal $P \supseteq 1$ and $a \notin P$. □

Corollary 2.7. A nearlattice S is distributive if and only if for $a, b \in S$ with $a \neq b$ there exists a prime ideal P containing exactly one of a and b. □

Corollary 2.8. A nearlattice S distributive if and only if every ideal is the intersection of all prime ideals containing it. □

3. Congruences.

An equivalence relation Θ of a nearlattice S is a congruence relation of the algebra $(S; \wedge)$ such that if $x_i \equiv y_i$ (Θ), for i=1,2, and both $x_1 \vee x_2$ and $y_1 \vee y_2$ exist, then $x_1 \vee x_2 \equiv y_1 \vee y_2$ (Θ).

The set c(S) of all congruences on S is an algebraic closure system on S x S and hence, when ordered by set inclusion, is an algebraic lattice.

Cornish and Hickman [14] showed that for an ideal I of a distributive nearlattice S, the relation $\Theta(I)$, defined by $x \equiv y$ $(\Theta(I))$ if and only if $(x] \vee I = (y] \vee I$ is the smallest conngruece having I as a congruence class. Moreover, the equivalence relation R(I), defined by $x \equiv y$ $(R(I))$ if and only if, for any $s \in S$, $x \wedge s \in I$ is equivalent to $y \wedge s \in I$, is the largest congruence having I as a congruence class.

Suppose S is distributive nearlattice and $x \in S$. We will use Θ_x as an abbreviation for $\Theta((x])$. Moreover, Ψ_I denotes the congruence, defined by $a \equiv b$ (Ψ_I) if and only if $a \wedge x = b \wedge x$.

Cornish and Hickman [14] also showed that for any two elements a,b of a distributive nearlattice S with a $\leq$ b, the smallest congruence identifying a and b is equal to $\Psi_a \cap \Theta_b$, and we denote it by $\Theta(a, b)$. Also, in a distributive nearlattice S, they observed that if S has a smallest element 0, then clearly $\Theta_x = \Theta(0, x)$ for any $x \in S$.

(i) $\Theta_a \vee \Psi_a = \iota$, the largest congruence of S

(ii) $\Theta_a \cap \Psi_a = \omega$, the smallest congruence of S and

(iii) $\Theta(a, b)' = \theta_a \vee \Psi_b$ where $a \leq b$ and $'$ denotes the complement.

Now suppose S is an arbitrary nearlattice and E(S) denotes its lattice of equivalence relations. For Φ_1, $\Phi_2 \in E(S)$, $\Phi_1 \vee \Phi_2$ denotes their supremum; $x \equiv y\ (\Phi_1 \vee \Phi_2)$ if and only if there exists $x = z_0, z_1, \ldots, z_n = y$ such that $z_{i-1} \equiv z_i\ (\Phi_1 \text{ or } \Phi_2)$ for $i = 1,2,..,n$.

The following result was stated by Grazter and Lakser in [20] without proof and a proof, different than given below, appears in Cornish and Hickman [14] ; but also see Hickman [22] and [23].

Theorem. 3.1. For any nearlattice S, c(S) is a distributive (complete) sublattice of E(S).

Proof: Suppose θ, $\Phi \in c(S)$. Define Ψ to be the supremum of θ and Φ in the lattice of equivalence relations E(S) on S. Let $x \equiv y\ (\Psi)$. Then there exists $x = z_0, z_1, .., z_n = y$ such that $z_{i-1} \equiv z_i (\theta \text{ or } \Phi)$. Thus, for any $t \in S$, $z_{i-1} \wedge t \equiv z_i \wedge t\ (\theta \text{ or } \Phi)$ as $\theta, \Phi \in c(S)$.

Hence, $x \wedge t \equiv y \wedge t (\Psi)$ and consequently Ψ is a semilattice congruence. Then, in particular $x \wedge y \equiv x\ (\Psi)$ and $x \wedge y \equiv y (\Psi)$. To show that Ψ is a congruence, let $x \equiv y(\Psi)$, with $x \leq y$, and choose any $t \in S$ such that both $x \vee t$ and $y \vee t$ exist. Then, there exists $z_0, \ldots, z_n$ such that $x = z_0$, $z_n = y$ and $z_{i-1} \equiv z_i\ (\theta \text{ or } \Phi)$.
Put $w_i = z_i \wedge y$ for all $i = 0, \ldots, n$. Then
$x = w_0$, $w_n = y$, $w_{i-1} \equiv w_i\ (\theta \text{ or } \Phi)$. Hence, by the upper bound property, $w_i \vee t$ exists for all $i = 0, \ldots, n$ (as $w_i, t \leq y \vee t$) and $w_{i-1} \vee t = w_i \vee t\ (\theta \text{ or } \Phi)$ for all $i = 1, \ldots, n$ (as $\theta, \Phi \in c(S)$), i.e., $x \vee t \equiv y \vee t (\Psi)$. Then by [15; lemma 2.3] Ψ is a congruence on S. Therefore, c(S) is a sublattice of the lattice E(S).

To show the distributivity of c(S), let
$x \equiv y\ (\theta \cap (\theta_1 \vee \theta_2))$. Then $x \wedge y \equiv y\ (\theta)$ and $(\theta_1 \vee \theta_2)$. Also, $x \wedge y \equiv x\ (\theta)$ and $(\theta_1 \vee \theta_2)$.

Since $x \wedge y \equiv y\ (\theta_1 \vee \theta_2)$, there exist $t_0, \ldots, t_n$ such that (as we have seen in the proof of the first part), $x \wedge y = t_0$, $t_n = y$, $t_{i-1} \equiv t_i\ (\theta_1 \text{ or } \theta_2)$ and $x \wedge y = t_0 \leq t_i \leq y$ for each $i=0,\ldots,n$. Hence, $t_{i-1} \equiv t_i\ (\theta)$ for all $i=1,\ldots,n$, and so $t_{i-1} \equiv t_i\ (\theta \cap \theta_1)$ or $(\theta \cap \theta_2)$. Thus, $x \wedge y \equiv y((\theta \cap \theta_1) \vee (\theta \cap \theta_2))$. By symmetry, $x \wedge y \equiv x((\theta \cap \theta_1) \vee (\theta \cap \theta_2))$ and the proof completes by transitivity of the congruences. □

In lattice theory it is well known that a lattice is distributive if and only if every ideal is a class of some congruence. Following theorem gives a generalization of this result in case of nearlattices.

This also characterizes the distributivity of a nearlattice, which is an extension of [14, Theorem 3.1].

Theorem 3.2. S is distributive if and only if every ideal is a class of some congruence.

Proof: Suppose S is distributive. Then by [14,Theorem 3.1] for each ideal I of S, $\theta(I)$ is the smallest congruence containing I as a class.

To prove the converse, let each ideal of S be a congruence class with respect to some congruence on S. Suppose S is not distributive. Then by Th.1.1.3 we have either N_5 (figure 1.2) or M_5 [figure 1.3] as a sublattice of S. In both cases consider I = (a] and suppose I is a congruence class with respect to θ. Since $d \in I$. $d \equiv a\ (\theta)$. Now $b = b \wedge c = b \wedge (a \vee c) \equiv b \wedge (d \vee c) = b \wedge c = d\ (\theta)$, i,e. $b \equiv d\ (\theta)$ and this implies $b \in I$, i.e., $b \leq a$ which is a contradiction. Thus S is distributive. □

To prove (ii) of the next theorem, the following lemma is needed. This lemma is also an extension of [14 Theorem 1.1].

Lemma 3.3. If $\{J_i\}$; $i \in A$ an indexed set, are ideals of a distributive nearlattice S, then $\vee Ji = \{j_{i_1} \vee, \ldots . \vee j_{i_n}$ where the supremum exists for some $i_1, \ldots . , i_n \in A$ and $j_{i_k} \in J_{i_k}$ }.

Proof: Let $x , y \in$ R.H.S. if $x \vee y$ exists then obviously it is also of the same form. That is $x \vee y \in$ R.H.S. Now let $x \in$ R.H.S and $y \leq x$. Then $x = j_{i_1} \vee \ldots . \vee j_{i_n}$ for some $j_{i_k} \in J_{i_k}$, $k = 1,2, \ldots . ,n$. So by the distributivity, $y = y \wedge x = (y \wedge j_{i_1}) \vee \ldots . . . . \vee (y \wedge j_{i_n})$.

Since $y \wedge j_{i_K} \in J_{i_K}$, this implies $y \in$ R.H.S. Thus R.H.S is an ideal of S. This clearly contains each J_i. Finally, let H be an ideal containing each J_i. Then for each $i_1,\ldots,i_n \in A$ and $j_{i_k} \in J_{i_k}$, $j_{i_1} \vee \ldots\ldots \vee j_{i_n} \in H$ if it exists and so R.H.S $\sqsubset$ H.

Therefore R.H.S. $= \vee J_i$. $\square$

We omit the proof of (i) of the following theorem as it is due to Cornish and Hickman in [14, theorem 3.6], while (ii) is an extension of a part of their result.

Theorem 3.4. Let S be a distributive nearlattice then,

(i) for ideals I and J, $\theta (I \cap J) = \theta (I) \cap \theta(J)$.

(ii) for ideals j_i, $i \in A$ an indexed set,

$$\theta (\vee J_i) = \vee \theta (J_i).$$

Proof: (ii) since for each $i \in A$, $J_i \sqsubset \vee J_i$, so $\theta (J_i) \sqsubset \theta (\vee J_i)$. Hence $\vee \theta (J_i) \sqsubset \theta (\vee J_i)$. To prove the reverse inequality, let $x \leq y$ and $x \equiv y \ \theta \ (\vee J_i)$. Then $(x] \vee (\vee J_i) = (y] \vee (\vee J_i)$, and so $y \in (x] \vee (\vee J_i)$. Then by the above lemma, $y = x \vee j_{i_1} \vee, \ldots, \vee j_{i_n}$ for some $i_1,..,i_n \in A$. Then $x \equiv x \vee j_{i_1} \theta (J_{i_1})$

$\equiv x \vee j_{i_1} \vee j_{i_2} \theta (J_{i_2})$

$\equiv \ldots\ldots \equiv x \vee j_{i_1} \vee \ldots\ldots \vee j_{i_n} = y \ \theta \ (J_{i_n})$.

Thus $x \equiv y \bigvee_{k=1}^{n} \theta(J_i)_k \sqsubset \bigvee \theta(J_i)$. This proves (ii). □

Following corollary is an immediate consequence of above theorem which is also a part of [14, Th.3.6].

Corollary 3.5. The mapping $I \longrightarrow \theta(I)$ is a homorphism from the lattice of ideals to the lattice of congruences. □

We now turn our attention to the permutability of the congruences in a distributive nearlattices. Two congruences θ and Φ in a nearlattice S is called permutable if for any $x, y, z \in S$ with $x \equiv y (\theta)$ and $y \equiv z(\Phi)$, there exists $t \in S$ such that $x \equiv t(\Phi)$ and $t \equiv z(\theta)$. It is well known that in distributive lattices the congruences of the form $\theta(I)$ and $\theta(J)$ always permutes for any ideals I and J. Unfortunately we are unable to establish such a result in distributive nearlattices. But the existence of medians plays a fundamental role in establishing such a result which is given in next theorem. Recall that a nearlattice S is medial if $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists for all $x, y, z \in S$. It is already mentioned in section 1. This is equivalent to the fact that for all $x , y , z \in S$,

$x \vee y \vee z$ exists whenever $x \vee y$, $y \vee z$ and $z \vee x$ exist.

Theorem 3.6. For any ideals I and J of a distributive medial nearlattice S, $\theta(I)$ and $\theta(J)$ permute if $i \vee j$ exists for all $i \in I, j \in J$.

Proof: Suppose $x \equiv y\ \theta(I)$ and $y \equiv z\ \theta(J)$. Then $(x] \vee I = (y] \vee I$ and $(y] \vee J = (z] \vee J$, and so $x = (x \wedge y) \vee (x \wedge i)$ and $z = (y \wedge z) \vee (z \wedge j)$ for some $i \in I$, $j \in J$. Consider $p = (x \wedge y \wedge z) \vee (x \wedge i) \vee (z \wedge j)$. This element exists as $i \vee j$ exists and S is medial. Now,

$z \wedge j \equiv x \wedge y \wedge j\ \theta(J)$ and $y \equiv z\ \theta(J)$.

Imply $p \equiv (x \wedge y) \vee (x \wedge i) \vee (x \wedge y \wedge j)\ \theta(J)$

$= (x \wedge y) \vee (x \wedge i) = x$. Again

$x \wedge i \equiv y \wedge z \wedge i\ \theta(I)$ and $x \equiv y\ \theta(I)$ imply that

$p \equiv (y \wedge z) \vee (y \wedge z \wedge i) \vee (z \wedge j)\ \theta(I)$

$= (y \wedge z) \vee (z \wedge j) = z$.

Therefore, $\theta(I)$ and $\theta(J)$ permute. □

Thus we have the following corollary.

Corollary 3.7. Let S be a distributive medial nearlattice. Then for $a, b \in S$, θ_a and θ_b are permutable if and only if $a \vee b$ exists. □

We conclude this section with the following corollary which is an immediate consequence of above corollary .

Corollary 3.8. The following conditions on a distributive medial nearlattice S are equivalent.

(i) S is a distributive lattice.

(ii) For any two ideals J and K, $\theta(J)$ and $\theta(K)$ are permutable.

(iii) For any $s,t \in S$, θ_s and θ_t are permutable. □

4. Semiboolean algebras.

An interesting class of distributive nearlattices is provided by those semilattices in which each principal ideal is a boolean algebra. These semilattices have been studied by Abbott [1], [2], [3] under the name of semiboolean algebras and mainly from the view of Abbott's implication algebras (an implication algebra is a groupoid $(I;\cdot)$ satisfying:

(i) $(ab)\,a = a$,

(ii) $(ab)\,b = (ba)\,a$,

(iii) $a\,(bc) = b\,(ac)$.

Abbott shows in [1, pp. 227-236] that each implication algebra determines a semiboolean algebra and conversely each semiboolean algebra determines an implication algebra.

Following result gives a characterization of semiboolean algebras which is due to Cornish and Hickman in their paper of weakly distributive semilattices [14] (such semilattices were first studied by Balbes [5] under the name of prime semilattices).

Theorem 4.1. [Cornish and Hickman [14, theorem 2.2]. A semilattice S is a semiboolean algebra if and only if the following conditions are satisfied.

(i) S has the upper bound property.

(ii) S is distributive.

(iii) S has a 0 and for any $x \in S$,

$(x]^* = \{ y \in S : y \wedge x = 0 \}$ is an ideal

and $(x] \vee (x]^* = S$. □

A nearlattice S is relatively complemented if each interval [x , y] in S is complemented. That is, for $x \leq t \leq y$ there exists t' in [x,y] such that $t \wedge t' = x$ and $t \vee t' = y$.

A nearlattice S is called sectionally complemented if [0 , x] is complemented for each $x \in S$. Of course every relatively complemented nearlattice S with 0 is sectionally complemented. It is not hard to see that S is semiboolean if and only if it is sectionally complemented and distributive. We denote P(S) by the set of all prime ideals of S.

There is a well known result in Lattice Theory due to Nachbin in 1937. c.f. [19, Theorem. 22] that a distributive lattice is boolean if and only if its prime ideals are unordered. Following theorem is a generalization to this result which is due to Cornish and Hickman in [14].

Theorem 4.2. For a distributive nearlattice S with 0, the following conditions are equivalent.

(i) S is semiboolean.

(ii) $J_f(S)$ is a generalized boolean algebra.

(iii) P(S), the set of all prime ideals is unordered by set inclusion. □

Now we extend the above result. For this we need a lemma which depends on theorem 1.2.5, the separation properties of nearlattices. This lemma was proved by Cornish in [8] for lattices. But in case of nearlattices the proof is bit tricky.In Cornish's proof, he has used the ideal that if T is a sublattice of a distributive lattice, then the ideal generated by T is exactly same as the hereditary subset generated by T. But this is not true in case of nearlattices.

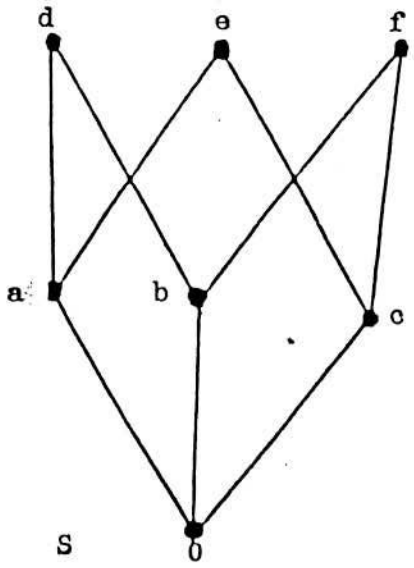


Figure 1.5

In Figure 1.5, observe that for the subnearlattice T = { d,b,f } of distributive nearlatiice S, hereditary subset generated by T is { 0, a,b,c,d,f } but (T] = S.

Lemma 4.3. If S_1 is a subnearlattice of a distributive nearlattice S and P_1 is a prime ideal in S_1, then there exists a prime ideal P in S such that $P_1 = S_1 \cap P$.

Proof. Let I be the ideal generated by P_1 in S. Then $I = (H]$ where H is the hereditary subset of S generated by P_1. Suppose $x \in I \cap (S_1 - P_1)$. Then $x \in I$ and $x \in S_1 - P_1$. Then by Th.1.2.2, $x = h_1 \vee \ldots \vee h_n$ for some $h_1, \ldots, h_n \in H$. Again, $h_i \in H$ implies $h_i \leq t_i$ for some $t_i \in P_1$, $i=1,2,\ldots,n$. Then $x = (x \wedge h_1) \vee \ldots \vee (x \wedge h_n) \leq (x \wedge t_1) \vee \ldots \vee (x \wedge t_n)$

(this exists by the upper bound property) $\leq x$. Thus, $x = (x \wedge t_1) \vee \ldots\ldots \vee (x \wedge t_n) \in P_1$ which gives a contradiction. Therefore, $I \cap (S_1 - P_1) = \Phi$. Then as $S_1 - P_1$ is a filter in S_1, $I \cap [S_1 - P_1) = \Phi$ where $[S_1 - P_1)$ is the filter generated by $S_1 - P_1$ in S. Then by Theorem 1.2.5, there is a prime ideal P in S such that $I \sqsubset P$ and $(S_1 - P_1) \cap P = \Phi$. Then $P_1 \sqsubset I \cap S_1 \sqsubset P \cap S_1$ and $P \cap S_1 \sqsubset P_1$. Hence $P_1 = P \cap S_1$. □

Theorem 4.4. Let S be a distributive nearlattice, S is relatively complemented if and only if P(S) is unordered.

Proof: Let S be relatively complemented and $P, Q \in P(S)$ with $P \subset Q$. Then there exists $q \in Q$ such that $q \notin P$. Also there exists $r \in S$ such that $r \notin Q$, as Q is prime. Consider the interval $[p \wedge q \wedge r, r]$ for some $p \in P$. Then $p \wedge q \wedge r \leq q \wedge r \leq r$. Since S is relatively complemented, there exists $t \in [p \wedge q \wedge r, r]$ such that $q \wedge r \wedge t = p \wedge q \wedge r \in P$ and $t \vee (q \wedge r) = r$. As P is prime and $q \wedge r \notin P$, so $t \in P$. This implies $t \vee (q \wedge r) = r \in Q$, which is a contradiction. Hence P(S) must be unordered.

Conversely, Suppose P(S) is unordered. Consider [a,b]. Let P_1, Q_1 be two prime ideals of [a,b]. Then by above lemma there exist prime ideals P and Q of S such

that $P_1 = P \cap [a,b]$ and $Q_1 = Q \cap [a,b]$. Since P_1 and Q_1 are prime, $b \notin P$, $b \notin Q$. Also P, Q are unordered. Then P_1 and Q_1 are also unordered. If not let $P_1 \sqsubset Q_1$. Then for any $x \in P$, $(x \wedge b) \vee a$ exists by the upper bound property as $x \wedge b$, $a \leq b$ and $(x \wedge b) \vee a \in P_1$. Then $(x \wedge b) \vee a \in Q_1$ and so $x \wedge b \in Q$. Since $b \notin Q$ and Q is prime, this implies $x \in Q$. This shows that $P \subset Q$ which is a contradiction. Thus, P_1 and Q_1 must be unordered. Then by [19,Theorem 22] [a, b] is complemented. Therefore S is relatively complemented. □

We conclude this chapter with the following result which is due to [14, Theorem 3.6]. This generalizes a well known result of Hashimoto in Lattice Theory [19, Theorem 9.8].

Theorem 4.5. For a nearlattice S with 0, S is semiboolean if and only if I(S) is isomorphic to c(S). □

Corollary 4.6. For a distributive nearlattice S with 0, following conditions are equivalent :

(i) S is semiboolean.

(ii) For all ideals I, $\theta(I) = R(I)$. □

CHAPTER –2

SKELETAL CONGRUENCES ON A DISTRIBUTRIBUTIVE NEARLATTICE

1. Introduction

Throughout this chapter we will be concerned with a distributive nearlattice S, with 0 as its smallest element. Skeletal congruences on distributive lattices have been studied extensively by Cornish in [11]. For any congruence θ of c(S), θ^* denotes the pseudocomplement of θ. The existence of θ^* is guaranteed by the fact that c(S) is a distributive algebraic lattice.The skeleton $Sc(S) = \{\theta \in c(S): \theta = \phi^* \text{ for some } \phi \in c(S)\} = \{\theta \in c(S): \theta = \theta^{**}\}$. For a distributive nearlattice S with 0, I(S) is pseudocomplemented. The pseudocomplement J^* of an ideal J is the annililator ideal $J^* = \{x \in S: x \wedge j = 0 \text{ for all } j \in J\}$. We also denote $KSc(S) = \{Ker\theta : \theta \in Sc(S)\}$.

The kernel of congruence θ is $ker\theta = \{x \in S: x \equiv 0(\theta)\}$. Of course, $ker\theta(J) = J$. For $a, b \in S$, $\langle a,b\rangle$ denotes the relative annihilator $\{x \in S: x \wedge a \leq b\}$. In presence of distributivity, it is easy to show that each relative annihilator is an ideal. Also note that $\langle a, b\rangle = \langle a, a \wedge b\rangle$. For relative

annihilator ideals of a distributive lattice we refer the reader to see [33].

A distributive lattice L with 0 is called disjunctive (weakly complemented and sectionally semicomplemented are alternative terms) if for $0 \leq a < b$ there is an element $x \in L$ such that $x \wedge a = 0$ and $0 < x \leq b$. For details on these lattices we refer the reader to consult [11],[26] and [50].

In section 1 we have studied skeletal congruences for distributive nearlattices. We have shown that for any $\theta \in c(S)$, $x \equiv y(\theta^{\ddagger})$ $(x, y \in S)$ if and only if for each $a, b \in S$ with $a \leq b$ and $a \equiv b(\theta)$, $(x \wedge b) \vee a = (y \wedge b) \vee a$. We have also shown that for any ideal J both $\theta(J)^{\ddagger}$ and $\theta(J^{\ddagger})$ have $J^{\ddagger}$ as their kernel. Moreover an ideal J is the kernel of skeletal congruence if and only if it is the intersection of relative annihilator ideals.

In section 2, we introduce the notion of disjunctive nearlattices. Then we give several characterizations of disjunctive nearlattices and semiboolean algebras using skeletal congruences. Finally we show that a distributive nearlattice is semiboolean if and only if $\theta \longrightarrow \ker\theta$ is a lattice isomorphism of Sc(S) onto KSc(S) whose inverse is the map $J \longrightarrow \theta(J)$.

2. Skeletal congruences of a distributive nearlattice.

Following theorems give a description of skeletal congruences of a distirbutive nearlattice which also extend several results due to Cornish [11] for distributive lattices.

Theorem 2.1. For a distributive nearlattice S with 0, the following conditions hold.

(i) For $a \leq b$ $(a, b \in S)$, $x \equiv y\ (\theta(a,b)')$ if and only if $(x \wedge b) \vee a = (y \wedge b) \vee a$, where $\theta(a,b)'$ is the complement of $\theta(a, b)$.

(ii) For any $\theta \in c(S)$, $x \equiv y\ (\theta^*)$ $(x,y \in S)$ if and only if for each $a,b \in S$ with $a \leq b$ and $a \equiv b(\theta)$, $(x \wedge b) \vee a = (y \wedge b) \vee a$.

Proof: (i) Define a relation θ_1 on S by $x \equiv y(\theta_1)$ if and only if $(x \wedge b) \vee a = (y \wedge b) \vee a$
(since $a \leq b$, $(x \wedge b) \vee a$ and $(y \wedge b) \vee a$ exist by the upper bound property of S). Here, θ_1 is obviously an equivalence relation. Now, let $x \equiv y\ (\theta_1)$ and $t \in S$. Then, $(x \wedge b) \vee a = (y \wedge b) \vee a$ and so $[(x \wedge t) \wedge b] \vee a$
$= [(x \wedge b) \vee a] \wedge [(t \wedge b) \vee a]$
$= [(y \wedge b) \vee a] \wedge [(t \wedge b) \vee a] = [(y \wedge t) \wedge b] \vee a$.

This implies $x \wedge t \equiv y \wedge t\ (\Theta_1)$. Again, if $x \vee t$, $y \vee t$ exist, then $[(x \vee t) \wedge b] \vee a$

$= [(x \wedge b) \vee a] \vee [(t \wedge b) \vee a]$

$= [(y \wedge b) \vee a] \vee [(t \wedge b) \vee a]$

$= [(y \vee t) \wedge b] \vee a$. i.e. $x \vee t \equiv y \vee t\ (\Theta_1)$.

Thus, Θ_1 is a congruence. Clearly, Θ_a and $\Psi_b \sqsubset \Theta_1$. Hence $\Theta_a \vee \Psi_b \sqsubset \Theta_1$.

Conversely, $x \equiv y\ (\Theta_1)$ implies

$x \equiv x \wedge b\ \Psi_b \equiv y \wedge b\ \Theta_a \equiv y\ (\Psi_b)$ i.e. $x \equiv y\ (\Theta_a \vee \Psi_b)$. Therefore, $\Theta_1 = \Theta_a \vee \Psi_b = \Theta(a,b)'$.

(ii) Since $\Theta = \vee \{\Theta(a,b);\ a \leq b,\ a \equiv b(\Theta)\}$, $\Theta^* = \cap \{\Theta(a,b)^*:\ a \leq b; a \equiv b(\Theta)\}$. But as $c(S)$ is distributive and $\Theta(a,b)$ is complemented, $\Theta(a,b)^* = \Theta(a,b)'$ and hence the result clearly follows from (i). □

Theorem 2.2. Let S be a distributive nearlattice with 0. Then for any $\Theta \in c(S)$. $x \equiv y\ (\Theta^*)$ if and only if $\Theta(0, x) \cap \Theta = \Theta(0, y) \cap \Theta$ if and only if $\Psi_x \cap \Theta = \Psi_y \cap \Theta$.

Proof: Define a relation Φ on S by $x \equiv y\ (\Phi)$ if and only if $\Psi_x \cap \Theta = \Psi_y \cap \Theta$. From Papert [46], Φ is the pseudocomplement of Θ in the lattice of congruence of the semilattice $(S; \wedge)$. We now show that Φ is a congruence.

Suppose $x \equiv y(\Phi)$ and $x \vee t$, $y \vee t$ exist for some $t \in S$. Then, because of distributivity of S, $\mathbb{F}_{x \vee t} = \mathbb{F}_x \cap \mathbb{F}_t$ and $\mathbb{F}_{y \vee t} = \mathbb{F}_y \cap \mathbb{F}_t$. Thus, $\mathbb{F}_x \cap \Theta = \mathbb{F}_y \cap \Theta$ implies $\mathbb{F}_{x \vee t} \cap \Theta = \mathbb{F}_{y \vee t} \cap \Theta$ and hence $x \vee t \equiv y \vee t$ (Φ). This implies $\Phi = \Theta^*$ in c(S).

Finally we know that Θ_x and $\mathbb{F}_x$ are the complementary and c(S) is distributive. Now, in a distributive lattice L, if a', b' are the complements of a and b respectively then obviously, $a \wedge c = b \wedge c$ if and only if $a' \wedge c = b' \wedge c$ for any $c \in L$. Thus, $x \equiv y(\Phi = \Theta^*)$ if and only if $\Theta_x \cap \Theta = \Theta_y \cap \Theta$. □

Theorem 2.3. For a distributive nearlattice S with 0, the following conditions hold.

(i) For any ideal J, $x \equiv y$ $(\Theta(J)^*)$ $(x, y \in S)$ if and only if $(x] \cap J = (y] \cap J$, i.e., if and only if $x \wedge j = y \wedge j$ for all $j \in J$.

(ii) For an ideal J, both $\Theta(J)^*$ and $\Theta(J^*)$ have J^* as their kernel.

Proof: (i) By theorem 3.4 of chapter 1, for any two ideals J_1 and J_2 of S, $\Theta(J_1 \cap J_2) = \Theta(J_1) \cap \Theta(J_2)$. Thus, $\Theta_x \cap \Theta(J) = \Theta((x]) \cap \Theta(J) = \Theta((x] \cap J)$. Hence, by

theorem 2.2.2, $x \equiv y\ (\theta(J)^*)$ if and only if $\theta((x] \cap J) = \theta((y] \cap J)$, i.e., if and only if $(x] \cap J = (y] \cap J$, i.e if and only if $x \wedge j = y \wedge j$ for all $j \in J$.

(ii) $x \in \ker(\theta(J)^*)$ if and only if $x \equiv 0\ (\theta(J)^*)$, i.e., if and only if $x \wedge j = 0$ for all $j \in J$ (by (i)), i.e., if and only if $x \in J^*$. Thus, $\ker(\theta(j))^* = J^*$. □

Theorem 2.4. In a distributive nearlattice S with 0, the following conditions hold.

(i) An ideal J is the kernel of a skeletal congruence if and only if it is the intersection of relative annihilator ideals.

(ii) Each principal ideal is an intersection of relative annihilator ideals.

Proof. (i) For any $\theta \in c(S)$, $\theta^* = \vee \{\theta(a, b): a \equiv b\ (\theta^*)\}$. If θ is skeletal, then $\theta = \theta^{**} = \cap \{\theta(a, b)^*: a \leq b\ ;\ a \equiv b\ (\theta^*)\}$ and hence $\ker\theta = \cap \{\ker(\theta(a, b)^*) : a \leq b;\ a \equiv b\ (\theta^*)\}$ $= \cap \{\langle b, a\rangle : a \leq b\ ;\ a \equiv b\ (\theta^*)\}$ by (i) of theorem 2.2.1, and this completes the proof.

(ii) Since $\theta_a = \theta((a])$ is complemented, (a] is the kernel of a skeletal congruence and hence the result follows from (i). □

3. Disjunctive nearlattices and semiboolean algebras.

A non empty subset T of a nearlattice S is called large if $x \wedge t = y \wedge t$ for all $t \in T$, $(x, y \in S)$ imply $x = y$, while T is called join-dense if each $z \in S$ is the join of its predecessors in T. Following result shows that two concepts coincide when T is a convex subsemilattice of a distributive nearlattice and hence an ideal of a nearlattice is large if and only it is join-dense.

Lemma 3.1. A convex subsemilattice J of a distributive nearlattice S is large if and only if it is join-dense in S.

Proof. Obviously, every join-dense subset of S is large in S. Thus, let J be large in S. Suppose $x \in S$ and $\{j_i\}$ are its predecessors in J. Let t be an upper bound of $\{j_i\}$. Clearly, for any $j \in J$, $j_i \wedge j \leq x \wedge j \leq j$ and so $x \wedge j \in J$ by the convexity of J. Thus, $x \wedge j = j_k$ for some k. Hence, $x \wedge j \leq t$ for all $j \in J$; it follows that $x \wedge j = x \wedge t \wedge j$ for all $j \in J$. Since J is large, $x \wedge t = x$, i.e., $x \leq t$. This implies that x is the supremum of $\{j_i\}$. □

Now, we give a characterization of join-dense ideals in terms of skeletal congruences.

Lemma 3.2. An ideal J of a distributive nearlattice S is join-dense if and only if $\theta(J)$ is dense in c(S), that is $\theta(J)^* = \omega$, the smallest element of c(S).

Proof. Suppose J is join-dense. Then by lemma 2.3.1, J is large. Let $x \equiv y\ (\theta(J^*))$, then by 2.2.3, $x \wedge j = y \wedge j$ for all $j \in J$. This implies $x = y$ as J is large. So $\theta\ (J)^* = \omega$. That is, $\theta(J)$ is dense.

Conversely, let $\theta\ (J)^* = \omega$. Suppose $x \wedge j = y \wedge j$ for all $j \in J$. Then again by theorem 2.2.3, $x \equiv y\ \theta\ (J)^*\ (=\omega)$ and so $x = y$. This implies J is large and so by lemma 2.3.1, it is join-dense. □

Recall that a distributive nearlattice S with 0 is disjunctive if $0 \leq a < b$ implies there is an element $x \in S$ such that $x \wedge a = 0$ where $0 < x \leq b$.

From section 3 of chapter 1 we know that for an ideal I of a distributive nearlattice S, the relation R(I) defined by $x \equiv y\ R(I)$ if and only if for all $r \in S$, $x \wedge r \in I$ is equivalent to $y \wedge r \in I$ is a

congruence of S. Moreover, it is the largest congruence of S containing I as a class.

Proposition: 3.3 For an ideal I of a distributive nearlattice S, S/R(I) is disjunctive.

Proof. If I is a prime ideal, then S/R(I) is a two element chain {I, S-I} and so it is disjunctive (in fact, Boolean).

Suppose I is not prime, consider the interval $I \sqsubset [x] \subset [y]$ in S/R(I), where $x, y \in S$.

We claim that there exists at least one $t \notin I$, such that $t \wedge x \in I$. If not, then for all $t \notin I$, $x \wedge t \notin I$ and since $[x \wedge t] \sqsubset [y \wedge t]$, so $y \wedge t \notin I$. This implies that $x \equiv y$ R(I) and so $[x] = [y]$, which is a contradiction. Moreover, there exists a $t \notin I$ such that $x \wedge t \in I$ but $y \wedge t \notin I$. For otherwise $x \equiv y$ R(I) would lead to another contradiction. Put $s = y \wedge t$. Then $I \subset [s] \sqsubset [y]$ and $[x] \wedge [s] = [x] \wedge [y \wedge t] = [x \wedge y \wedge t] = I$ and this implies that S/R(I) is disjunctive. □

Following theorem gives characterizations of disjunctive nearlattices.

Theorem 3.4. For a distributive nearlattice S with 0, the following conditions are equivalent.

(i) S is disjunctive.

(ii) For all $a \in S$, $(a] = (a]^{**}$.

(iii) $R((0]) = \omega$.

Proof. (i) implies (ii). Suppose S is disjunctive. For any $a \in S$. Obviously, $(a] \sqsubset (a]^{**}$. To prove the reverse inequality, let $x \in (a]^{**}$. If $x \notin (a]$, then $x \nleq a$ i.e., $x \neq x \wedge a$. Then $0 \leq x \wedge a < x$. Since S is disjunctive there exists t with $0 < t \leq x$ such that $t \wedge x \wedge a = 0$ i.e, $t \wedge a = 0$. This implies $t \in (a]^{*}$. Since $x \in (a]^{**}$, so $x \wedge t = 0$, i,e. $t = 0$, which gives a contradiction. Hence $x \in (a]$. In otherwords $(a] = (a]^{**}$ for all $a \in S$.

(ii) implies (iii). Suppose (ii) holds and $x \equiv y$ R((0]) for some $x, y \in S$. If $x \neq y$, then either $x \wedge y < y$ or $x \wedge y < x$. Suppose $x \wedge y < y$. Then $(y]^{*} \sqsubset (x \wedge y]^{*}$. Since $(a] = (a]^{**}$ for all $a \in S$, $(y]^{*} \neq (x \wedge y]^{*}$. Thus, $(y]^{*} \subset (x \wedge y]^{*}$. So there exists $t \in (x \wedge y]^{*}$ such that $t \notin (y]^{*}$. Then $t \wedge x \wedge y = 0$

but $t \wedge y \neq 0$, which implies $x \wedge y \not\equiv y\ R((0])$, and so $x \not\equiv y\ R((0])$, which is a contradiction. Therefore, $R((0]) = \omega$.

(iii) Implies (i). Suppose $R((0]) = \omega$. Let $0 \leq x < y$ $(x, y \in S)$. Since $R((0]) = \omega$, there exists $t \in S$ such that $t \wedge x = 0$ but $t \wedge y \neq 0$. For otherwise $x \equiv y\ R((0])$, which implies $x = y$ and there is a contradiction to our assumption. Thus we have $0 < t \wedge y \leq y$, such that $x \wedge t \wedge y = 0$, and so S is disjunctive. □

In chapter 1, we have already denoted the set of all finitely generated ideals of a nearlattice S by $I_f(S)$. Of course $I_f(S)$ is a join semilattice of I(S). In [23] Hickman exhibited a nearlattice S for which $I_f(S)$ is a meet semilattice. But in [14] Cornish and Hickman have shown that if S is distributive then $I_f(S)$ is a distributive sublattice of I(S), the lattice of ideals. Following lemma was suggested to the author by supervisor Dr. Noor.

Lemma 3.5. A distributive nearlattice S with 0 is disjunctive if and only if $I_f(S)$ is disjunctive.

Proof. Let S be disjunctive and
$(a_1,.....,a_r] \subset (b_1,.....,b_t]$ in $I_f(S)$.
Choose $x \in (b_1,.....,b_t] - (a_1,.....,a_r]$.
Then $(a_1 \wedge x,.....,a_r \wedge x] = (a_1,.....,a_r] \cap (x] \subset (x]$.
Now, by the upper bound property of S,
$(a_1 \wedge x) \vee.....\vee (a_r \wedge x) = e$ (say) exists and
$0 \leq e < x$. Since S is disjunctive, there exists $d \in S$ such that $0 = d \wedge e$ and $0 < d \leq x$. Thus $(d] \cap (e] = (0]$ and so $(d] \cap (a_1,.....,a_r] \cap (x] = (0]$. This implies that $(d] \cap (a_1,.....,a_r] = (0]$. Of course,
$(0] \neq (d] \sqsubset (x] \sqsubset (b_1,.....,b_t]$ and hence, $I_f(S)$ is disjunctive.

Conversely, let $I_f(S)$ be disjunctive and suppose $0 \leq c < d$; $c, d \in S$. Then, $(0] \sqsubset (c] \sqsubset (d]$. Since $I_f(S)$ is disjunctive, there exists $(a_1,.....,a_r]$ in $I_f(S)$ such that $(c] \cap (a_1,.....,a_r] = (0]$, where
$(0] \neq (a_1,.....,a_r] \sqsubset (d]$. Now, by the upper bound property of S, $a_1 \vee.....\vee a_r = f$ (say) exists. Thus, we have $c \wedge f = 0$ and $0 < f \leq d$, and which proves that S is disjunctive. □

The following theorem is an extension of Theorem. 2.1. of Cornish [11].

Theorem 3.6. In a distributive nearlattice S with 0, the following conditions are equivalent.

(i) S is disjunctive.

(ii) Each dense ideal J (i,e. $J^{\ddagger} = (0]$) is join-dense.

(iii) For each dense ideal J, $\theta(J^{\ddagger}) = \theta(J)^{\ddagger}$.

(iv) For each dense ideal J, $\theta(J^{\ddagger\ddagger}) = \theta(J)^{\ddagger\ddagger}$.

Proof. Since $J^{\ddagger} = (0]$ if and only if $J^{\ddagger\ddagger} = S$ and J is join- dense if and only if $\theta(J)^{\ddagger} = \omega$, obviously (ii), (iii) and (iv) are equivalent.

(i) Implies (ii). Suppose J is a dense ideal and $x \wedge j = y \wedge j$ $(x, y \in S)$ for all $j \in J$. If $x \neq y$, then either $x \wedge y < x$ or $x \wedge y < y$. Without loss of generality suppose $x \wedge y < x$. Since S is disjunctive, there exists a $(\neq 0) \in S$, $a \leq x$ such that $a \wedge x \wedge y = 0$. Then, $0 = a \wedge x \wedge y \wedge j = a \wedge x \wedge j$ for all $j \in J$. Hence, $a \wedge x = 0$ as J is dense ; i.e., $a = 0$ which is a contradiction. Thus J is join-dense.

(ii) implies (i). For any $a \in S$, $(a] \vee (a]^*$ is always a dense ideal. Thus, with (ii) holding, $(a] \vee (a]^*$ is join-dense. Then by lemma 2.3.2, $\omega = \Theta((a] \vee (a]^*)^* = (\Theta(a] \vee \Theta(a]^*)^* = \Theta((a])^* \cap \Theta((a]^*)^*$. Thus, $\Theta((a]^*)^* \sqsubset \Theta((a])^{**} = \Theta_a$. Taking the kernel on both sides we have $(a]^{**} \sqsubset (a]$ by using Th.2.2.3 (ii). It follows that $(a] = (a]^{**}$ and hence S is disjunctive. □

Next theorem is an extension of 2.2 of Cornish [11]. We omit the proof as this can be proved exactly in a similar way the corresponding result of [11] was proved.

Theorem 3.7. For a distributive nearlattice S wih 0, the following conditions are equivalent.

(i) S is disjunctive.

(ii) For each congruence Φ, $\Phi^* = \Theta(\ker\Phi)^*$.

(iii) For each ideal J, $R(J)^* = \Theta(J)^*$

(iv) For each congruence Φ, $\ker(\Phi^*) = (\ker\Phi)^*$.

(v) For each congruence Φ, $\ker(\Phi^{**}) = (\ker\Phi)^{**}$.

(vi) The kernel of each skeletal congruence is an annihilator ideal. □

According to section 4 of chapter 1 a nearlattice S with 0 is called semiboolean if it is distributive and [0, x] is complemented for all $x \in S$. By 1.4.5 we know that the lattice of all ideals of a nearlattice is isomorphic to the lattice of congruences if and only if S is semiboolean. Using this result we get the following theorem, which is an extension of 2.3 of (11).

Theorem 3.8. The following conditions are equivalent for a distributive nearlattice S with 0.

(i) S is semiboolean.

(ii) For each congruence Φ, $\Phi^{*} = \Theta(\ker \Phi^{*})$.

(iii) For each ideal J, $\Theta(J^{*}) = \Theta(J)^{*}$.

(iv) For each ideal J, $\Theta(J^{**}) = \Theta(J)^{**}$.

Proof. (i) implies (ii). Suppose S is semiboolean. Then by 1.4.5 I(S) is isomorphic to c(S). Hence for any congruence Ψ, $\Psi = \Theta(\ker\Psi)$. Taking $\Psi = \Phi^{*}$, we see that (i) implies (ii).

(ii) implies (iii) follows from Th.2.2.3 (ii) and (iii) ===> (iv) is obvious.

(iv) implies (i). Suppose (iv) holds. Put $J = (a] \vee (a]^{*}$. Then $J^{*} = (0]$ and so $J^{**} = S$.

Then by (iv), $\Theta((a] \vee (a]^*)^{**} = \iota$. It follows that $\Theta((a])^* \cap \Theta((a]^*)^* = \omega$ and so $\Theta((a]^*)^* \sqsubset \Theta((a])^{**} = \Theta_a^{**} = \Theta_a$. Since $\ker\Psi_a = (a]^*$, we have $\Theta((a]^*) \sqsubset \Psi_a = \Theta_a^*$ and so $\Theta_a = \Theta_a^{**} \sqsubset \Theta((a]^*)^*$. Thus $\Theta((a]^*)^* = \Theta_a$. But $(a]^* = (a]^{***}$. Now, by (iv), $\Theta((a]^*)^{**} = \Theta((a]^{***}) = \Theta((a]^*)$. But $\Theta_a^* = \Theta((a]^*)^{**}$, and so $\Theta((a]^*) = \Theta_a^* = \Psi_a$. Now if $0 \leq a \leq b$, then $a \equiv b\ (\Psi_a)$ and so $a \equiv b\ (\Theta((a]^*))$. Then $(a] \vee (a]^* = (b] \vee (a]^*$ and so $b = a \vee j$ for some $j \in (a]^*$. Then $j \wedge a = 0$, and so $[0, b]$ is complemented. Hence S is semiboolean. □

The skeleton $Sc(S) = \{\Theta \in c(S) ; \Theta = \Phi^* \text{ for some } \Phi \in c(S)\} = \{\Theta \in c(S) ; \Theta = \Theta^{**}\}$ is a complete Boolean lattice. The meet of a set $\{\Theta_i\} \sqsubset Sc(S)$ is $\cap \Theta_i$; as in $c(S)$, while the join is given by $\bigvee \Theta_i = (\vee \Theta_i)^{**} = (\cap \Theta_i^*)^*$ and the complement of $\Theta \in Sc(S)$ is Θ^*. The fact that $Sc(S)$ is complete follows from the fact that $Sc(S)$ is precisely the set of closed elements associated with the closure operation $\Theta \dashrightarrow \Theta^{**}$ on the complete lattice $c(S)$ and $Sc(S)$ is Boolean because of Glivenko's theorem, c.f. Gratzer [19. Th.4,p.58].

The set $KSc(S) = \{\ker\Theta ; \Theta \in Sc(S)\}$ is closed under arbitrary set-theoretic intersections and hence is a complete lattice. We will use the symbol $\bigvee$ to denote the

join in Sc(S) and in KSc(S). We also denote $A(S) = \{J: J \in I(S) ; J = J^{\ddagger\ddagger}\}$, which is a complete Boolean lattice.

The following theorems are extensions of 2.4 and 2.5 of Cornish [11] to nearlattices.

Theorem 3.9. For a distributive nearlattice S with 0, the following conditions are equivalent.

(i) S is disjunctive.

(ii) The map θ ----> $\ker\theta$ of Sc(S) onto KSc(S) is one-to-one.

(iii) The map θ ----> $\ker\theta$ of Sc(S) onto KSc(S).

(iv) The map θ ----> $\ker\theta$ is a lattice isomorphism of Sc(S) onto KSc(S), whose inverse is the map J ---> $\theta(J)^{\ddagger\ddagger}$.

Proof. (i) implies (iv). Suppose S is disjunctive. Then by Th.2.3.7 (vi) KSc(S) = A(S). By 2.3.7 (ii), $\Phi = \Phi^{\ddagger\ddagger} = \theta(\ker\Phi)^{\ddagger\ddagger}$ for any $\Phi \in Sc(S)$. Thus, the map θ ---> $\ker\theta$ is one-to-one. Clearly it preserves meet.

Now using 2.3.7 (iv), for θ, $\Phi \in Sc(S)$, $\ker(\theta \vee \Phi)$

$= \ker((\theta^{\ddagger} \cap \Phi^{\ddagger})^{\ddagger}) = (\ker(\theta^{\ddagger} \cap \Phi^{\ddagger}))^{\ddagger} = (\ker\theta^{\ddagger} \cap \ker \Phi^{\ddagger})^{\ddagger} = ((\ker\theta)^{\ddagger} \cap (\ker \Phi)^{\ddagger})^{\ddagger} = \ker\theta \veebar \ker\Phi$ as $KSc(S) = A(S)$. Thus $\theta \dashrightarrow \ker\theta$ is a lattice isomorphism. Moreover, by 2.3.7, $\ker(\theta(J)^{\ddagger\ddagger}) = (\ker\theta(J))^{\ddagger\ddagger} = J^{\ddagger\ddagger} = J$ for all $J \in A(S) = KSc(S)$, while $\theta(\ker\Phi)^{\ddagger\ddagger} = \Phi^{\ddagger\ddagger} = \Phi$ for all $\Phi \in Sc(S)$. Therefore $J \dashrightarrow \theta(J)^{\ddagger\ddagger}$ is the inverse of $\theta \dashrightarrow \ker\theta$.

(iv) implies (ii) is trivial.

(ii) implies (iii). If $\theta \dashrightarrow \ker\theta$ is one-to-one, then it is a meet isomorphism of the lattice $Sc(S)$ onto the lattice $KSc(S)$, then of course it is a lattice isomorphism and so (iii) holds.

Finally we shall show that (iii) implies (i). If (iii) holds, then of course $\theta \dashrightarrow \ker\theta$ is a lattice homorphism of $Sc(S)$ onto $KSc(S)$. Hence $KSc(S)$ must be Boolean. Since for all $a \in S$, $(a] = \ker(\theta_a)$, the map $a \dashrightarrow (a]$ embeds S, as a join-dense subnearlattice, into the complete Boolean lattice $KSc(S)$. Therefore S must be disjunctive. □

We conclude this chapter with the following theorem which is also a generalization of [11. Th. 2.5].

Theorem 3.10. A distributive nearlattice S is semiboolean if and only if the map θ ---> kerθ is a lattice isomorphism of Sc(S) onto KSc(S), whose inverse is the map J ---> θ (J).

Proof. If S is semiboolean, then of course it is disjunctive and so by Theorem 2.3.9 the inverse of θ ---> kerθ is J ---> $\theta(J)^{\ddagger\ddagger}$. Now by 2.3.8 $\theta(J)^{\ddagger\ddagger} = \theta(J^{\ddagger\ddagger})$ for any $J \in KSc(S)$. Since by Th. 2.3.7, $J \in A(S)$ so $J = J^{\ddagger\ddagger}$. Thus J ---> $\theta(J)$ is the inverse.

Conversely, suppose J ---> θ (J) is the inverse of θ --> kerθ. Then by 2.3.9 S is disjunctive and so $\ker(\theta\ (K)^{\ddagger\ddagger}) = (\ker\theta\ (K))^{\ddagger\ddagger} = K^{\ddagger\ddagger}$ for any ideal K. This implies $K^{\ddagger\ddagger} \in KSc(S)$. Then using the description of the inverse, $\theta\ (K^{\ddagger\ddagger}) = \theta(\ker(\theta\ (K)^{\ddagger\ddagger})) = \theta\ (K)^{\ddagger\ddagger}$. Hence by 2.3.8, S is semiboolean. □

CHAPTER – 3

NORMAL NEARLATTICE

1.Introduction.

Normal lattices have been studied by several authors including Cornish [8] and Monteiro [34] [35]; while n-normal lattices have been studied by Cornish [9] and Davey [16].On the other hand Cignoli in [6] and [7] introduced the notions of k-normal and k-completely normal lattices.

A distributive lattice L with 0 is called normal if each prime ideal of L contains a unique minimal prime ideal. Equivalently, L is called normal if each prime filter of L is contained in a unique ultrafilter (maximal and proper) of L. L is called n-normal if each filter is contained in at most n ultrafilters of it.

In this chapter we have defined normal and n-normal nearlattices in the same manner. Then we have generalized several results of Cornish [8] [9] and Davey [16].In proving some of the results we have used principle of localization [Th. 2.6], which is an extension of lecture

note of Dr. Noor on localization. For some ideas on localization see section 5 of Cornish [13].This technique is very interesting and quite different from those of the previous authors.

2. Normal nearlattices.

Throughout this chapter all nearlattices are assumed to be distributive.

For an ideal J in a nearlattice S with 0
$J^{\ddagger} = \{ y \in S : y \wedge x = 0 \text{ for all } x \in J \}$.

Ideals I and J of a nearlattice S are said to be comaximal if $I \vee J = S$.

If P is a prime ideal in a nearlattice S with 0 then O(P) is used to denote the ideal $\{ y \in S ; y \wedge x = 0$ for some $x \in S-P \}$. Clearly $O(P) \sqsubset P$.

A prime ideal P is said to be a minimal prime ideal belonging to ideal I, if (i) $I \sqsubset P$ and (ii) there exists no prime ideal Q such that $Q \neq P$ and $I \sqsubset Q \sqsubset P$. In lattice theory some authors called it minimal prime divisor of I

A minimal prime ideal belonging to the zero ideal of a nearlattice with 0 is called a minimal prime ideal. For the theory of minimal prime ideals in a general setting see Cornish [12].

Lemma 2.1. Let P be a prime ideal in a nearlattice S with 0. Then each minimal prime ideal belonging to O(P) is contained in P.

Proof: Let Q be a minimal prime ideal belonging to O(P).

If $Q \not\subseteq P$ then choose $y \in Q-P$. Then from [27, lemma 3.1] and by the distributivity of S it follows that $y \wedge z \in O(P)$ for some $z \notin Q$. Hence $y \wedge z \wedge x = 0$ for a suitable $x \notin P$. As P is prime, $y \wedge x \notin P$ so $z \in O(P) \sqsubset Q$. This is a contradiction. Hence $Q \sqsubset P$. □

Proposition. 2.2. If P is a prime ideal in a nearlattice with 0, then the ideal O(P) is the intersection of all the minimal prime ideals contained in P i,e $O(P) = \cap \{ Q ; Q \sqsubset P, Q \text{ is a minimal prime ideal} \}$.

Proof. If Q is prime and $Q \sqsubset P$, then $O(P) \sqsubset O(Q) \sqsubset Q \sqsubset P$. Again, if Q is a minimal prime ideal belonging to O(P) then Q is a minimal prime ideal inside P by the lemma 3.2.1.

Thus, $\{ Q : Q \text{ is minimal prime and } Q \subset P \} = \{ Q ; Q$ is minimal prime ideal belonging to O(P) $\}$. Since L is distributive O(P) is the intersection of all minimal prime ideals belonging to O(P) (c.f. corollary 1.2.8) this establishes the proposition. □

Let F be a filter of a distributive nearlattice S. It can be easily shown that the relation Ψ_F on S, defined by $x \equiv y \ (\Psi_F)$ ($x , y \in S$) if and only if $x \wedge f = y \wedge f$,

for some $f \in F$ is a congruence on S. Let us denote $S/\Psi(F)$ by S_F (the quotient lattice) Then $\Psi_F : S \longrightarrow S_F$ is the natural epimorphism.

Lemma 2.3. S_F is a distributive lattice.

Proof: Clearly, S_F is a lower semilattice. Now, let $p, q \in S_F$. Then there exists $x, y \in S$ such that $p = \Psi_F(x)$ and $q = \Psi_F(y)$, as Ψ_F is an epimorphism. Clearly, $x \equiv x \wedge f\ (\Psi_F)$ and $y \equiv y \wedge f\ (\Psi_F)$ for any $f \in F$.

So, $\Psi_F(x) = \Psi_F(x \wedge f)$ and $\Psi_F(y) = \Psi_F(y \wedge f)$. Now, $(x \wedge f) \vee (y \wedge f)$ always exists in S, due to the upper bound property of S. Thus, $p \vee q$ exists. Moreover $p \vee q = \Psi_F(x \wedge f) \vee \Psi_F(y \wedge f) = \Psi_F((x \wedge f) \vee (y \wedge f))$. Hence S_F is a lattice. The distributivity of S_F clearly follows from the distributivity of S. □

Lemma 2.4. Let F be any filter of a distributive nearlattice S. For any ideals I and J of S, the following hold.

(i) $\Psi_F(I)$ is an ideal of S_F .

(ii) $\Psi_F(I)$ is a proper ideal (i.e. $\neq$ whole lattice) if and only if $I \cap F = \Phi$.

(iii) $\Psi_F(I) \vee \Psi_F(J) = \Psi_F(I \vee J)$.

(iv) $\Psi_F(I) \cap \Psi_F(J) = \Psi_F(I \cap J)$.

Proof: (i) For $i, j \in I$, $\Psi_F(i) \vee \Psi_F(j)$ $= \Psi_F((i \wedge f) \vee \Psi_F(j \wedge f)) = \Psi_F[(i \wedge f) \vee (j \wedge f)]$ for any $f \in F$. Thus, $\Psi_F(I)$ is closed under finite supremum. Now, suppose $t \in S_F$ and $t \leq \Psi_F(i)$ for some $i \in I$. Then, $t = \Psi_F(x)$ for some $x \in S$, and $t = \Psi_F(x) \wedge \Psi_F(i)$ $= \Psi_F(x \wedge i) \in \Psi_F(I)$. Therefore, $\Psi_F(I)$ is an ideal of S_F.

(ii) If $\Psi_F(I)$ is proper , then there exists $x \in S$, such that $\Psi_F(x)$ does not belong to $\Psi_F(I)$. Suppose $I \cap F \neq \Phi$ and $r \in I \cap F$. Since $r \in F$, $x \equiv x \wedge r(\Psi_F)$. But $x \wedge r \in I$, and this implies $\Psi_F(x) \in \Psi_F(I)$, which is a contradiction. Hence, $I \cap F = \Phi$.

Conversely, if $\Psi_F(I)$ is not proper, then for any $f \in F$, $\Psi_F(f) \in \Psi_F(I)$. Thus, $\Psi_F(f) = \Psi_F(i)$ for some $i \in I$. Then, $f \wedge f_1 = i \wedge f_1$ for some $f_1 \in F$ and this implies $f \wedge f_1 \in I \cap F$, and so $I \cap F \neq \Phi$.

(iii) and (iv) are trivial. □

Theorem 2.5. Suppose F is a filter of a distributive nearlattice S. Then for any ideal J of S, $\Psi_F^{-1}\ \Psi_F\ (J) = \{x \in S : x \wedge f \in J \text{ for some } f \in F\} = \cap \{P ; P \text{ is a (minimal) prime ideal belonging to J in S such that } P \cap F = \Phi\}$.

Proof: $\Psi_F^{-1}\ \Psi_F\ (J) = \{ y \in S ; \Psi_F\ (y) \in \Psi_F\ (J) \} = \{ y \in S : y \equiv x\ (\Psi_F) \text{ for some } x \in J \} = \{ y \in S : y \wedge f = x \wedge f \text{ for some } f \in F, x \in J \} = \{ y \in S ; y \wedge f \in J \text{ for some } f \in F \}$. Now we consider two cases:

Case 1. Let $J \cap F \neq \Phi$. Then there exists $x \in J \cap F$ and for any prime ideal P belonging to J, $P \cap F \neq \Phi$. Thus, $\{ P : P \text{ is a prime ideal belonging to J and } P \cap F = \Phi \} = \Phi$, and so $\cap \{ P : P \text{ is a prime ideal belonging to J and } P \cap F = \Phi\} = S = \{y \in S : y \wedge x \in J, x \in J \cap F \}$.

Case 2. suppose $J \cap F = \Phi$. Clearly, $\{ y \in S ; y \wedge f \in J \text{ for some } f \in F \} \sqsubset \cap \{ P : P \text{ is a prime ideal belonging to J and } P \cap F = \Phi \}$. Let $x \in S$ be such that $x \wedge f \notin J$ for all $f \in F$, and let $G = [x) \vee F$. If $J \cap G \neq \Phi$, then there exists $t \in J$ and $t \geq x_1 \wedge f$ for some $x_1 \geq x$ and for some $f \in F$. This implies $x \wedge f \leq x_1 \wedge f \leq t$ and consequently $x \wedge f \in J$, which is a contradiction. Thus, $J \cap G = \Phi$. Then by Birkhoff Stone theorem, there exists

a prime ideal P of S such that $J \sqsubset P$ and $G \cap P = \Phi$. In effect, $x \notin P$ and $F \cap P = \Phi$ as $F \sqsubset G$. This completes the proof. □

Theorem 2.6. Suppose F is a filter of a distributive nearlattice S. Also, suppose $\mathcal{Q} = \{ P : P$ is a prime ideal of S, such that $P \cap F = \Phi \}$ and $\mathcal{P} = \{ \bar{P} : \bar{P}$ is a prime ideal of $S_F \}$. Then $\mathcal{Q}$ and $\mathcal{P}$ are order isomorphic posets.

Proof: Let $P \in \mathcal{Q}$. Then $\nabla_F(P) \neq S_F$ by 3.2.4. Also, $\nabla_F(x) \wedge \nabla_F(y) \in \nabla_F(P)$ implies $\nabla_F(x \wedge y) = \nabla_F(q)$ for some $q \in P$. Then, $x \wedge y \wedge f = q \wedge f$ for some $f \in F$ and so either $x \in P$ or $y \in P$. Hence, $\nabla_F(x) \in \nabla_F(P)$ or $\nabla_F(y) \in \nabla_F(P)$, showing that $\nabla_F(P)$ is a prime ideal of S_F. Thus, ∇_F is a map from $\mathcal{Q}$ to $\mathcal{P}$ and it is clearly isotone. Again, for any $\bar{p} \in \mathcal{P}$ it is very easy to show that $\nabla_F^{-1}(\bar{p}) \in \mathcal{Q}$ and $\nabla_F^{-1} : \mathcal{P} \dashrightarrow \mathcal{Q}$ is obviously isotone. As $\nabla_F : S \dashrightarrow S_F$ is onto, $\nabla_F \nabla_F^{-1} = I_{\mathcal{P}}$. Moreover by 3.2.5 $\nabla_F^{-1} \nabla_F(Q) = Q$ for any $Q \in \mathcal{Q}$, and hence $\nabla_F^{-1} \nabla_F = I_{\mathcal{Q}}$. Therefore, $\mathcal{P}$ and $\mathcal{Q}$ are order isomorphic. □

In the above theorem, $S-P \supseteq F$ for all $P \in \underline{Q}$. Of course in any nearlattice S, the map $P \dashrightarrow S-P$ is an order reversing isomorphism between the poset of prime ideals and the poset of prime filters of S. Thus, we have the

following important corollary which is an immediate consequence of above theorem.

Corollary 2.7. For a distributive nearlattice S, the set of prime filters of S containing a given filter F of S is order isomorphic to the set of prime filters of S_F. □

Principle of localization.

Theorem 2.8. Let S be a distributive nearlattice. Then for each ideal J of S, $J = \bigcap_F (\Psi_F^{-1} \Psi_F (J))$ where F ranges over the prime F filters of S.

Hence for any ideals I and J of S, $\Psi_F (I) = \Psi_F (J)$ for all prime filters F of S implies I = J.

Proof: For any filter F of S. Clearly $\Psi_F^{-1} \Psi_F (J) \supseteq J$. Hence, $J \sqsubset \bigcap (\Psi_F^{-1} \Psi_F (J))$ where F ranges over the prime filters F of S. Now, let $x \in \bigcap_F (\Psi_F^{-1} \Psi_F (J))$. Then, $x \in \Psi_F^{-1} \Psi_F (J)$ for all prime filters F of S. But, for any filter F of S, $\Psi_F^{-1} \Psi_F (J) = \{ y \in S : y \wedge f \in J$ for some $f \in F \}$ by 3.2.5. Thus, for any prime filter F of S, $x \wedge f_1 \in J$ for some $f_1 \in F$. If $x \notin J$, then by Birkhoff Stone theorem, there is a prime ideal Q of S

such that x does not belong to Q and $J \sqsubset Q$. Then for any $f \in S-Q$, $x \wedge f$ does not belongs to $J \sqsubset Q$ which is a contradiction as Q is a prime ideal of S. Hence $x \in J$. □

Suppose S is a distributive nearlattice. For any $x, y \in S$, we define $\langle x, y\rangle = \{ s \in S : s \wedge x \leq y \}$ and $\langle x, J\rangle = \{ s \in S : s \wedge x \in J \}$ for any ideal J of S. It is easily seen that $\langle x, y\rangle$ and $\langle x, J\rangle$ are ideals of S. Moreover, $\langle x, y\rangle$ is known as the relative annihilator ideal c.f. Mandelker [33]. For any x in a nearlattice S with 0, we denote $(x]^* = \{ y \in S : y \wedge x = 0 \}$.

The following proposition is needed for the further development of this chapter. We omit the proof as it is easily verifiable.

Proposition.2.9. Suppose F is a filter of a distributive nearlattice S with 0. Then the following condition hold.

(i) $\Psi_F((x]) = (\Psi_F(x)]$

(ii) For any ideal J of S, $\Psi_F(\langle x, J\rangle) = \langle \Psi_F(x), \Psi_F(J)\rangle$.

(iii) $\Psi_F((x]^*) = (\Psi_F(x)]^*$.

(iv) $\Psi_F(\langle x, y\rangle) = \langle \Psi_F(x), \Psi_F(y)\rangle$. □

Recall that a distributive nearlattice S with 0 is normal if each prime ideal contains a unique minimal prime ideal. Equivalently, a nearlattice S with 0 is called normal if each prime filter of S is contained in a unique ultrafilter.(i,e, maximal and proper filter) of S.

The following theorem contains the main result of this section. This generalizes the result of Th. 2.4. of Cornish [8].

Theorem 2.10. Let S be a distributive nearlattice with O. Then the following conditions are equivalent.

(i) Any two distinct minimal prime ideals are comaximal.

(ii) S is normal.

(iii) O(P) is a prime ideal for each prime ideal P.

(iv) For all $x , y \in S$, $x \wedge y = 0$ implies $(x]^* \vee (y]^* = S$.

(v) $(x \wedge y]^* = (x]^* \vee (y]^*$.

Moreover, if $1 \in S$ so that S is a lattice, then for all $x , y \in S$, $x \wedge y = 0$ implies there exists $x_1 , y_1 \in S$, such that $x \wedge x_1 = 0 = y \wedge y_1$ and $x_1 \vee y_1 = 1$.

Proof: (i) <===> (ii) is trivial and (ii) <===> (iii) hold by proposition 3.2.2.

(ii) implies (iv). Suppose (ii) holds. Then by corollary 3.2.7, for any prime filter F of S, S_F has a unique ultrafilter. Thus S_F has a unique minimal prime ideal. But the zero ideal of S_F (as $0 \in S$) is the intersection of all minimal prime ideals of S_F. Hence by uniqueness, it is a (minimal) prime ideal of S_F. Now suppose $x , y \in S$ such that $x \wedge y = 0$, and so $\Psi_F(x) \wedge \Psi_F(y) = \bar{o}$. Then, either $\Psi_F(x) = \bar{o}$ or $\Psi_F(y) = \bar{o}$. Thus $[\Psi_F(x)]^* \vee (\Psi_F(y))^* = S_F$. Then by 3.2.9 $\Psi_F((x]^* \vee (y]^*) = \Psi_F(S)$ and hence by 3.2.8 $(x]^* \vee (y]^* = S$.

(iv) implies (ii). Let P ,Q be distinct ultrafilters of S containing a prime filter F of S. Then $P \vee Q = S$ otherwise $P \vee Q$ will be a proper filter of S, which contradicts the fact that P,Q are ultrafilters. Thus, there exist $x \in P-Q$ and $y \in Q-P$ such that $x \wedge y = 0$. Let $t \in (x]^*$. Then, $t \wedge x = 0$. Thus, $t \in S-P$ (otherwise if $t \in P$, then $0 = t \wedge x \in P$ which is impossible) and $S-P \sqsubset S-F$. That is, $(x]^* \sqsubset S-F$. Similarly, $(y]^* \sqsubset S-F$. Therefore, $S = (x]^* \vee (y]^* \sqsubset S-F$, which is a contradiction.

(ii) implies (v). Suppose (ii) holds. Then for any prime filter F of S, the zero-ideal of S_F is prime (This has been already shown in (ii) ==> (iv)). For any $x , y \in S$ consider the following two cases.

Case 1. $\Psi_F (x \wedge y) = \bar{o}$. Then, either $\Psi_F (x) = \bar{o}$ or $\Psi_F (y) = \bar{o}$. Hence, $(\Psi_F (x \wedge y)]^* = S_F$ and either $(\Psi_F (x)]^* = S_F$ or $(\Psi_F (y)]^* = S_F$. Thus, $(\Psi_F (x \wedge y)]^* = (\Psi_F (x)]^* \vee (\Psi_F (y)]^*$. Then, by 3.2.9, $\Psi_F ((x \wedge y]^*) = \Psi_F ((x]^* \vee (y]^*)$ and so $(x \wedge_s y]^* = (x]^* \vee (y]^*$ by 3.2.8.

Case 2. $\Psi_F (x \wedge y) \neq \bar{o}$. Then, $\Psi_F (x), \Psi_F (y) \neq \bar{o}$. Hence $(\Psi_F (x \wedge y)]^*$, $(\Psi_F (x)]^*$ and $(\Psi_F (y)]^*$ are equal to zero ideal of S_F (as zero ideal is prime), and so the result follows trivially.

(v) implies (iv) is obvious.

Finally, (iv) and the stated condition are trivially equivalent. □

A nearlattice S with O is called dense if $(x]^* = (0]$ for each $x \neq 0$ in S. The following theorem is an extension of 4.1 of Cornish [8].

Theorem 2.11. For a nearlattice S with 0, the following hold.

(i) If S is normal, then S_F is normal for any filter F of S.

(ii) S is normal if and only if for each prime filter F of S, S_F is a dense lattice.

Proof: (i) Let $\Psi_F(x)$, $\Psi_F(y) \in S_F$ be such that $\Psi_F(x) \wedge \Psi_F(y) = \bar{o}$. Then, $x \wedge y \equiv 0(\Psi_F)$, which implies $x \wedge y \wedge f = 0$ for some $f \in F$. Since S is normal, $(x]^* \vee (y \wedge f]^* = S$ by 3.2.10.

Hence $(\Psi_F(x)]^* \vee (\Psi_F(y)]^*$

$= (\Psi_F(x)]^* \vee (\Psi_F(y \wedge f)]^*$

$= \Psi_F((x]^* \vee (y \wedge f]^*) = \Psi_F(S) = S_F$

Thus, by 3.2.10 S_F is normal.

(ii) Suppose S is normal. Let $\Psi_F(x) \neq \bar{o}$ and $\Psi_F(q) \in (\Psi_F(x)]^*$. Then $\Psi_F(q) \wedge \Psi_F(x) = \bar{o}$. But we already know from the proof of (ii)===>(iv) in 3.2.10 that the zero ideal of S_F is prime. Hence, $\Psi_F(q) = \bar{o}$, showing that S_F is dense.

Conversely, let S_F be dense for each prime filter F of S. Suppose x , y $\in$ S are such that $x \wedge y = 0$. Then, $\Psi_F(x \wedge y) = \Psi_F(0) = \bar{o}$.

That is, $\Psi_F(x) \wedge \Psi_F(y) = \bar{o}$ which implies $\Psi_F(x) = \bar{o}$ or $\Psi_F(y) = \bar{o}$ as S_F is dense. Hence, either $(\Psi_F(x)]^* = S_F$ or $(\Psi_F(y)]^* = S_F$. Thus, $\Psi_F((x]^* \vee (y]^*) = S_F = \Psi_F(S)$, and so by 3.2.8 $(x]^* \vee (y]^* = S$. Therefore, S is normal.

3.Relatively normal nearlattices.

Definition 3.1. A distributive nearlattice S is called relatively normal if each interval [x,y] with $x < y$ is a normal lattice.

Definition 3.2. A nearlattice S with 0 is called sectionally normal if each interval [0,x] with $0 < x$ is a normal lattice.

Katrinák [28, lemma 9, P.135] has shown that a normal lattice is sectionally normal. Cornish in [8, Th. 3.3] has improved that result. Our following theorem is a nice generalization of their results.

Theorem 3.3. Let S be a nearlattice with 0. Then the following are equivalent.

(i) S is normal.

(II) Each ideal $J \neq S$ is a normal subnearlattice.

(iii) S is sectionally normal.

Proof: (i) implies (ii). If J is an ideal and $x, y \in S$ with $x \wedge y = 0$ then $(x]^* \vee (y]^* = S$ because of theorem 3.2.10. Hence $J = J \cap S = (J \cap (x]^*) \vee (J \cap (y]^*)$.

But $J \cap (x]^*$ and $J \cap (y]^*$ are respectively $\{z \in J ; z \wedge x = 0\}$ and $\{z \in J ; z \wedge y = 0\}$ and it follows from theorem 3.2.10 that J is normal.

(ii) implies (iii) is trivial.

(iii) implies (i). Let $x, y \in S$ with $x \wedge y = 0$. Let $r \in S$, then $(r \wedge x) \wedge (r \wedge y) = 0$.

Since S is sectionally normal, so $(r]$ is a normal nearlattice. Then $r = (r \wedge x]^* \vee (r \wedge y]^*$ and so $r = p \vee q$ for some $p \in (r \wedge x]^*$ and $q \in (r \wedge y]^*$. Then $p \wedge r \wedge x = 0$ and $q \wedge r \wedge y = 0$ i.e. $p \wedge x = 0$ and $q \wedge y = 0$. This implies $p \in (x]^*$ and $q \in (y]^*$. Therefore $r \in (x]^* \vee (y]^*$ and so $(x]^* \vee (y]^* = S$. □

For non-empty subsets A and B of a nearlattice S, $< A, B >$ denotes $\{ x \in S ; x \wedge a \in B \text{ for all } a \in A \}$. $< a, b >$ denotes $< \{a\}, \{b\} >$. As observed by Mandelker [33] $< a, b >$ is an ideal due to distributivity of S. When A and B are ideals clearly $< A, B >$ is an ideal. Moreover, $< (a], (b] > = < a, b >$. For any ideal J of S and $x \in S$ we write $< x, J > = \{ y \in S \mid x \wedge y \in J \}$. The following lemma summarizes some useful informations.

Lemma 3.4. Let S be a nearlattice. Then the following hold.

(i) $\langle x, J \rangle = \bigvee_{y \in J} \langle x, y \rangle$, the supremum of ideals $\langle x, y \rangle$ in the lattice of ideals of S, for any $x \in S$ and any ideal J in S.

(ii) $\{ \langle x, a \rangle \vee \langle y, a \rangle \} \cap [a, b] = \{ \langle x, a \rangle \cap [a, b] \vee \{ \langle y, a \rangle \cap [a, b] \}$, for any $x, y \in [a, b]$, $a < b$.

Proof: (i). Let $p \in \langle x,y \rangle$ where $y \in J$.

Then $p \wedge x \leq y$

$\Longrightarrow p \wedge x \in J \Longrightarrow p \in \langle x, J \rangle$

$\Longrightarrow \langle x, y \rangle \sqsubset \langle x, J \rangle$

$\Longrightarrow \bigvee_{y \in J} \langle x, y \rangle \sqsubset \langle x, J \rangle$

Suppose $t \in \langle x, J \rangle \Longrightarrow t \wedge x \in J$.

Now $t \in \langle x, t \wedge x \rangle$ where $t \wedge x \in J$.

Hence $t \in \bigvee_{y \in J} \langle x, y \rangle$ and so (i) holds.

(ii) Let z be a member of the left hand side of (ii). Then $a \leq z = c \vee d \leq b$ with $c \wedge x \leq a$ and $d \wedge y \leq a$. Then $(c \vee a) \wedge x = (c \wedge x) \vee (a \wedge x)$ $= (c \wedge x) \vee a \leq a \vee a \leq a$ and similarly $(d \vee a) \wedge y \leq a$. Thus $c \vee a \in \langle x, a \rangle \cap [a, b]$ and $(d \vee a) \in \langle y, a \rangle \cap [a, b]$, so $z = (c \vee a) \vee (d \vee a)$ is

a member of the right hand side of (ii). The reverse inequlity is clear and (ii) follows. □

The following theorem gives a characterization of a relatively normal nearlattice which is also a generalization of cornish [8, Th. 3.7].

Theorm 3.5. Let S be a distruibutive nearlattice. The following conditions are equivalent.

(i) S is relatively normal.

(ii) For all $x, y \in S$ $\langle x, y \rangle \vee \langle y, x \rangle = S$

(iii) For all $x, y, z \in S$,

$$\langle x \wedge y, z \rangle = \langle x, z \rangle \vee \langle y, z \rangle.$$

(iv) For any ideal J of S

$$\langle x \wedge y, J \rangle = \langle x, J \rangle \vee \langle y, J \rangle.$$

Proof: (i) implies (ii). Let $x, y \in S$. For any $a \in S$, consider $I = [\, x \wedge y \wedge a, a \,]$ in S. Now, $x \wedge y \wedge a = (x \wedge a) \wedge (y \wedge a)$. Since I is normal, so by 3.2.10 there exist $r, s \in I$ such that $x \wedge a \wedge r = x \wedge y \wedge a = y \wedge a \wedge s$ and $r \vee s = a$. Since $r, s \leq a$, we have $x \wedge y \wedge a = x \wedge r = y \wedge s$. Thus $x \wedge r \leq y$ and $y \wedge s \leq x$. This implies $a = r \vee s \in \langle x, y \rangle \vee \langle y, x \rangle$ and (ii) holds.

(ii) implies (iii). Suppose $b \in \langle x \wedge y, z \rangle$. Then by (ii) $b = c \vee d$ where $c \in \langle x, y \rangle$ and $d \in \langle y, x \rangle$. Thus $x \wedge c = x \wedge y \wedge c \leq x \wedge y \wedge b \leq z$. Hence $c \in \langle x, z \rangle$. Similarly $d \in \langle y, z \rangle$. It follows that $b = c \vee d \in \langle x, z \rangle \vee \langle y, z \rangle$.

The reverse inequality always holds and so, **(iii)** is established.

(iii) ===> (i). Let $a, b \in S$, $(a < b)$. Suppose $x, y \in [a, b]$. Such that $x \wedge y = a$. Then by (iii)

$$[a, b] \cap (\langle x, a \rangle \vee \langle y, a \rangle) = [a, b] \cap \langle x \wedge y, a \rangle$$
$$= [a, b] \cap \langle a, a \rangle$$
$$= [a, b]$$

Hence by 3.3.4 and 3.2.10 S is relatively normal.

(iv) ====> (iii) is trivial as

$$\langle x \wedge y, z \rangle = \langle x \wedge y, (z] \rangle.$$

(iii) ===> (iv). By lemma 3.3.4 (i)

$$\langle x \wedge y, J \rangle = \bigvee_{t \in J} \langle x \wedge y, t \rangle.$$

$= \bigvee_{t \in J} (\langle x, t \rangle \vee \langle y, t \rangle)$. Then applying lemma 3.3.4 (i) again, $\langle x \wedge y, J \rangle = \langle x, J \rangle \vee \langle y, J \rangle$. i.e., (iv) holds. □

Theorem 3.6. Let a , b and c be arbitrary elements of a nearlattice S. Let A,B and C be arbitrary ideals in S. Then the following are equivalent.

(i) $\langle c, a \vee b \rangle = \langle c, a \rangle \vee \langle c, b \rangle$ whenever $a \vee b$ exists.

(ii) $\langle C, A \vee B \rangle = \langle C, A \rangle \vee \langle C, B \rangle$

Proof: (i) ===> (ii). Let $t \in \langle C, A \vee B \rangle$. Then for any $c \in C$, $t \wedge c \in A \vee B$. Thus $t \wedge c = p \vee q$ for some $p \in A$ and $q \in B$. This implies $t \in \langle c, p \vee q \rangle = \langle c, p \rangle \vee \langle c, q \rangle$ by (i), $\sqsubset \langle C, A \rangle \vee \langle C, B \rangle$.

i.e., $\langle C, A \vee B \rangle \sqsubset \langle C, A \rangle \vee \langle C, B \rangle$

Reverse inequality is trivial.So (ii) holds.

(ii) ====> (i). Let $a, b, c \in S$ with $a \vee b$ exists, then

$$\langle c, a \vee b \rangle = \langle (c], (a \vee b] \rangle$$
$$= \langle (c], (a] \vee (b] \rangle$$
$$= \langle (c], (a] \rangle \vee \langle (c], (b] \rangle$$
$$= \langle c, a \rangle \vee \langle c, b \rangle.$$ ∎

Lemma 3.7. A distributive nearlattice S is relatively complemented if and only if for all $x, y \in S$, $(x] \vee \langle x, y \rangle = S$, where $\langle x, y \rangle = \{ z \in S ; z \wedge x \leq y \}$.

Proof: Suppose S is relatively complemented. For $x, y, z \in S$. Consider the interval $[\, x \wedge y \wedge z, z\,]$. Let w be the relative complement of $x \wedge z$ in $[\, x \wedge y \wedge z, z]$. Then $x \wedge z \wedge w = x \wedge y \wedge z$ and $(x \wedge z) \vee w = z$. Now $x \wedge z \wedge w = x \wedge y \wedge z \leq y$ implies $z \wedge w \in \langle x, y \rangle$. Hence $z = (x \wedge z) \vee w = (x \wedge z) \vee (w \wedge z) \in (x] \vee \langle x, y \rangle$.

Conversely, let $c \in [\, a, b\,]$, $a \leq b$. Then $b \in (c] \vee \langle c, a \rangle = S$ and so $b = c \vee d$, $d \in \langle c, a \rangle$. Then $d \wedge c \leq a$ and so $(d \vee a) \wedge b$ is the relative complement of c in [a, b]. Here $d \vee a$ exists by the upper bound property as both $d, a \leq b$. □

Lemma 3.8. The set of all prime ideals of a distributive nearlattice S is unordered if and only if for all x, y in S, $(x] \vee \langle x, y \rangle = S$.

Proof: Suppose the prime ideal are unordered and there exist $x, y \in S$ such that $(x] \vee \langle x, y \rangle \neq S$. Therefore $(x] \vee \langle x, y \rangle \sqsubset P$ for some prime ideal P. Since the primes are unordered, S-P is a maximal filter. But $x \notin S\text{-}P$ and hence $[x) \vee (S\text{-}P) = S$ and so $y \in [x) \vee (S\text{-}P)$.

Therefore $y = x_1 \wedge q$ for some $x_1 \geq x$ and $q \in S\text{-}P$. Then $x \wedge q \leq x_1 \wedge q = y$ and so $q \in \langle x, y \rangle \sqsubset P$ which is

a contradiction.

Conversely, suppose (x] $\vee$ < x, y > = S for all x, y in S. Let P and Q be primes such that P $\subset$ Q and P $\neq$ Q. Choose a $\in$ Q-P and b $\in$ P. Now, (a] $\wedge$ < a, b > = (a $\wedge$ b] and b $\in$ P implies a $\wedge$ b $\in$ P. Thus (a] $\wedge$ < a, b > $\sqsubset$ P and a $\notin$ p. This implies < a, b > $\sqsubset$ P as P is prime. Therefore < a, b > $\sqsubset$ Q and (a] $\sqsubset$ Q and so, S = (a] $\vee$ < a, b > $\sqsubset$ Q. Which is a contradiction. □

Corollary. 3.9. (Gratzer and Schmidt [21a].

A distributive nearlattice S is relatively complemented if and only if its prime ideals are unordered. □

Following theorem generalizes Th-3.5, Th.3.7, and Th.4.3 of Cornish [8] also c.f. [57 , section 5, p-83] and Mandelker [33, Th.4, p-380].

Theorem 3.10. let S be a distributive nearlattice. The following conditions are equivalent.

(i) S is relatively normal.

(ii) The set of all prime ideals contained in a prime ideal is a chain.

(iii) Any two incomparable prime ideals are comaximal.

(iv) The set of all prime filters of S containing a prime filter is a chain.

(v) S_F is a chain for each prime filter F of S.

Proof: (i) ====> (ii). Suppose (i) holds. Then by Th. 3.3.5 $< x, y > \vee < y, x > = S$, for all $x, y \in S$. If (ii) does not hold, then there exist prime ideals P, Q, R with $P \supset Q, R$; and Q and R are incomparable. Let $x \in Q-R$ and $y \in R-Q$. Then $< x, y > \sqsubset R$ and $< y, x > \sqsubset Q$.
Thus $S = < x, y > \vee < y, x > \sqsubset Q \vee R \sqsubset P \neq S$, which is a contradiction. Hence (ii) holds.

(ii) <===> (iii) is trivial.

(ii) <===> (iv) is also trivial.

(iv) ===> (v). Suppose (iv) holds. Then by 3.2.7 the prime filters of S_F form a chain for any prime filter F of S. But, in a distributive lattice if the set of prime filters form a chain, then the lattice itself is a chain. Therefore S_F is a chain for each prime filter F of S.

(v) ===> (i). let F be any prime filter of S. By (v)

S_F is a chain, and so for any x, y in S, we have either $\mathbb{T}_F(x) \leq \mathbb{T}_F(y)$ or $\mathbb{T}_F(y) \leq \mathbb{T}_F(x)$. In either case, $\langle \mathbb{T}_F(x), \mathbb{T}_F(y) \rangle \vee \langle \mathbb{T}_F(y), \mathbb{T}_F(x) \rangle = S_F$ i.e., $\mathbb{T}_F(\langle x, y \rangle \vee \langle y, x \rangle) = \mathbb{T}_F(S)$, and so by the principle of localization, $\langle x, y \rangle \vee \langle y, x \rangle = S$. Hence by Th.3.3.5, S is relatively normal. □

Theorem 3.11. If F is a filter in a relatively normal nearlattice, then $S/\mathbb{T}(F)$ is relatively normal.

Proof: Suppose S is relatively normal.

Let $\mathbb{T}_F(x), \mathbb{T}_F(y) \in S_F$.

Then by 3.2.9, $\langle \mathbb{T}_F(x), \mathbb{T}_F(y) \rangle \vee \langle \mathbb{T}_F(y), \mathbb{T}_F(x) \rangle$

$= \mathbb{T}_F\langle x, y \rangle \vee \mathbb{T}_F\langle y, x \rangle$

$= \mathbb{T}_F[\langle x, y \rangle \vee \langle y, x \rangle]$

$= \mathbb{T}_F(S)$ as S is relatively normal.

$= S_F$

Hence by theorem 3.3.5 S_F is relatively normal. □

4. n- Normal nearlattices.

Recall that an n-normal nearlattice is a distributive nearlattice with 0 such that each prime ideal contains at most n minimal prime ideals. Equivalently a distributive nearlattice with 0 is n-normal if each prime filter is contained in at most n ultrafilters.

n-Normal lattices have been studied by Cornish in [9] and Davey in [16]. Davey called these lattices as B_n-lattices. To prove our main result we need the following lemma 4.1 which is an extension of 2.3 of Cornish [9]. Since the proof of the lemma follows easily from Cornish's proof, we omit details.

Lemma 4.1 Let J be an ideal of a distributive nearlattice S. For a given positive integer $n > 1$, the following conditions are equivalent.

(i) For any $x_0, x_1, \ldots\ldots, x_n \in S$, which are "pairwise in J" i.e. $x_i \wedge x_j \in J$ for any $i \neq j$, there exists k such that $x_k \in J$.

(ii) J is the intersection of at most n distinct prime ideals. □

Following theorem provides a characterization of n-normal nearlattices which also generalizes some of the results of Cornish [9] and Davey [16].

Theorem 4.2. For a distributive nearlattice S with 0, the following conditions are equivalent:

(i) Each prime filter of S is contained in at most n ultrafilters of S, i.e. S is n-normal.

(ii) For any $x_0, x_1, \ldots\ldots, x_n \in S$ such that $x_i \wedge x_j = 0$ for $(i \neq j)$, $i = 0,1,2,\ldots\ldots,n$; $j = 0,1,2,\ldots,n$, $(x_0]^* \vee (x_1]^* \vee \ldots \vee (x_n]^* = S$.

(iii) For any distinct n+1 minimal prime ideals $P_0, P_1, \ldots\ldots, P_n$, $P_0 \vee P_1 \vee \ldots \vee P_n = S$.

Proof. (i) implies (ii). Suppose (i) holds. Then by 3.2.7 for any prime filter F of S, S_F has at most n ultrafilters and so S_F has atmost n-minimal prime ideals. Since every ideal is the intersection of all of its minimal prime divisors, the zero ideal of S_F is the intersection of at most n minimal(distinct) prime ideals.

Now, let $x_0, x_1, \ldots, x_n \in S$ be such that $x_i \wedge x_j = 0$ for $i \neq j$, $i = 0,1,\ldots,n$, $j = 0,1,\ldots,n$. Then $\Psi_F(x_i) \wedge \Psi_F(x_j) = \bar{o}$ (zero of S_F), for $i \neq j$. Hence by lemma 4.1 above, there exists k, $0 \leq k \leq n$ such that $\Psi_F(x_k) = 0$. Consequently, $(\Psi_F(x_k))^* = S_F$. Then

$$\Psi_F((x_0]^* \vee (x_1]^* \vee \ldots\ldots\ldots \vee (x_n]^*)$$
$$= \Psi_F(x_0]^* \vee \ldots\ldots\ldots \vee \Psi_F(x_n]^*$$
$$= (\Psi_F(x_0))]^* \vee \ldots\ldots \vee (\Psi_F(x_n))]^* = S_F = \Psi_F(S).$$

Thus by 3.2.8 $(x_0]^* \vee (x_i]^* \vee \ldots\ldots \vee (x_i]^* = S$.

(ii) ====> (i). Suppose (ii) holds and F is any prime filter of S. If (i) does not hold then let $F \sqsubset Q_0, \ldots\ldots, Q_n$, where Q_i are ultrafilters of S. Notice that $Q_i \vee Q_j = S$ for $i \neq j$. Thus for each Q_i, Q_j ; $i \neq j$. there exist $x_i \in Q_i$ and $x_j \in Q_j$ such that $x_i \wedge x_j = 0$. Then it is not hard to find elements $y_0, y_1, \ldots\ldots\ldots, y_n$ with $y_i \in Q_i$, such that $y_i \wedge y_j = 0$ whenever $i \neq j$. Then by (ii), $(y_0]^* \vee (y_1]^* \vee \ldots\ldots \vee (y_n]^* = S$.

Now, if $t \in (y_k]^*$ for some k ; $0 \leq k \leq n$, then $t \wedge y_k = 0$. This implies $t \notin Q_k$, otherwise $0 \in Q_k$ as $y_k \in Q_k$. Thus $t \in S - Q_k \sqsubset S-F$, and so $(y_k]^* \sqsubset S-F$ for each $k : 0 \leq k \leq n$. Hence $S = (y_0]^* \vee (y_1]^* \vee \ldots\ldots \vee (y_n]^* \sqsubset S-F$, which is a contradiction. Therefore, (i) holds.

(i) ====> (iii). Suppose (i) holds and $P_0 \vee P_1 \vee \ldots\ldots \vee P_n \neq S$. Since each proper ideal in a distributive nearlattice is contained in some prime ideal, there exists a prime ideal P of S containing $P_0 \vee P_1 \vee \ldots\ldots \vee P_n$. Then S-P is a prime filter which is contained in n+1 ultrafilters $S-P_0, \ldots\ldots, S-P_n$. This contradicts (i) and so $P_0 \vee P_1 \vee \ldots\ldots \vee P_n = S$.

(iii) =====> (i). Suppose (iii) holds. If (i) does not hold, there exists a prime filter F which is contained in atleast n+1 ultrafilters $Q_0, \ldots\ldots\ldots, Q_n$ (say) of S. Then $S-Q_0, \ldots\ldots\ldots, S-Q_n$ are n+1 distinct minimal prime ideals of S and $(S-Q_0) \vee, \ldots, \vee (S-Q_n) \sqsubset S-F$, which is a contradiction to (iii). Therefore (i) holds. □

Notice that the above theorem plays an important role in case of pseudocomplemented lattices. For the class of pseudocomplemented B_n-lattices (ii) of the above theorem reduces to the condition of Gratzer and Lakser [21, lemma 8].

Definition. Let S be a nearlattice with 0, S is called sectionally n-normal for $n > 1$ if each initial segment $[0,x]$, $x \in S$ is an n-normal lattice.

Following result generalizes theorem 3.6 of Cornish [9]. Here proof of (i) ====> (ii) is bit tricky as the nearlattices are not that well behaved like lattices, while the rest follows easily from Cornish's proof.

Theorem 4.3. For a nearlattice S with 0 the following conditions are equivalent.

(i) S is sectionally n-normal.

(ii) S is n-normal.

(iii) Each ideal J in S is an n-normal subnearlattice.

Proof. (i) =====> (ii). Suppose that (i) holds. Let $x_0, x_1, \ldots\ldots, x_n \in S$ be such that $x_i \wedge x_j = 0$ for $i \neq j$. Choose any $y \in S$. Consider $I = [0,y]$. Now $y \wedge x_0, y \wedge x_1, \ldots\ldots\ldots, y \wedge x_n \in I$ and $(y \wedge x_i) \wedge (y \wedge x_j) = y \wedge (x_i \wedge x_j) = 0$ for $i \neq j$. Since I is n-normal, $I = (y \wedge x_0]^{\dagger} \vee \ldots\ldots \vee (y \wedge x_n]^{\dagger}$, by 3.4.2. Where $(y \wedge x_i]^{\dagger} = \{ t \in I : t \wedge y \wedge x_i = 0 \}$. So $y \in (y \wedge x_0]^{\dagger} \vee \ldots\ldots \vee (y \wedge x_n]^{\dagger}$, and hence $y \in (y \wedge x_0]^{\ddagger} \vee \ldots\ldots \vee (y \wedge x_n]^{\ddagger}$. Thus $y = t_0 \vee \ldots\ldots \vee t_n$

where $t_i \in (y \wedge x_i]^*$. Then $t_i \wedge y \wedge x_i = 0$, and so $t_i \wedge x_i = 0$ as $t_i \wedge y = t_i$. This implies $t_i \in (x_i]^*$, and so $y = t_0 \vee \ldots\ldots \vee t_n \in (x_0]^* \vee \ldots\ldots \vee (x_n]^*$. Hence, $(x_0]^* \vee \ldots\ldots \vee (x_n]^* = S$, and so by 3.4.2, S is n-normal.

(ii) =====> (iii). Let J be an ideal in S and suppose $x_0, x_1, \ldots\ldots, x_n \in J$ are such that $x_i \wedge x_j = 0$ for all $i \neq j$. Let $(x_i]^* = \{ y \in J : y \wedge x_i = 0 \}$. Clearly, $(x_i]^* = (x_i]^* \cap J$ By 3.4.2, $(x_0]^* \vee \ldots\ldots \vee (x_n]^* = S$ and so

$$J = J \cap S = J \cap ((x_0]^* \vee \ldots\ldots \vee (x_n]^*)$$
$$= (J \cap (x_0]^*) \vee \ldots\ldots \vee (J \cap (x_n])^*$$
$$= (x_0]^* \vee \ldots\ldots \vee (x_n]^*.$$

Consequently, J is n-normal.

(iii) ====> (i) is trivial. □

Following theorem extends theorem 3.5 of Davey [16].

Theorem 4.4. For a distributive nearlattice S with 0 the following hold.

(i) If S is n-normal, then S_F is n-normal for any filter F of S.

(ii) S is n-normal if and only if for each prime filter F of S, S_F has at most n minimal prime ideals.

Proof (i). Let $\Psi_F(x_0), \ldots\ldots, \Psi_F(x_n) \in S_F$ be such that $\Psi_F(x_i) \wedge \Psi_F(x_j) = \bar{0}$ for all $i \neq j$, $i = 0, \ldots, n$, $j = 0, \ldots, n$. Then $x_i \wedge x_j \equiv 0 \ (\Psi_F)$ for each

i, j $(i \neq j)$. This implies

$x_i \wedge x_j \wedge f_{ij} = 0$ for some $f_{ij} \in F$. Set $f = \bigwedge_{i \neq j} f_{ij}$, where $i = 0,\ldots\ldots,n$; $j = 0,\ldots\ldots,n$. Then $x_i \wedge x_j \wedge f = 0$.

Since S is n-normal so by theorem 3.4.2,

$$(x_0 \wedge f]^* \vee (x_1 \wedge f]^* \vee \ldots\ldots\ldots \vee (x_n \wedge f]^* = S.$$

Hence $(\Psi_F(x)]^* \vee (\Psi_F(x_1)]^* \vee \ldots\ldots \vee (\Psi_F(x_n)]^*$

$$= (\Psi_F(x_0 \wedge f)]^* \vee (\Psi_F(x_1 \wedge f)]^* \vee \ldots$$

$$\ldots\ldots \vee (\Psi_F(x_n \wedge f)]^*$$

$$= \Psi_F((x_0 \wedge f]^*) \vee \Psi_F((x_1 \wedge f]^*) \vee \ldots\ldots$$

$$\ldots \vee \Psi_F((x_n \wedge f]^*)$$

by 3.2.9

$$= \Psi_F[(x_0 \wedge f]^* \vee \ldots\ldots\ldots \vee (x_n \wedge f]^*]$$

$$= \Psi_F(S) = S_F .$$

Hence S_F is n-normal by 3.4.2.

(ii). This is trivial by Th.3.2.7. □

We conclude this section with the following result which was given by Cornish in [9] and Davey in [16] for lattices.

Theorem 4.5. For any n+1 elements $x_0, x_1, \ldots\ldots, x_n$ in an n-normal nearlattice S

$$(x_0 \wedge \ldots\ldots \wedge x_n]^{\ddagger} = \bigvee_{0 \leq i \leq n} (x_0 \wedge \ldots\ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots\ldots \wedge x_n]^{\ddagger}$$

Proof. Let $b_i = x_0 \wedge \ldots\ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots\ldots \wedge x_n$ for each $0 \leq i \leq n$. Suppose that $x \in (x_0 \wedge \ldots\ldots\ldots \wedge x_n]^{\ddagger}$ Then $x \wedge x_0 \wedge \ldots\ldots\ldots\ldots \wedge x_n = 0$ so that for $i \neq j$, $(x \wedge b_i) \wedge (x \wedge b_j) = 0$. From the Theorem 3.4.2 $x \in (x \wedge b_0]^{\ddagger} \vee \ldots\ldots \vee (x \wedge b_n]^{\ddagger}$ so that $x = a_0 \vee \ldots\ldots \vee a_n$, for some $a_i \in S$, such that $a_i \wedge x \wedge b_i = 0$. Then $x = (a_0 \wedge x) \vee \ldots\ldots \vee (a_n \wedge x)$ and $a_i \wedge x \in (b_i]^{\ddagger}$ and so

$$(x_0 \wedge \ldots\ldots \wedge x_n]^{\ddagger} \sqsubseteq \bigvee_{0 \leq i \leq n} (x_0 \wedge \ldots\ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots\ldots \wedge x_n]^{\ddagger}$$

The reverse inclusion is trivial. □

5. Relatively n-normal nearlattices.

Recall that a relatively n- normal nearlattice S is a distributive nearlattice such that for each $x < y$ $(x, y \in S)$ $[x, y]$ is an n-normal lattice. For relatively n- normal lattices we refer the reader to consult Cornish [9] while Davey [16] preferred to call them as relative B_n -lattices.

We start this section with the following characterization of relatively n-normal nearlattices which will be needed in our next theorem.

Theorem 5.1. Let S be a distributive nearlattice, the following conditions are equivalent.

(i) S is relatively n-normal.

(ii) For all $x_0, x_1, \ldots\ldots\ldots, x_n \in S$,

$$< x_1 \wedge x_2 \wedge \ldots\ldots \wedge x_n, x_0 > \vee < x_0 \wedge x_2 \wedge \ldots$$
$$\ldots \wedge x_n, x_1 > \vee \ldots\ldots\ldots \vee < x_0 \wedge x_1 \wedge \ldots\ldots$$
$$\ldots\ldots \wedge x_{n-1}, x_n > = S.$$

(iii) For all $x_0, x_1, \ldots\ldots\ldots, x_n, z \in S$,

$$< x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_n, z > = < x_1 \wedge x_2 \wedge \ldots$$
$$\ldots \wedge x_n, z > \vee < x_0 \wedge x_2 \wedge \ldots \wedge x_n, z > \vee \ldots$$
$$\ldots\ldots \vee < x_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_{n-1}, z >$$

Proof: (i) ====> (ii). Let $a \in S$, consider the interval $I = [\, x_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_n \wedge a,\ a\,]$ in S.

For $0 \leq i \leq n$, the set of elements

$t_i = x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots\ldots \wedge x_n \wedge a$, are obviously pairwise disjoint in the interval I. Since I is n-normal, so by 3.4.2 $(t_0]^{\dagger} \vee (t_1]^{\dagger} \vee \ldots\ldots\ldots \vee (t_n]^{\dagger} = I$, where $(t_i\,]^{\dagger} = (t_i]^{\dagger} \cap I$. Since I is n-normal.

$$(t_0]^{\dagger} \vee \ldots\ldots\ldots \vee (t_n]^{\dagger} = I$$

$$\text{So, } a \in (t_0]^{\dagger} \vee \ldots\ldots\ldots \vee (t_n]^{\dagger}$$

Thus, $a = p_0 \vee \ldots\ldots\ldots \vee p_n$

Where $p_0 \wedge t_0 = p_1 \wedge t_1 = \ldots\ldots\ldots = p_n \wedge t_n$

$= 0$ of I

$= x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_n \wedge a$

Now, $p_0 \wedge t_0 = x_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_n \wedge a$ implies

$p_0 \wedge t_0 \leq x_0$.

Again $p_0 \wedge t_0 = p_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_n \wedge a$

$= p_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_n$ as $p_0 \leq a$.

This implies $p_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_n \leq x_0$ and

so $p_0 \in \langle x_1 \wedge \ldots\ldots\ldots\ldots \wedge x_n,\ x_0 \rangle$

Similarly, $p_1 \in \langle x_0 \wedge x_2 \wedge \ldots\ldots\ldots \wedge x_n,\ x_1 \rangle$

. . . . . .

. . . . . .

$p_n \in \langle x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_{n-1},\ x_n \rangle$.

Therefore $a \in \langle x_1 \wedge \ldots\ldots\ldots \wedge x_n , x_0 \rangle \vee \ldots\ldots$

$\ldots\ldots \vee \langle x_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_{n-1}, x_n \rangle$ and

hence $S = \langle x_1 \wedge \ldots\ldots\ldots \wedge x_n , x_0 \rangle \vee \ldots\ldots$

$\ldots\ldots \vee \langle x_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_{n-1}, x_n \rangle$

(ii) ====> (iii). Suppose $b \in \langle x_0 \wedge \ldots\ldots \wedge x_n , z \rangle$.

Then by (ii) $b = s_0 \vee \ldots\ldots\ldots \vee s_n$,

for some $s_0 \in \langle x_1 \wedge \ldots\ldots \wedge x_n , x_0 \rangle$

$s_1 \in \langle x_0 \wedge x_2 \wedge \ldots\ldots \wedge x_n , x_1 \rangle$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$s_n \in \langle x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_{n-1} , x_n \rangle$.

Thus, $x_1 \wedge \ldots\ldots \wedge x_n \wedge s_0 \leq x_0$

$x_0 \wedge x_2 \wedge \ldots\ldots \wedge x_n \wedge s_1 \leq x_1$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$x_1 \wedge \ldots\ldots \wedge x_{n-1} \wedge s_n \leq x_n$.

Then $x_1 \wedge x_2 \wedge \ldots\ldots \wedge x_n \wedge s_0$

$= x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_n \wedge s_0 \leq x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_n \wedge b \leq z$.

Hence, $s_0 \in \langle x_1 \wedge x_2 \wedge \ldots\ldots \wedge x_n , z \rangle$

Similarly, $s_1 \in \langle x_0 \wedge x_2 \wedge \ldots\ldots \wedge x_n , z \rangle$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$s_n \in \langle x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_{n-1} , z \rangle$.

Therefore b $\in$ $< x_1 \wedge \ldots \wedge x_n , z > \vee < x_0 \wedge x_2 \wedge \ldots \wedge x_n, z > \vee \ldots\ldots \vee < x_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_{n-1} , z >$.

Since the reverse inequality always holds. Therefore $< x_0 \wedge \ldots \wedge x_n , z > = < x_1 \wedge \ldots \wedge x_n, z > \vee < x_0 \wedge x_2 \wedge \ldots \ldots\ldots \wedge x_n, z > \vee \ldots\ldots\ldots \vee < x_0 \wedge \ldots \wedge x_{n-1} , z >$.

(iii) ======> (i). Let a, b $\in$ S , with a < b.

Let $x_0, \ldots, x_n \in [a, b]$ such that

$x_i \wedge x_j = a$ for all $i \neq j$.

Let $d_0 = x_1 \vee x_2 \vee \ldots\ldots \vee x_n$

$d_1 = x_0 \vee x_2 \vee \ldots\ldots \vee x_n$.

.....

.....

$d_n = x_0 \vee x_1 \vee \ldots\ldots \vee x_{n-1}$.

Note that $d_0, d_1, \ldots\ldots, d_n$ exist by the upper bound property of S .Then $a \leq d_i \leq b$ for all i. Now using $x_i \wedge x_j = a$ for all $i \neq j$. We can easily show by some routine calculations that

$x_0 = d_1 \wedge d_2 \wedge \ldots\ldots\ldots \wedge d_n$.

$x_1 = d_0 \wedge d_2 \wedge \ldots\ldots\ldots \wedge d_n$.

.....

.....

$x_n = d_0 \wedge d_1 \wedge \ldots\ldots\ldots \wedge d_{n-1}$.

Then $[a, b] \cap \{ < x_0, a > \vee < x_1, a > \vee \ldots \vee < x_n, a >\}$

$= [a, b] \cap \{ < d_1 \wedge d_2 \wedge \ldots \wedge d_n, a > \vee < d_0 \wedge d_2 \wedge \ldots$

$\ldots\ldots \wedge d_n, a > \vee < d_0 \wedge d_1 \wedge \ldots\ldots \wedge d_{n-1}, a > \}.$

$= [a, b] \cap < d_0 \wedge d_1 \wedge \ldots\ldots\ldots \wedge d_n, a >$ (by (iii))

$= [a, b] \cap < a, a >$

$= [a, b] \cap S = [a, b]$

Hence by 3.4.2 [a, b] is n-normal. Therefore S is relatively n-noraml. □

Following characterization on relatively n-normal nearlattices are extension of some work of Cornish [9] and Davey [16].

Theorem.5.2. For a distributive nearlattice S with 0, the following conditions are equivalent:

(i) S is relatively n-normal.

(ii) For any (n+1) pairwise incomparable prime ideals $P_0, \ldots\ldots, P_n$. $P_0 \vee \ldots\ldots \vee P_n = S$.

(iii) Any prime ideal of S contains at most n mutually incomparable prime ideals.

Proof: (i) <===> (ii). Suppose S is relatively n-normal. Let $P_0, \ldots\ldots\ldots, P_n$ be (n+1) pairwise incomparable prime

ideals. Then there exists $x_0, \ldots\ldots, x_n \in S$, such that

$$x_j \in P_j - \bigcup_{\substack{i=1 \\ i \neq j}}^{n} P_i.$$

Since S is relatively n-normal.

So by theorem 3.5.1 $\langle x_1 \wedge \ldots\ldots\ldots \wedge x_n, x_0 \rangle \vee \langle x_0 \wedge \ldots$ $\ldots\ldots \wedge x_n, x_1 \rangle \vee \ldots\ldots\ldots \vee \langle x_0 \wedge \ldots\ldots\ldots \wedge x_{n-1}, x_n \rangle = S$.

Let $t_0 \in \langle x_1 \wedge \ldots\ldots\ldots \wedge x_n, x_0 \rangle$.

Then $t_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_n \leq x_0 \in P_0$.

Thus $t_0 \wedge x_1 \wedge \ldots\ldots\ldots \wedge x_n \in P_0$. Since P_0 is prime and $x_1 \wedge \ldots\ldots \wedge x_n \notin P_0$, so $t_0 \in P_0$. Therefore $\langle x_1 \wedge \ldots \wedge x_n, x_0 \rangle \sqsubset P_0$.

Similarly $\langle x_0 \wedge \ldots\ldots \wedge x_n, x_1 \rangle \sqsubset P_1$.

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\langle x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_{n-1}, x_n \rangle \sqsubset P_n$.

Hence $P_0 \vee \ldots\ldots\ldots \vee P_n = S$.

Conversely, let any (n+1) pairwise incomparable prime ideals in S are co-maximal. Consider an interval [a, b] of S. Let $P'_0, \ldots\ldots\ldots, P'_n$ be (n+1) distinct minimal prime ideals of [a, b]. Then by 1.4.3 there exists prime ideals $P_0, \ldots\ldots\ldots, P_n$ of S such that

$P'_0 = P_0 \cap [a, b]$,

.....

.....

$P'_n = P_n \cap [a, b]$,

Since $P'_0, \ldots\ldots, P'_n$ are incomparable, so $P_0, \ldots, P_n$ are incomparable. Now by (ii) $P_0 \vee \ldots \vee P_n = S$. Hence, $P'_0 \vee \ldots\ldots \vee P'_n = (P_0 \vee \ldots\ldots \vee P_n) \cap [a, b]$

$= S \cap [a, b]$

$= [a, b]$

Therefore [a, b] is n-normal and so S is relatively n-normal.

(ii) ======> (iii) is trivial.

Finally we extend a result of Davey [16, Th. 3.6].

Theorem. 5.3. If S is a relatively n-normal nearlattice, then S_F is also relatively n-normal for each filter F.

Proof: Suppose S is relatively n-normal.

Choose $\Psi_F (x_0), \ldots\ldots, \Psi_F (x_n) \in S$.

Then $< \Psi_F (x_1) \wedge \ldots\ldots \wedge \Psi_F (x_n), \Psi_F (x_0) >$

$\vee < \Psi_F (x_0) \wedge \Psi_F (x_2) \wedge \ldots \wedge \Psi_F (x_n), \Psi_F (x_1) > \vee \ldots\ldots$

$\ldots\ldots \vee < \Psi_F (x_0) \wedge \Psi_F (x_1) \wedge \ldots \wedge \Psi_F (x_{n-1}), \Psi_F (x_n) >$

$= \Psi_F (< x_1 \wedge x_2 \wedge \ldots \wedge x_n, x_0 > \vee < x_0 \wedge x_2 \wedge \ldots\ldots$

$\ldots \wedge x_n, x_1 > \vee \ldots\ldots \vee < x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_{n-1}, x_n >)$ (by 3.2.9)

$= \Psi_F(S) = S_F$ by theorem 3.5.1.

Therefore by theorem 3.5.1, again, S_F is relatively n-normal. □

CHAPTER - 4

MULTIPLIER EXTENSION OF A DISTIRBUTIVE NEARLATTICE.

1. Introduction.

Multipliers on semilattices and lattices have been previously studied by several authors including Szasz [54] [55], Szasz and Szendrei [56], Kolibiar [29], Cornish [10], and by Nieminen [37] [38] on a lattice. Analogues on multipliers have been studied by many other workers in various branches of algabra ; for references we suggest the readers to consult the bibliographies of Petrich [47] and Cornish [10]. In a more recent paper, Noor and Cornish in [39] studied them on nearlattices.

Let S be a nearlattice and Φ a mapping of S into itself. Then Φ is called a multiplier on S if $\Phi(x \wedge y) = \Phi(x) \wedge y$ for each $x, y \in S$ c.f. [39]. Each multiplier on S has the following properties, $\Phi(x) \leq x$, $\Phi(\Phi(x)) = \Phi(x)$ and $x \leq y$ implies $\Phi(x) \leq \Phi(y)$. Each $a \in S$ induces a multiplier μ_a defined by $\mu_a(x) = a \wedge x$ for each $x \in S$, which is called an inner multiplier. The identity function on S, which will be denoted by ι is always a multiplier. M(S) denotes the set of all multipliers on S. It is obvious that M(S) has a zero, denoted by ω if and only if S has a 0.

In section 1, we have given a description of multipliers on nearlattices. Here we have mentioned several results given by Noor and Cornish [39] and Nieminen [38]. Then we give a categorical result, where we see that the multiplier extension has a functorial character which is entirely diferent from that of Lattice Theory c.f. Cornish [10, Theorem 2.4].

In section 2 we studied multipliers on sectionally pseudocomplemented distributive nearlattices and also on distributive nearlattices which are sectionally in B_n, $-1 \leq n \leq \omega$ and generalized a number of results of [10]. We showed that S is sectionally in B_n if and only if M (S) is in B_n. We also showed that for $1 \leq n < \omega$, above conditions are also equivalent to the condition that S is sectionally pseudocomplemented and for any n+1 minimal prime ideals $P_1, \ldots\ldots\ldots, P_{n+1}$,

$P_1 \vee \ldots\ldots\ldots \vee P_{n+1} = S$.

2. Multipliers on distributive nearlattices.

let S be a nearlattice and Φ a mapping of S into itslf. Recall that Φ is a multiplier on S, if $\Phi (x \wedge y) = \Phi (x) \wedge y$ for each $x , y \in S$. For a multiplier Φ on S, $M_\Phi = \{ x \in S \mid \Phi(x) = x \}$ is clearly and ideal of S. By Szasz [55, Theorem 3] M_Φ determines Φ uniquely.

The following result is due to Niemineen [38, lemma 1]. It is also a generalization of a part of proposition 2.1 of Cornish [10].

Lemma. 2.1. An ideal I of a nearlattice S generates a multiplier Φ on S, that is , $M_\Phi = I$, if and only if for each $a \in S$ there is an element $b \in I$ such that $I \cap (a] = (b]$, and moreover, $b = \Phi (a)$. □

If Φ and λ are multipiers on a nearlattice S, then $\Phi \wedge \lambda$ and $\Phi \vee \lambda$ are defined by $(\Phi \wedge \lambda) (x) = \Phi (x) \wedge \lambda (x)$ and $(\Phi \vee \lambda) (x) = \Phi (x) \vee \lambda (x)$. Notice that $\Phi (x) \vee \lambda (x)$ always exists by the upper bound property of S, as $\Phi (x), \lambda (x) \leq x$, though $\Phi \vee \lambda$ is not necessarily a multiplier. Also, $\Phi (\lambda (x)) = \Phi (\lambda (x \wedge x)) = \Phi (\lambda (x) \wedge x) = \Phi (x) \wedge \lambda (x)$.

As shown by Szasz and Szendrei [56, Theorem 3], M(S) is a meet semilattice.

The following result is due to Nieminen [38].

Proposition 2.2. Let Φ and λ be two multipliers on a nearlattice S. Then, $\Phi \vee \lambda$ is a multiplier on S if and only if $(M_\Phi \vee M_\lambda) \cap (x] = (M_\Phi \cap (x]) \vee (M_\lambda \cap (x])$ for each $x \in S$. □

Next result is due to Noor and Cornish [39]. For the idea of standard ideals in lattices we refer the reader to consult [18] and [21b] , while a complete description of these ideals in nearlattices can be found in [15].

Proposition 2.3. [Noor and Cornish 39, corollary 3.3]. Let Φ be a multiplier on a nearlattice S. The mapping $\Phi \vee \lambda$ is a multiplier on S for each $\lambda \in M(S)$ if and only if M_Φ is a standard ideal of S.

Following result involves the ideas on direct summands of a nearlattice given by Noor and Cornish in [39]. For direct summand of a lattice we suggest the reader to consult F. Maeda and S. Maeda [25] and M.F. Janowitz [27].

Proposition 2.4. [Theorem 3.4. Noor and Cornish 39].

A nearlattice S with 0 has a decomposition into a direct summand if and only if there are at least two multipliers Φ and λ on S such that $\Phi \vee \lambda = \iota$ and $\Phi \wedge \lambda = \omega$, and both Φ and λ have a supremum with each multiplier on S. □

Next theorem is due to Nieminen [38, Theorem 3] also see [39]. This is also a generalization of a part of Cornish [10, Theorem 2.2].

Theorem 2.5. In a nearlattice S, the following conditions are equivalent.

(i) The meet semilattice of all multipliers on S is a lattice (in fact, distributive lattice).

(ii) Each multiplier on S is a join-partial endomorphism of S.

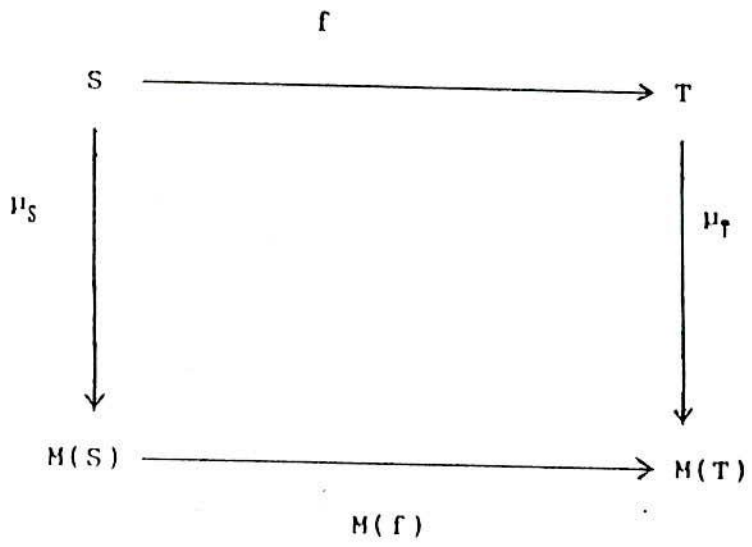
(iii) (x] is a distributive sublattice of S for each $x \in S$. In other words, S is distributive. □

The next result was also mentioned by Nieminen in [38, Theorem 4] without proof. A complete proof of this has been given by Noor and Cornish [39,Theorem 3.6].

Theorem. 2.6. Let S be a nearlattice. Each multiplier Φ on S has the property that $\Phi (\Phi (y) \vee z) = \Phi (y) \vee \Phi (z)$ when $\Phi (y) \vee z$ exists in S, if and only if (x] is a modular sublattice of S for each $x \in S$. □

A subset T of a nearlattice S is called finitely join-dense in S if each $x \in S$ is the join of a finite numbers of its predecessors in T. Now we give the following categorical result.

Theorem. 2.7. Let S and T be distributive nearlattices and $f : S \dashrightarrow T$ be a join-partial homorphism such that f(S) is finitely join dense in T. Then the following diagram is commutative, where $\mu (s) = \mu_s$ and $\mu(t) = \mu_t$



for all $s \in S$ and $t \in T$, and for $\Phi \in M(S)$, $M(f)(\Phi)(t) = f(\Phi(s_1)) \vee \ldots\ldots\vee f(\Phi(s_n))$ for $t \in T$ where $t = f(s_1) \vee\ldots\ldots\vee f(s_n)$; $s_1,\ldots\ldots\ldots,s_n \in S$.

Moreover, $M(f) : M(S)\text{-----}> M(T)$ is an isomorphism when f is one to one.

Proof. Let $t_1 \leq t_2$ in T. Suppose $t_1 = f(a_1) \vee \ldots\vee f(a_n)$ and $t_2 = f(b_1) \vee\ldots\ldots\vee f(b_m)$ where $a_1,\ldots,a_n, b_1,\ldots\ldots,b_m \in S$. Since $t_1 \leq t_2$, so for any $\Phi \in M(S)$, $f(\Phi(a_i)) \leq f(a_i) \leq t_1 \leq t_2$ for all i, $i = 1,\ldots,n$.

Then
$$\begin{aligned} f(\Phi(a_i)) &= f(\Phi(a_i)) \wedge t_2 \\ &= f(\Phi(a_i)) \wedge (f(b_1) \vee\ldots\vee f(b_m)) \\ &= (f(\Phi(a_i)) \wedge f(b_1)) \vee\ldots\ldots\vee (f(\Phi(a_i)) \wedge f(b_m)) \\ &= f(\Phi(a_i) \wedge b_1) \vee \ldots\ldots\vee f(\Phi(a_i) \wedge b_m) \\ &= f(a_i \wedge \Phi(b_1)) \vee\ldots\ldots\ldots \vee f(a_i \wedge \Phi(b_m)) \\ &= f(a_i) \wedge [\, f(\Phi(b_1)) \vee\ldots\ldots\ldots\ldots\vee f(\Phi(b_m))] \\ &= f(a_i) \wedge M(f)(\Phi)(t_2). \end{aligned}$$

That is, $f(\Phi(a_i)) \leq M(f)(\Phi)(t_2)$ for each i ; $i = 1,\ldots\ldots,n$. Thus, $M(f)(\Phi)(t_1) \leq M(f)(\Phi)(t_2)$ and hence $M(f)(\Phi)$ is well defined for every $\Phi \in M(S)$. Also, it can be easily seen $M(f)(\Phi)$ is a multiplier on T.

Now for any $a \in S$, $M(f)(\mu)(a) = M(f)(\mu_a)$. Then for

any $t \in T$, $M(f)(\mu_a)(t) = f(\mu_a(s_1)) \vee \ldots\ldots\ldots \vee f(\mu_a(s_n))$, where $t = f(s_1) \vee \ldots\ldots\ldots \vee f(s_n)$; $s_1, \ldots\ldots, s_n \in S$.

Thus, $M(f)(\mu_a)(t) = f(a \wedge s_1) \vee \ldots\ldots\ldots \vee f(a \wedge s_n)$

$= f(a) \wedge [\, f(s_1) \vee \ldots\ldots\ldots \vee f(s_n) \,]$

$= f(a) \wedge t = \mu_{f(a)}(t) = \mu(f(a))(t)$.

Thus, $M(f)(\mu)(a) = \mu f(a)$, i,e., the diagram is commulative.

Finally, suppose f is 1-1. Then without loss of generality we can regard S as a finitely join-dense subset of T. Define $M(f)^{-1} : M(T) \dashrightarrow M(S)$ by $M(f)^{-1}(\lambda) = \lambda|_S$ (restriction to S). Here $M(f)^{-1}$ is clearly isotone. Now, $M(f)^{-1}(M(f)(\Phi))(s) = M(f)(\Phi)|_S(s)$ $= \Phi(s)$ for all $s \in S$. That is $M(f)^{-1} M(f) = id_{M(S)}$.

Again, for $a \in T$, if $a = a_1 \vee \ldots\ldots\ldots \vee a_n$ with $a_i \in S$, then $(M(f)\, M(f)^{-1})(\lambda)(a) = M(f)(M(f)^{-1}(\lambda))(a)$

$= M(f)^{-1}(\lambda)(a_1) \vee \ldots\ldots\ldots \vee M(f)^{-1}(\lambda)(a_n)$

$= \lambda(a_1) \vee \ldots\ldots \vee \lambda(a_n) = \lambda(a_1 \vee \ldots\ldots \vee a_n)$

(by 4.2.5)

$= \lambda(a)$. Thus, $M(f)M(f)^{-1} = id_{M(T)}$ and hence M(f) is an isomorphism. □

We refer a join-partial homorphism of the Theorem 2.7 above, as finitely join-dense homomorphism. Now, suppose S is a distributive nearlattice. Notice that the

map ϵ ; S----> $\mathfrak{I}_f$ (S) (the lattice of finitely generated ideals of S) defined by $\epsilon(s) = (s]$ is clearly a monomorphism. Also, it is easily seen that $\epsilon(S)$ is finitely join dense in $I_f(S)$. Thus, we have the following result which is trivial from 4.2.7.

Corollary. 2.8. For a distributive nearlattice S, M(S) is isomorphic to $M(J_f(S))$. □

Remark 2.9 : Suppose f : S ------> T and g : T -----> R are two finitely join-dense homomorphisms (S, R, T are distributive nearlattices). Let $r \in R$ and $\Phi \in M(S)$, and so $M(f)(\Phi) \in M(T)$. Then, $r = g(t_1) \vee \ldots\ldots \vee g(t_n)$ where $t_i \in T$, whereby each $t_i = f(s_{1i}) \vee \ldots\ldots \vee f(s_{n_i i})$ for suitable $s_{1i}, \ldots, s_{n_i i}$ in S; i = 1,.....,m. Here, it is not hard to see that gf is also finitely join-dense.

Now, as $\Phi(s_{ji}) \leq s_{ji}$ for all $j = 1, \ldots\ldots, n_i$, $i = 1, \ldots, m$, $f(\Phi(s_{1i})) \vee \ldots\ldots\ldots \vee f(\Phi(s_{n_i i}))$ exists in T for all i = 1,.,m, and is equal to $M(f)(\Phi)(t_i)$. But, $M(gf)(\Phi)(r) = \bigvee_i [(gf)(\Phi(s_{1i})) \vee \ldots \vee (gf)(\Phi(s_{n_i i}))] = \bigvee_i g[f(\Phi(s_{1i})) \vee \ldots\ldots\ldots \vee f(\Phi(s_{n_i i}))] = \bigvee_i g(M(f)(\Phi)(t_i)) = M(g)(M(f)(\Phi))(r)$.

Hence, $M(gf)(\Phi) = M(g)(M(f)(\Phi))$ as r is arbitrary in S.

Since Φ is also arbitrary in M(S), M(gf) = M(g)M(f). This shows that M is a functor (which is different from that of Lattice Theory, c.f. Cornish [10, Theorem 2.4]) from the category *A* to the category *B*. The objects of *A* are distributive nearlattices and the morphisms are the join-partial homomorphisms such that if f : S----->T (f, S, T $\in$ *A*), then f (S) is finitely join-dense in T. On the other hand, *B* contains distributive lattices as its objects and the morphisms are usual lattice homomorphisms.

In conclusion, we would like to note that in the commuting diagram of 4.2.7, μ is not a natural transformation, as it does not have finitely join-dense components.

3. Multipliers on distributive nearlattices which are sectionally in B_n.

Lee in [31] has determined the lattice of all equational subclasses of the class of all pseudocomplemented distributive lattices. They are given by $B_{-1} \subset B_0 \subset B_1 \subset \ldots\ldots \subset B_n \subset \ldots \subset B_\omega$ where all the inclusions are proper and B_ω is the class of all pseudocomplemented distributive lattices, B_{-1} consists of all one element algabras, B_0 is the variety of Boolean algabras while B_n, for $1 \leq n < \omega$ consists of all algabras satisfying the equation

$$(x_1 \wedge x_2 \wedge \ldots\ldots\ldots \wedge x_n)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots\ldots\ldots \wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge \ldots\ldots\ldots \wedge x_n)^* = 1$$

where x^* denotes the pseudocomplement of x. Thus B_1 consists of all stone algabras.

A distributive nearlattice S with 0 is called sectionally pseudocomplemented if each interval $[0,x]$, $x \in S$ is pseudocomplemented. Moreover, S is said to be sectionally in B_n, $-1 \leq n \leq \omega$, if each interval $[0,x]$, $x \in S$ is in B_n.

Proposition 3.1, proposition 3.2 and Theorem 3.4 were proved by Cornish in [10] for lattices. Here we extend those results for nearlattices.

Proposition 3.1. If S is a sectionally pseudocomplemented distributive nearlattice with 0, then M(S) is pseudocomplemented.

Proof. For each $\sigma \in M(S)$ and $x \in S$, $\sigma(x) \in [0,x]$. Suppose $\sigma(x)^*$ denotes the pseudocomplemented of $\sigma(x)$ in $[0,x]$. Define $\sigma^* : S \dashrightarrow S$ by $\sigma^*(x) = \sigma(x)^*$ for each $x \in S$. If $a, b \in S$, then $(\sigma^*(a) \wedge b) \wedge (\sigma(a \wedge b))$ $= \sigma^*(a) \wedge b \wedge \sigma(a) \wedge b = 0$ implies $\sigma^*(a) \wedge b \leq \sigma(a \wedge b)^*$ $= \sigma^*(a \wedge b)$. On the other hand, $\sigma^*(a \wedge b) \wedge \sigma(a)$ $= \sigma(a \wedge b)^* \wedge \sigma(a) = \sigma(a \wedge b)^* \wedge \sigma(a) \wedge b$ $= \sigma(a \wedge b)^* \wedge \sigma(a \wedge b) = 0$ implies $\sigma^*(a \wedge b) \leq \sigma(a)^* = \sigma^*(a)$. Since $\sigma^*(a \wedge b) \leq b$, so $\sigma^*(a \wedge b) \leq \sigma^*(a) \wedge b$. Therefore, $\sigma^*(a \wedge b) = \sigma^*(a) \wedge b$, and so $\sigma^* \in M(S)$.

Now $(\sigma \wedge \sigma^*)(x) = \sigma(x) \wedge \sigma^*(x) = 0 = \omega(x)$ implies $\sigma \wedge \sigma^* = \omega$. If $\sigma \wedge \tau = \omega$, then $\sigma(x) \wedge \tau(x) = 0$ for each $x \in S$. Since $\sigma(x), \tau(x) \in [0,x]$, so $\tau(x) \leq \sigma(x)^* = \sigma^*(x)$. This implies $\tau \leq \sigma^*$, and so σ^* is the pseudocomplement of σ in M(S). Therefore, M(S) is pseudocomplemented. □

Proposition 3.2. For a distributive nearlattice S with 0, if M(S) is pesudocomplemented then S is sectionally pseudocomplemented.

Moreover, for each $\sigma \in M(S)$ and $x \in S$ $\sigma^*(x)$ is the relative pseudocomplemented of $\sigma(x)$ in $[0, x]$.

Proof. Consider any interval $[0,y]$ in S. Suppose $x \in [0,y]$. Then $0 = \omega(y) = (\mu_x \wedge \mu^*_x)(y) = \mu_x(y) \wedge \mu^*_x(y) = x \wedge y \wedge \mu^*_x(y) = x \wedge \mu^*_x(y)$ Now, if $x \wedge t = 0$ for some $t \in [0,y]$, then for all $p \in S$, $(\mu_x \wedge \mu_t)(p) = x \wedge t \wedge p = 0$, and so $\mu_x \wedge \mu_t = \omega$. This implies $\mu_t \leq \mu^*_x$. Thus, $\mu_t(y) \leq \mu^*_x(y)$, and so $t = t \wedge y \leq \mu^*_x(y)$. Hence, $\mu^*_x(y)$ is the relative pseudocomplement of x in $[0,y]$. Therefore, S is the sectionally pseudocomplemented.

Finally, for each $x \in S$, $\sigma(x) \wedge \sigma^*(x) = 0$. Also, $\sigma^*(x) \in [0,x]$. Now, let $t \wedge \sigma(x) = 0$ for some $t \in [0,x]$. Then for any $p \in S$, $(\mu_t \wedge \sigma)(p) = \mu_t(p) \wedge \sigma(p) = t \wedge p \wedge \sigma(p) = t \wedge \sigma(p) = t \wedge x \wedge \sigma(p) = t \wedge p \wedge \sigma(x) = 0 = \omega(p)$. This implies $\mu_t \wedge \sigma = \omega$, and so $\mu_t \leq \sigma^*$. Then $\mu_t(x) \leq \sigma^*(x)$. Thus, $t = t \wedge x \leq \sigma^*(x)$. This shows that $\sigma^*(x)$ is the pseudocomplement of $\sigma(x)$ in $[0,x]$. □

Corollary 3.3. Suppose S is a sectionally pseudocomplemented distributive nearlattice with 0. If x^* is the pseudocomplement of x in [0,y], then $x^* = \mu^*_I(y)$. □

Recall from chapter I that a distributive nearlattice S with 0 is semiboolean if each interval [0,x], $x \in S$ is boolean.

Theorem. 3.4. Let S be a distributive nearlattice with 0. For given n such that $-1 \leq n \leq \omega$, the following conditions are equivalent:

(i) S is sectionally in B_n.

(ii) M(S) is in B_n.

Proof. (i) implies (ii). The case n = -1 is trivial. The case $n = \omega$ follows from proposition 4.3.1.

For n = 0, S is semiboolean. Then by proposition 4.3.1, M(S) is pseudocomplemented and for $\sigma \in M(S)$, $\sigma^*(x) = \sigma(x)^*$ for each $x \in S$, where $\sigma(x)^*$ is the pseudocomplement of $\sigma(x)$ in [0,x]. Since S is semiboolean, $\sigma(x)^*$ is also the relative complement of $\sigma(x)$ in [0,x]. Then $(\sigma \vee \sigma^*)(x) = \sigma(x) \vee \sigma^*(x) = \sigma(x) \vee \sigma(x)^* = x = \iota(x)$. This implies $\sigma \vee \sigma^* = \iota$

and so σ^* is also the complement of σ in M(S). Therefore M(S) is boolean.

Now suppose S is sectionally in B_n ; $1 \leq n < \omega$. For $\sigma_1, \ldots\ldots\sigma_n \in M(S)$ and for each $x \in S$, using proposition 4.3.1.

$$[(\sigma_1 \wedge \ldots\ldots \wedge \sigma_n)^* \vee \bigvee_{i=1}^{n} (\sigma_1 \wedge \ldots\ldots \wedge \sigma^*_i \wedge \ldots\ldots \wedge \sigma_n)^*](x)$$

$$= (\sigma_1 \wedge \ldots \wedge \sigma_n)^* (x) \vee \bigvee_{i=1}^{n} (\sigma_1 \wedge \ldots \wedge \sigma^*_i \wedge \ldots\ldots \wedge \sigma_n)^* (x)$$

$$= ((\sigma_1 \wedge \ldots \wedge \sigma_n)(x))^* \vee \bigvee_{i=1}^{n} ((\sigma_1 \wedge \ldots \wedge \sigma^*_i \wedge \ldots \wedge \sigma_n)(x))^*$$

$$= (\sigma_1(x) \wedge \ldots \wedge \sigma_n(x))^* \vee \bigvee_{i=1}^{n} (\sigma_1(x) \wedge \ldots \wedge \sigma^*_i(x) \wedge \ldots \wedge \sigma_n(x))^*$$

$$= (\sigma_1(x) \wedge \ldots \wedge \sigma_n(x))^* \vee \bigvee_{i=1}^{n} (\sigma_1(x) \wedge \ldots \wedge \sigma_i(x)^* \wedge \ldots\ldots \wedge \sigma_n(x))^*$$

$$= x = \iota(x)$$

Hence, $(\sigma_1 \wedge \ldots\ldots \wedge \sigma_n)^* \vee (\sigma^*_1 \wedge \ldots\ldots \wedge \sigma_n)^* \vee \ldots\ldots \vee (\sigma_1 \wedge \ldots\ldots \wedge \sigma^*_n)^* = \iota$, and so M(S) in is B_n.

(ii) implies (i). The case $n = \omega$ follows from proposition 4.3.2. For $n = 0$, M(S) is boolean. Then by proposition 4.3.2, S is sectionally pseudocomplemented.

Suppose $x \in [0,y]$. Then the pseudocomplement μ_x^* of μ_x is also the complement of μ_x. Thus, $\mu_x \vee \mu_x^* = \iota$. If x^* is the pseudocomplement of x in $[0,y]$, then by corollary 4.3.3, $y = \iota(y) = (\mu_x \vee \mu_x^*)(y) = \mu_x(y) \vee \mu_x^*(y) = (x \wedge y) \vee x^* = x \vee x^*$. This implies x^* is the relative complement of x in $[0,y]$ and hence S is semiboolean.

Now suppose M(S) is in B_n ; $1 \leq n < \omega$. Let $x_1, \ldots, x_n \in [0,y]$. Then using proposition 4.3.1.

$$y = \iota(y) = [(\mu_{x_1} \wedge \ldots \wedge \mu_{x_n})^* \vee \bigvee_{i=1}^{n} (\mu_{x_1} \wedge \ldots \wedge \mu_{x_i}^* \wedge \ldots \wedge \mu_{x_n})^*](y)$$

$$= (\mu_{x_1} \wedge \ldots \wedge \mu_{x_n})(y)^* \vee \bigvee_{i=1}^{n} (\mu_{x_1} \wedge \ldots \wedge \mu_{x_i}^* \wedge \ldots \wedge \mu_{x_n})(y)^*$$

$$= (x_1 \wedge \ldots \wedge x_n \wedge y)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge \mu_{x_i}^*(y) \wedge \ldots \wedge x_n \wedge y)^*$$

$$= (x_1 \wedge \ldots \wedge x_n)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge (\mu_{x_i}(y))^* \wedge \ldots \wedge x_n)^*$$

$$= (x_1 \wedge \ldots \wedge x_n)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge x_i^* \wedge \ldots \wedge x_n)^*$$

Which implies $[0,y]$ is in B_n and so S is sectionally in B_n.

Following lemmas are needed for further development of this chapter. We omit the proof of 3.5 as it is trivial.

Lemma 3.5. (i) Let S be a distributive nearlattice with 0. If $0 \leq x \in S$ and the interval $[0,x]$ is pseudocomplemented, where y^* is the pseudocomplement of $y \in [0,x]$, then in the lattice of ideals of S, $(y^*] = (y]^* \cap (x]$ and $(y^{**}] = (y]^{**} \cap (x]$.

(ii) If S is a distributive nearlattice with 0 and $0 \leq x \in S$ is such that $(y]^* \cap (x]$ is principal for each $y \in [0,x]$, then $[0,x]$ is pseudocomplemented and $(y]^* \cap (x] = (y^*]$. □

Lemma 3.6. Let S be a distributive nearlattice with 0. For any $r \in S$ and any ideal I,

$$((r] \cap I)^* \cap (r] = I^* \cap (r].$$

Proof. Obviously R.H.S $\sqsubset$ L.H.S. To prove the reverse inequality, let $t \in ((r] \cap I)^* \cap (r]$. Then $t \leq r$ and $t \wedge r \wedge i = 0$ for all $i \in I$. This implies $t \wedge i = 0$ and so $t \in I^*$. Thus, $t \in I^* \cap (r]$ and this completes the proof. □

Recall that a prime ideal P of a nearlattice S with 0 is called minimal prime ideal if there exists no prime ideal Q such that $Q \sqsubset p$.

Followling lemma will also be needed for the proof of the next theorem. This is an improvement of 1.4.3 and we omit the proof as it can be done in a similar way.

Lemma 3.7. If S_1 is a subnearlattice of a distributive nearlattice S and P_1 is a minimal prime ideal in S_1, then there exists a minimal prime ideal P in S such that $P_1 = S_1 \cap P$. □

We conclude this chapter with the following theorem which is a nice extension of [10, Th. 4.5].

Theorem 3.8. Let S be a distributive nearlattice with 0, For given n such that $1 \leq n < \omega$, the following conditions are equivalent :

(i) S is sectionally in B_n ;

(ii) M(S) is in B_n ;

(iii) For any $y \in S$, and for $x_1, \ldots\ldots, x_n \in (y]$,

$(y] \sqsubset ((x_1] \wedge \ldots\ldots \wedge (x_n])^* \vee ((x_1]^* \wedge \ldots\ldots$

$\ldots\ldots \wedge (x_n])^* \vee \ldots\ldots\ldots \vee ((x_1] \wedge \ldots\ldots \wedge (x_n]^*)^*$;

(iv) For any $x_1, \ldots\ldots, x_n \in S$,

$((x_1] \wedge \ldots\ldots \wedge (x_n])^* \vee ((x_1]^* \wedge \ldots\ldots \wedge (x_n])^* \vee \ldots\ldots \vee ((x_1] \wedge \ldots\ldots \wedge (x_n]^*)^* = S$;

(v) S is sectionally pseudocomplemented and each prime ideal contains at most n minimal prime ideals.

(vi) S is sectionally pseudocomplemented and for any n+1 distinct minimal prime ideals $P_1, \ldots\ldots P_{n+1}$, $P_1 \vee \ldots\ldots \vee P_{n+1} = S$.

Proof. (i) <===> (ii) have already been proved in Theorem 4.3.4.

(i) implies (iii). Suppose $2 \leq n$. Let x^*_i be the pseudocomplement of x_i in $[0,y]$. By lemma 4.3.5,

$$\begin{aligned} & (x_1] \wedge \ldots\ldots \wedge (x_i]^* \wedge \ldots\ldots \wedge (x_n] \\ = & (x_1] \wedge \ldots\ldots \wedge (x_i]^* \wedge (y] \wedge \ldots\ldots \wedge (x_n] \\ = & (x_1] \wedge \ldots\ldots \wedge (x^*_i] \wedge \ldots\ldots \wedge (x_n] \\ = & (x_1 \wedge \ldots\ldots \wedge x^*_i \wedge \ldots\ldots \wedge x_n]. \end{aligned}$$

Since (i) holds, so

$$(y] = ((x_1 \wedge \ldots\ldots \wedge x_n)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots\ldots \wedge x^*_i \wedge \ldots\ldots \wedge x_n)^*]$$

$$=((x_1 \wedge \ldots\ldots \wedge x_n)^{\dagger}] \vee \bigvee_{i=1}^{n} ((x_1 \wedge \ldots\ldots \wedge x^{\dagger}_i \wedge \ldots\ldots \wedge x_n)^{\dagger}]$$

$$=((x_1 \wedge \ldots \wedge x_n]^{\ddagger} \wedge (y]) \vee \bigvee_{i=1}^{n} ((x_1 \wedge \ldots \wedge x^{\dagger}_i \wedge \ldots \wedge x_n]^{\ddagger} \wedge (y])$$

$$\sqsubset ((x_1] \wedge \ldots\ldots \wedge (x_n])^{\ddagger} \vee \bigvee_{i=1}^{n} ((x_1] \wedge \ldots\ldots \wedge (x_i^{\dagger}] \wedge \ldots \ldots \wedge (x_n])^{\ddagger}$$

$$= ((x_1] \wedge \ldots\ldots \wedge (x_n])^{\ddagger} \vee \bigvee_{i=1}^{n} ((x_1] \wedge \ldots \wedge (x_i]^{\ddagger} \wedge \ldots \wedge (x_n])^{\ddagger},$$

by lemma 4.3.5 and as each $x_i \leq y$. If $n = 1$, then by (i) and using lemma 4.3.5, we have

$$\begin{aligned} (y] &= (x_1^{\dagger} \vee x_1^{\dagger\dagger}] \\ &= (x_1^{\dagger}] \vee (x_1^{\dagger\dagger}] \\ &= ((x_1]^{\ddagger} \wedge (y]) \vee ((x_1]^{\ddagger\ddagger} \wedge (y]) \\ &\sqsubset (x_1]^{\ddagger} \vee (x_1]^{\ddagger\ddagger} \end{aligned}$$

(iii) Implies (iv). Firstly suppose $2 \leq n$.

Let $x_1, \ldots\ldots, x_n \in S$. Choose any $r \in S$. Then by (iii).

$$(r] \sqsubset ((r \wedge x_1] \wedge \ldots\ldots \wedge (r \wedge x_n])^{\ddagger} \vee \bigvee_{i=1}^{n} ((r \wedge x_1] \wedge \ldots \wedge (r \wedge x_i]^{\ddagger} \wedge \ldots \wedge ((r \wedge x_n])^{\ddagger},$$

and so

$$(r] = (((r \wedge x_1] \wedge \ldots \wedge (r \wedge x_n])^{\ddagger} \wedge (r]) \vee \bigvee_{i=1}^{n} (((r \wedge x_1] \wedge \ldots\ldots \wedge (r \wedge x_i]^{\ddagger} \wedge \ldots\ldots \wedge ((r \wedge x_n])^{\ddagger} \wedge (r]))$$

Now, by lemma 4.3.6.

$$((r \wedge x_1] \wedge \ldots \wedge (r \wedge x_n])^{\ddagger} \wedge (r] = ((x_1] \wedge \ldots \ldots \wedge (x_n])^{\ddagger} \wedge (r].$$

Again for each $1 \leq i \leq n$, $r \wedge x_i \leq x_i$ implies

$$(r \wedge x_1]^{\ddagger} \supseteq (x]^{\ddagger}.$$

Thus , $(r \wedge x_1] \wedge \ldots \wedge (r \wedge x_i]^{\ddagger} \wedge \ldots \ldots \wedge (r \wedge x_n]$

$\supseteq (r \wedge x_1] \wedge \ldots \wedge (x_i]^{\ddagger} \wedge \ldots \ldots \wedge (r \wedge x_n]$, and so

$((r \wedge x_1] \wedge \ldots \wedge (r \wedge x_i]^{\ddagger} \wedge \ldots \wedge (r \wedge x_n])^{\ddagger} \wedge (r]$

$\sqsubset ((r \wedge x_1] \wedge \ldots \wedge (x_i]^{\ddagger} \wedge \ldots \ldots \wedge (r \wedge x_n])^{\ddagger} \wedge (r]$

$= ((x_1] \wedge \ldots \wedge (x_i]^{\ddagger} \wedge \ldots \ldots \wedge (x_n])^{\ddagger} \wedge (r],$

by using lemma 4.3.6 again.

Therefore, $(r] \sqsubset ((x_1] \wedge \ldots \wedge (x_n])^{\ddagger} \vee ((x_1]^{\ddagger} \wedge \ldots \ldots \wedge (x_n])^{\ddagger} \vee \ldots \vee ((x_1] \wedge \ldots \ldots \wedge (x_n]^{\ddagger})^{\ddagger}$,

Which implies that

$$((x_1] \wedge \ldots \wedge (x_n])^{\ddagger} \vee ((x_1]^{\ddagger} \wedge \ldots \wedge (x_n])^{\ddagger} \vee \ldots \vee ((x_1] \wedge \ldots \ldots \ldots \wedge (x_n]^{\ddagger})^{\ddagger} = S.$$

If $n = 1$, then for any $r \in S$, we have by (iii) that

$$(r] \sqsubset (r \wedge x_1]^{\ddagger} \vee (r \wedge x_1]^{\ddagger\ddagger}$$

Thus, $(r] = ((r \wedge x_1]^{\ddagger} \cap (r]) \vee ((r \wedge x_1]^{\ddagger\ddagger} \cap (r])$

$= ((x_1]^{\ddagger} \cap (r]) \vee ((r \wedge x_1]^{\ddagger\ddagger} \cap (r])$

(by lemma 4.3.6)

$\sqsubset (x_1]^{\ddagger} \vee (x_1]^{\ddagger\ddagger}$,and hence

$(x_1]^{\ddagger} \vee (x_1]^{\ddagger\ddagger} = S$

(iv) implies (i) follows exactly from the same proof of [10, Th.4.5(iv) ===> (i)].

(v) implies (vi). Suppose (v) holds, and $P_1,.....,P_{n+1}$ are distinct minimal prime ideals. If $P_1 \vee .....\vee P_{n+1} \neq S$, then by 1.2.5, there exists a prime ideal P containing $P_1,.....,P_{n+1}$, which contradicts (v).

(vi) implies (v). Suppose (vi) holds. If (v) does not holds, then there exists a prime ideal P which contains more than n minimal prime ideals. Then by (vi) P = S which is impossible.

(iv) implies (vi). We omit this proof, as it can be proved exactly in a similar way that Cornish has proved (iv) ===> (vi) in [10, Th.4.5].

(vi) implies (i). Suppose (vi) holds and $a \in S$. Let $Q_1,.....,Q_{n+1}$ be n+1 distinct minimal prime ideals in [0,a]. By corollary 4.3.7, there are minimal prime ideals P_i in S such that $Q_i = [0, a] \cap P_i$ for each $1 \leq i \leq n+1$. Since Q_i are distinct, all P_i are also distinct. By (vi), $(a] = (a] \wedge (P_1 \vee.....\vee P_{n+1}) = ((a] \wedge P_1) \vee.....\vee ((a] \wedge P_{n+1}) = Q_1 \vee .....\vee Q_{n+1}$.

Since each interval [0,a] is pseudocomplemented, so [0,a] $\in B_n$ by [31, Th.1], and hence S is sectionally in B_n.

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