STUDY OF
CERTAIN TOPICS IN FUZZY SUPRA TOPOLOGICAL SPACES

By

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Declaration

This is to certify that the thesis work entitled "Study of Certain Topics in Fuzzy Supra Topological Spaces" has been carried out by Md. Yahia Molla in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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Dedicated

to

My Parents

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ABSTRACT

American Mathematician Lotfi A. Zadeh in 1965 first introduced the concept of fuzzy set. He interpreted a fuzzy set on a set as a mapping from the set into the unit interval I= [0, 1], which is a generalization of the characteristic function of the set. Many mathematicians throughout the world used this set to fuzzify different areas of mathematics. Fuzzy supra topology is one of the outcomes of such fuzzification of the usual topology. In this thesis, we have studied and have introduced several results on fuzzy supra topological spaces. At first we have discussed the standard definitions and properties of fuzzy supra R_0 and R_1 topological spaces, which are found in the literatures. Then we have introduced some new definitions and properties for these spaces. We have also studied the Fuzzy supra T_0, T_1, T_2 and Fuzzy supra regular topological spaces and obtained the following properties, such as, Good extension, Initial, Reciprocal, Productivity, Hereditary and Homeomorphism, etc. Moreover we have discussed compactness of Fuzzy Supra Topological Spaces and have proposed some new definitions, theorems and proofs.

CONTENTS

INTRODUCTION

This thesis is a study of Fuzzy Supra Topological Spaces.

Fuzzy Mathematics

The concept of fuzzy set was introduced in 1965, by the American Cyberneticist Zadeh, L.A., in his classical paper [66] as the generalizations of the concept of the characteristic function of a set to allow its grade of membership functions (grade of membership of x in A) representing $\mu_A(x)$, belonging into the unit interval [0. 1]. That is the nearer the value of $\mu_A(x)$ is to 1, the higher is the grade of membership of x in A. A set A in ordinary sense of the term membership function can take only two values 0 and 1, with $\mu_A(x)=1$ or 0 according as x does and does not belong to A. This concept led to the 'fuzzification' of many areas of mathematics and also Fuzzy Mathematics has found numerous applications in different fields such as Robotics, Pattern Recognition, Military control, Medical diagnosis Psychology, Taxonomy, Economics etc.

Fuzzy Topology

General topology is one of the most important branch of mathematics in which the notation of fuzzy set has been applied systematically. In 1968, C. L Chang [21] did 'fuzzification' of topology by replacing subsets in the definition of fuzzy topology by fuzzy sets. Since then a large body of mathematicians have been working in this area, such as, C. K Wong[63, 64], Lowen, R., [33,34], Hutton, B., and Reilly, I., [30], and others.

Several definitions of R_0, R_1, T_0, T_1, T_2 and Regular topological spaces were introduced and studied by many mathematicians in fuzzy topological spaces. In this thesis we try to show that these definitions are equally and significantly working in fuzzy supra topological spaces also. For our whole work we prefer the concept of fuzzy topology given by Chang, C.L., [21].

Fuzzy Supra Topology

In 1983 Mashour, A.S., Allam, A.A., and Khedr, F.S., [38] introduced the concepts of Supra Topological spaces and studied S-continuous and continuous functions in those Spaces. In 1987, Abd El-Monsef, M.E., and Ramadan, A.E., [1] introduced the fuzzy-supra topological spaces and studied fuzzy supra-continuous functions and obtained a number of characterizations. Also fuzzy-supra topological spaces are generalization of supra topological spaces. In 1996, Min, W.K., [40, 41] introduced fuzzy s-continuous, fuzzy s-open and fuzzy s-closed maps and established a number of characterizations. In 2003, Mukherjee, A., and Bhattacharya B., [43] introduce s- Induced L- Supra Topological Spaces In 2004, Gupta, M.K., and Singh, R.P., [27] introduced the concepts of fuzzy pairwise s-open mapping and fuzzy pairwise s-closed mappings and study the implications that exists between them. The main purpose of our study is to extend the concepts of Fuzzy Supra topological spaces and to obtain some results regarding them.

Summary of the Thesis

The thesis is divided into eight chapters.

The preliminary definitions and results which are used in the succeeding chapters are given in the first chapter. Due references are given wherever necessary. Some of the preliminary results which are relevant to each chapter have been given at the beginning of the corresponding chapter.

The concept of Separation axioms is one of the most important concepts in topology. In this thesis from chapter two to chapter seven, we investigate several types of separation axioms in Fuzzy Supra Topological Spaces.

Chapter two is a study of Fuzzy Supra R_0 Topological Spaces. In this chapter we list possible definitions of Fuzzy Supra R_0 Topological Spaces and investigate implications and non-implications among these concepts. We also investigate good extension, reciprocal, Initial, Productivity, hereditary Properties' and homeomorphic properties in Fuzzy Supra R_0 Topological Spaces.

Chapter three is a study of Fuzzy Supra R_1 Topological Spaces. In this chapter we study and investigate several properties of FSR_1 as in Chapter two.

Chapter four is a study of Fuzzy Supra T_0 Topological Spaces. This chapter contains three sections; first section is on different types of definitions, implications and non-implications among these definitions with some lemmas and counter- examples, second section is on good extension property of supra T_0 topological spaces. The third sections are on subspace, heredity and productive properties and on homeomorphic property of fuzzy supra T_0 topological spaces.

Chapter five is a study of Fuzzy Supra T_1 Topological Spaces. We find several results on Fuzzy Supra T_1 spaces in a similar way as in chapter four.

In Chapter six, we study Fuzzy Supra T_2 Topological Spaces. Here, we study the fuzzy Hausdorffness concepts of Gantner, T.E., Steinlage, R.C., and Warren, R.H.,[24] Srivastava, R. Lal S.N.,and Srivastava, A.K.,[58], Sarkar , M., [52], Ali, D.M., and Srivastava, A.K., [5], Ghanim, M.H., Mashhour, A.S., and Fath Alla, M.A., [25]. We show the implications and non-implications among these concepts. We study Good extension, Initiality, hereditary and productivity of fuzzy supra Hausdorff spaces and also investigate the mapping between two Fuzzy Supra T_2 Topological Spaces, and study some α- type fuzzy supra Hausdorff spaces.

Fuzzy Supra Regular Topological Spaces have been studied in Chapter seven, we study the fuzzy Regular concepts of Hutton, B., and Reilly, I., [30], Sarkar , M., [52], Ali, D.M., [10], Ghanim, M.H., Keree, E.E.,and Mashhour, A.S., [26], Benchalli, S.S., and Malghan, S.R., [19], Wang, G.J., [60],in Fuzzy supra topological spaces with some other properties of the same.

Chapter eight is a study of Compactness in Fuzzy Supra Topological Spaces. Many topologists studied the concept of fuzzy compactness such as Gantner, T.E. et. al [24] , Lowen, R[33, 34], Wang, G.J.[60]. α-compactness is studied by Choubey, A. and Srivastava, A.K.[22], Shi, F.G. [55], also studied in [24]. In 1988 Mao-Kang, L. [37], introduce the S^*- paracompactness concept in fuzzy topological spaces. In this chapter we study fuzzy supra compactness, Fuzzy supra α - compactness, Fuzzy supra paracompactness ■

CHAPTER-I

Preliminaries

1. Introduction:

This chapter containing concepts and results of the Fuzzy sets, Fuzzy Topological spaces, and Fuzzy Supra Topological spaces. Several preliminary definitions and results which are used in the succeeding chapters are being given here. Due references are given wherever necessary. Most of the results are quoted from various research papers. Through the sequel, we make use of the following notations.

1.1 Symbol :

$I = [0, 1]$	:	Closed unit interval.
J =	:	An index set.
$I_1 = [0.1)$	:	Right open unit interval.
$I_0 = (0.1]$	:	Left open unit interval.
$I_{0,1} = (0,1)$	:	Open unit interval.
λ, μ, ν, u, v	:	Fuzzy set.
(X, t)	:	Fuzzy topological space.
$(X. t^*)$	:	Fuzzy supra topological space.
(X, T^*)	:	Supra topological space.

$I(t^*) = \{u^{-1}(0,1], u \in t^*)\}$: Supra topology on X

1.2. Fuzzy Set:

This thesis is a study of fuzzy supra topological spaces. To present our work in a systematic way, we consider in this chapter various concepts and results from the theories of fuzzy sets. Fuzzy topological spaces and Fuzzy supra topological spaces scattered in various research papers. For this, we start with.

1.2.1. Definition: Let X be a non – empty set and $A \subseteq X$, now the characteristic function of A is a function, that declares which elements of X are members of the set A and which are not. It is denoted by κ_A or 1_A. The function κ_A or $1_A : X \to [0, 1]$ is defined by

$$1_A(x) = 1 \text{ if } x \in A \\ \quad = 0, \text{ if } x \notin A.$$

Set A can be represented for all element x ∈X by its characteristic function κ_A (x); such sets are called crisp set. Throughout this work, we use, if needed, 1_A to denote the characteristic function of a set A .

1.2.2. Definition: Let u: X→ I, then the set {x∈X, u(x)>0} is called the support of u and is denoted by u_0 or supp(u).If $A \subseteq X$, then 1_A denotes the characteristic function of A. The characteristic function of a singleton set {x} is denoted by 1_x. **[10]**

1.2.3. Definition: Let X be a non - empty set and let I= [0, 1]. A fuzzy set in X is a function ; λ: X→I which assigns to each x ∈ X, its grade of membership λ(x) ∈ I. **[66]**

1.2.4. Definition: Let I^X denote the set of all mappings λ: X→I. A member of I^X is called fuzzy subset of X.

1.2.5. Definition: Let I^X denote the set of all mappings λ: X→I. A member of I^X is called fuzzy subset of X, but the name fuzzy set is now is almost universally use.

1.2.6. Definition: A fuzzy subset is empty if and only if its grade of membership is identically zero on X; it is denoted by 0.

1.2.7. Definition: A fuzzy subset is whole if and only if its grade of membership is identically one on X; it is denoted by 1.

1.2.8. Definition: For any two members $\lambda$and μ of I^X, $\lambda \geq \mu$ if λ (x) $\geq \mu$ (x) for each x ∈X. and in this case λ is said to contain μ and is denoted by $\lambda \supseteq \mu$ or μ is said to be contained in λ.**[66]**

1.2.9. Definition: Let X be a set and u and v be two fuzzy subset of X. Then u is said to be the complement of v if v(x) =1-u(x), for every x ∈X. It is denoted by $\bar{u}$ or u^c, Obviously $(u^c)^c$ =u . **[65]**

1.2.10. Definition: Let X be a set and let I= [0, 1]. Let I^X denote the set of all mapping: α: X→ I. A member of I^X is called a fuzzy subset of X, and unions and intersections of fuzzy sets are denoted by ∨ and ∧ respectively, and defined by

$\vee \mu_i$ =sup { μ_i (x) : i ∈J and x∈X}

$\wedge \mu_i$ = inf { μ_i (x) : i ∈J and x∈X}.**[40]**

1.2.11. Definition : A fuzzy point in X is a special type of fuzzy set in X with membership function

$\mu(x) = r$, for x∈X and

$\mu(y) = 0$, for y≠x where $0 < r < 1$, ∀ y∈X.

This fuzzy point is said to have support x and value α is denoted by x_α or $\alpha 1_x$.

1.2.12. Definition: Let $\alpha 1_x$ be a fuzzy point in X and u be a fuzzy set in X. Then $\alpha 1_x \in u$ if and only if $\alpha < u(x)$. **[42]**

1.2.13. Definition: For all fuzzy points $\alpha 1_x$ and for all fuzzy set u, v in X, we have

(i) $u \subseteq v$ if and only if $\alpha 1_x \in u \rightarrow \alpha 1_x \in v$.

(ii) u = v if and only if $\alpha 1_x \in u \Leftrightarrow \alpha 1_x \in v$.

(iii) $\alpha 1_x \in u \vee v$ if and only if $\alpha 1_x \in u$ or $\alpha 1_x \in v$.

(iv) $\alpha 1_x \in u \wedge v$ if and only if $\alpha 1_x \in u$ and $\alpha 1_x \in v$.

(v) $\overline{\alpha 1_x} = \alpha \vee \overline{1_x}$

1.2.14. Definition: A fuzzy set u in X is the union of all its fuzzy points, i. e. u = $\bigvee\limits_{\alpha I_x \in u}^{n} \alpha 1_x$. **[42]**

1.2.15. Definition: Let $\lambda : X \rightarrow I$ be a fuzzy set. If there exist $\alpha \in I$, $\alpha \neq 0$, such that $\lambda(x) > 0, \Leftrightarrow \lambda(x) \geq \alpha$ ∀ x∈X, then λ is called Pseudo crisp set.

1.2.16. Definition: A Partition of a nonempty fuzzy set λ is a collection of fuzzy sets $\mathcal{A}$ such that $\mathcal{A} = \{ \lambda_i | i \in J,\ \lambda_i \subseteq \lambda$ and $\lambda_i \neq 0 \}$ where $i \neq J$ $\lambda_i \wedge \lambda_J = 0$ and $\bigvee\limits_{i \in J} \lambda_i = \lambda$.

1.3. Mapping and Fuzzy Subsets induced by mappings:

1.3.1. Definition: Let f: X→Y be a mapping and u be fuzzy set in X. Then the image f(u) is a fuzzy set in Y which is defined as

$$f(u)(y) = \begin{cases} \sup \{u(x) : f(x)=y\} & \text{if } f^{-1}(y) \neq \varphi \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$ **[4]**

1.3.2. Definition: Let f: X→Y be a mapping and u be fuzzy set in X. Then the inverse image $f^{-1}(u)$ is a fuzzy set in x which is defined by $f^{-1}(u)(x) = u(f(x)) \ \forall x \in X$. **[21]**

Here we mention some properties of fuzzy subsets induced by mappings.

Let f: X→Y be a mapping, then

(a) $u_1 \leq u_2 \Rightarrow f(u_1) \leq f(u_2) \ \forall u_1, u_2 \in I^X$,

(b) $u_1 \leq u_2 \Rightarrow f^{-1}(u_1) \leq f^{-1}(u_2) \ \forall u_1, u_2 \in I^Y$,

(c) $u \leq f^{-1}(f(u))$, where u be a fuzzy set in X and if f is one- one then $u = f^{-1}(f(u))$,

(d) $f(f^{-1}(u)) \leq u$ where u be a fuzzy set in X and if f is onto then $f(f^{-1}(u)) = u$,

(e) $f^{-1}(1-u) = 1 - f^{-1}(u)$,

(f) If $f(u) \leq v$ then $u \leq f^{-1}(v)$ where u and v are fuzzy set in X,

(g) $f(\wedge_j u_j) \leq \wedge_j f(u_j)$,

(h) $f^{-1}(\wedge_j u_j) = \wedge_j f^{-1}(u_j)$,

(i) $f(\vee_j u_j) = \vee_j f(u_j)$ and

(j) $f^{-1}(\vee_j u_j) = \vee_j f^{-1}(u_j)$.

(k) Let f be a function from X into Y and g be a function from Y into Z. Then$(g \circ f)^{-1}(w) = f^{-1}(g^{-1}(w))$, for any fuzzy subset w in Z, where $(g \circ f)$ is the composition of g and f.

1.3.1. Proposition: Let f: X→Y be a function, and $u \in I^X$, $v \in I^Y$, then the following hold:

(i) If x_α is a fuzzy point in X, then $f(x_\alpha) = [f(x)]_\alpha$ is a fuzzy point in Y.

(ii) If x_α is a fuzzy point in $u \in I^X$, then $f(x_\alpha)$ is a fuzzy point in $f(u) \in I^Y$

(iii) If $f(x_\alpha)$ is a fuzzy point in $u \in I^Y$, then x_α is a fuzzy point in $f^{-1}(u) \in I^X$

(iv). If x_α is a fuzzy point in Y, then $f^{-1}(x_\alpha)$ need not to be a fuzzy point in X. However, if f is injective and $x_\alpha \in f(X)$, then $f^{-1}(x_\alpha)$ is a fuzzy point in X and is defined as $f^{-1}(x_\alpha) = [f^{-1}(x)]_\alpha$. **[21]**

1.4. Fuzzy Topological Spaces:

Before define fuzzy topological space we would like to define General topological space.

1.4.1. Definition: Let X be a non-empty set and T be a collection of subsets of X satisfying the following axioms

(i) X, φ∈T (ii) if $U_i \in T\ \forall\ i \in T$ then $\underset{i\in J}{\cup} U_i \in T$, (iii) if $U_1, U_2 \in T$ hen $U_1 \cap U_2 \in T$.

Then T is called topology on X and (X, T) will be called topological space.

Chang C. L. defined a fuzzy topological space as follows:

1.4.2. Definition: Let X be a non-empty set, and I= [0, 1], and I^X be the collection of all mappings from X into I, i.e. the class of all fuzzy sets in X. A fuzzy topology on X, is defined as a family t of members of I^X, satisfying the following conditions.

(a) 0, 1∈t,

(b) If u_i for all i∈J then $\underset{i\in J}{\vee} u_i \in t$ (c) If u, v∈t then u∧v∈t.

The pair (X, t) is called a fuzzy topological space. In short, fuzzy topological space is denoted by fts. The members of t are called t- open (or open) fuzzy sets. A fuzzy set v is called a t-closed (or closed) fuzzy set if 1-v ∈ t. **[21]**

1.4.3. Definition: Lowen R. modified the definition of a fuzzy topological space defined by Chang C.L. **[21]** changing of condition (a) namely 0, 1∈t, to (a)$'$ all constants $\alpha \in$ t. In the sense of Lowen R. the definition of a fuzzy topological space as follows:

Let X be a set and t⊂ I^X is a fuzzy topology on X iff

(i) all constants $\alpha \in$ t.

(ii) $\forall\ \mu, \nu \in t \Rightarrow \mu \wedge \nu \in t$.

(iii) $\forall (\mu_j)_{j\in J} \subset t \Rightarrow$ Sup $(\mu_j)_{j\in J} \in t$ **[33]**

In the present thesis; we use the concept of fuzzy topology due to Chang C.L. **[21]**, or occasionally also due to Lowen, R., **[33]**.

1.4.4. Definition.: Let (X, t) be a fts. A fuzzy set λ in X is a neighborhood of a fuzzy set μ in X iff there is $\gamma \in$t such that $\mu \leq \gamma \leq \lambda$.**[60]**

1.4.5. Definition : A fuzzy set u in a fts (X , t) is called a neighborhood of a fuzzy point x_r if and only if there exist a fuzzy set $u_1 \in$ t such that $x_r \in u_1 \subseteq u$. A neighborhood u is called an open neighborhood if u is open. The family consisting of all the neighborhoods of x_r is called the system of nhds of x_r. **[42]**

1.4.1. Proposition: Let u and v be fuzzy sets in a fts (x, t) . Then

(i). $\bar{u} \vee \bar{v} = \overline{u \vee v}$ and $\bar{u} \wedge \bar{v} = \overline{u \wedge v}$

(ii). $u^{o} \wedge v^{0} = (u \wedge v)^{o}$ and $u^{o} \vee v^{0} \leq (u \vee v)^{o}$

(iii). $(1 - u)^{o} = 1 - \bar{u}$.

(iv). $\overline{1 - u} = 1 - u^{o}$ **[60]**

1.4.6. Definition.. Let (X, t) be a fts and let $A \subset X$. The fuzzy topology t_A is called the relative fuzzy topology on A or the fuzzy topology on A induced by the fuzzy topology t on X. Also, (A, t_A) is called a subspace of (X, t). **[40]**

1.4.7. Definition: The function $f : (X , t) \longrightarrow (Y , s)$ is called fuzzy continuous if and only if for every $v \in s$, $f^{-1}(v) \in t$, the function f is called fuzzy homeomorphic if and only if f is bijective and both f and f^{-1} are fuzzy continuous. **[47]**

1.4.8. Definition: The function $f : (X , t) \longrightarrow (Y , s)$ is called fuzzy open if and only if for each open fuzzy set u in (X , t) , f(u) is open fuzzy set in (Y , s) **[20].**

1.4.9. Definition: The function $f : (X , t) \longrightarrow (Y , s)$ is called fuzzy closed if and only if for each closed fuzzy set u in (X , t), f(u) is closed fuzzy set in (Y , s). **[47]**

1.4.2. Proposition ([47]Theorem 1.1) :- Let $f : (X , t) \rightarrow (Y , s)$ be a fuzzy continuous function , then the following properties hold :

(i) For every s – closed v, $f^{-1}(v)$ is t – closed.

(ii) For each fuzzy point p in X and each neighborhood u of f(u), then there exist a neighborhood v of p such that f(v) = u .

(iii) For any fuzzy set u in X, $f(\overline{u}) \subset (\overline{f(u)})$.

(iv) For any fuzzy set v in Y , $(\overline{f^{-1}(v)}) \subset f^{-1}\overline{(v)}$.

1.4.3. Proposition ([20]Theorem 3.1) :- Let $f:(X,t) \longrightarrow (Y,s)$ be a fuzzy open function , then the following properties hold:

(i) $f(u^{o}) \subseteq (f(u))^{o}$, for each fuzzy set u in X .

(ii) $(f^{-1}(v))^{o} \subseteq f^{-1}(v^{o})$, for each fuzzy set v in Y.

1.4.4. Proposition ([20] Theorem 1.5) :- Let $f:(X,t) \longrightarrow (Y,s)$ be a function. Then generated by f is closed if and only if $f(\overline{u}) \subseteq \overline{f(u)}$ for each fuzzy set u in X.

1.4.10. Definition:- A fuzzy topological property FP is said to be an initial property if for each family of functions $\{f_j : X \to (X_j, t_j) ; j \in J\}$, whenever each fts $(X_j, t_j) ; j \in J$, has FP, then (X, t) also FP, t being the initial fuzzy topology on X induced by the family $\{f_j; j \in J\}$.**[11]**

1.4.11. Definition:- If (X_1, t_1) and (X_2, t_2) be two fuzzy topological space and $X = X_1 \times X_2$ be the usual product and t be the coarsest fuzzy topology on X, then each projection $\pi_i : X \to X_i$, $i = 1, 2$. is fuzzy continuous. The pair (X , t) is called the product space of the fuzzy topological spaces (X_1, t_1) and (X_2, t_2) . **[41]**

1.4.12. Definition: Let $\{(X_i, t_i)_{i\in J}\}$ be a collection of fuzzy topological spaces.

Let $X = \prod_{i\in k} X_i$ be their Cartesian product and $p_i : X \to X_i$ be the projection map. Then the fuzzy topology on X generated $\{p_i^{-1}(u_i) : i \in J, u_i \in t_i\}$ is called the product fuzzy topology on X and the pair (X, t) is called the product fuzzy topological space. It can be verified that $p_i^{1}(u_i)$, $i \in J$, as defined above, can be expressed as $\prod_{ki\in J} \lambda_k$ where $\lambda_k = u_i$ if k=i and $\lambda_k = X_k$ if $k \neq i$. **[10]**

1.4.13. Definition: Let $a \in [0, 1)$.A fuzzy topological spaces(X,δ) is called a-compact if and only if each a-shading family in δ has a finite a-shading subfamily. **[46]**

1. 5. Fuzzy Supra Topological Spaces

1.5.1. Definition: Let X be a non-empty set and T^* be a collection of subsets of X s.t. (i) $\varphi \in T^*$ (ii) $X \in T^*$. Then (X, T^*) will be simply called a space. If (X, T^*) satisfies the condition if $U_i \in T^* \ \forall\ i \in T^*$ then $\underset{i\in J}{\cup} U_i \in T^*$, (X, T^*) will be called supra topological space .

1.5.2. Definition.: Let X be a non-empty set, and I= [0, 1] and I^X be the collection of all mappings from X into I, i.e. the class of all fuzzy sets in X. A subfamily t^* of I^X is said to be fuzzy supra topology on X, if

(a) $0, 1 \in t^*$ (b) $\alpha_i \in t^*$ For all $i \in J$ then $\vee\ \alpha_i \in t^*$

(X, t^*) is called a fuzzy supra topological space. In short, fuzzy supra topological space is denoted by FSTS. The elements of t^* are called fuzzy supra open sets in (X, t^*). A fuzzy set λ is supra closed if and only if complement of λ, i. e $\lambda^c = 1 - \lambda$ is a fuzzy supra open set in (X, t^*).We abbreviate topo. spaces for topological spaces.**[1]**

1.5.1. Example: let X= {a, b, c, d,} with a fuzzy supra topology.

t^*={1, 0, {(a, 0), (b, .5), (c, 1), (d, 0)}, {(a, .5), (b, .25), (c, 0), (d, 1)}, {(a, .5), (b, .5), (c, 1), (d, 1)}}on X then the class of supra closed sets t^* are

t^{*c}={0, 1, {(a, 1), (b, .5), (c, 0), (d, 1)}, {(a, .5), (b, .75), (c, 1), (d, 0)}, {(a, .5), (b, .5), (c, 0), (d, 0)}}.

Note: It is clear that every fuzzy topological space is fuzzy supra topological space but the converse may not true.

1.5.2. Example: Let X= {a, b}, and $\alpha, \beta, \gamma \in I^X$. Let $\alpha(a) = .2$, $\alpha(b) = .3$; $\beta(a) = .4$, $\beta(b) = .1$; $\gamma(a) = .4$, $\gamma(b) = .3$ then $t^* = \{0, 1, \alpha, \beta, \gamma\}$ is a fuzzy supra topology on X but t^* is not fuzzy topology on X.

1.5.3. Definition: Let X be a set and t^* be the class of all fuzzy sets in X and t^* satisfies the axioms of fuzzy supra topological space on X. This fuzzy supra topology t^* on X is called the discrete fuzzy supra topology and the pair (X, t^*) is called the discrete fuzzy supra topological space.

1.5.4. Definition: Let X be a set and t^* be the fuzzy supra topology on X consist of the fuzzy sets 0 and 1 alone, then t^* is called the indiscrete fuzzy supra topology and the pair (X, t^*) is called the indiscrete fuzzy supra topological space.

1.5.5. Definition: The supra closure of a fuzzy set λ is denoted by $\overline{\lambda}$, or λ^{sc} or $Scl(\lambda)$, and given by

$Scl(\lambda) = \wedge\{s \mid s$ is a fuzzy supra- closed set and $\lambda \leq s\}$.**[39]**

1.5.6. Definition: The supra interior of a fuzzy set λ is denoted by λ^o or λ^{si} or Si(λ), and given by

$$Si(\lambda) = \vee\{s \mid s \text{ is a fuzzy supra- open set and } s \leq \lambda\}.$$ **[1]**

1.5.7. Definition The τ_i-supra closure of a fuzzy set λ is denoted by $\tau_i - scl(\lambda)$ and defined as

$$\tau_i - scl(\lambda) = \wedge\{\mu : \mu \text{ is a } \tau_i\text{-fuzzy supra closed set and } \lambda \leq \mu\}$$

The τ_i-supra interior of a fuzzy set λ is denoted by $\tau_i - \sin t(\lambda)$, and defined as

$$\tau_i - \sin t(\lambda) = \vee\{\mu : \mu \text{ is a } \tau_i\text{-fuzzy supra open set and } \mu \leq \lambda\}$$

Note: In fuzzy topological space, we have $\overline{\mu} \cup \overline{\lambda} = (\mu \cup \lambda)^-$ and $\mu^o \cap \lambda^o = (\mu \cap \lambda)^o$. But this is not satisfied in fuzzy supra topological space. **[27]**

1.5.8. Definition: Let (X, t^*) be a fuzzy supra topological space. Let A be an ordinary subset of X. Then the relative fuzzy supra topology of A can be defined in the following way; the subset A of X (in the ordinary sense) has a characteristic function say μ_A such that

$\mu_A(x) = 1$ if $x \in A$

$\mu_A(x) = 0$ if $x \notin A$.

Let $t_A^* = \{\lambda \wedge A : \lambda \in t^*\}$, then t_A^* is called a fuzzy supra subspace topology on A.

1.5.9. Definition: Let (X, t^*) be a fuzzy supra topological space. A fuzzy supra topological property is said to be hereditary if whenever a space has that property, then so does every subspace of it.

1.5.10. Definition: Let (X, t_1^*) and (X, t_2^*) be two fuzzy supra topological spaces and two fuzzy supra topologies t_1^* and t_2^* be such that $t_1^* \subset t_2^*$, we say that t_2^* is finer than t_1^* and is t_1^* coarser than t_2^*.

1.5.11. Definition: Let (X, t_1^*) and (Y, t_2^*) be two fuzzy supra topological spaces. A mapping f: (X, t_1^*) $\rightarrow$ (Y, t_2^*) is called fuzzy supra continuous if the inverse image of each fuzzy supra open set in (Y, t_2^*) is t_1^* fuzzy supra open in X.

1.5.12. Definition: Let (X, t_1^*) be a fuzzy supra topological spaces and (Y, t_2) be a fuzzy topological space. A mapping f: (X, t_1^*) → (Y, t_2) is called fuzzy **s**- continuous if the inverse image of each fuzzy open set in (Y, t_2) is t_1^* fuzzy supra open in X. **[41]**

1.5.13. Definition: Let (X, t_1^*) and (Y, t_2^*) be two fuzzy supra topological spaces. A mapping f: (X, t_1^*) → (Y, t_2^*) is called fuzzy supra continuous if $f^{1}(t_2^*) \subset t_1^*$. **[1]**

1.5.14. Definition: Let (X, t_1^*) and (Y, t_2^*) be two fuzzy supra topological spaces. A mapping f: (X , t_1^*) → (Y, t_2^*) is called fuzzy supra open map if the image of each fuzzy supra open in t_1^* is t_2^* fuzzy supra open in Y. **[40]**

1.5.15. Definition: Let (X, t_1^*) and (Y, t_2^*) be two fuzzy supra topological spaces. A mapping f: (X, t_1^*) → (Y, t_2^*) is called fuzzy supra closed map if the image of each fuzzy supra closed in t_1^* is t_2^* fuzzy supra closed in Y.

1.5.16. Definition : For each i ∈ J, Let $f_i : X \to (Y_i, \tau^*_i)$ are the functions from a set X into fsts (Y_i, τ^*_i) then the smallest fuzzy supra topology on X for which the functions f_i, i ∈ J are fuzzy continuous is called initial fuzzy supra topology on X generated by the collection of functions $\{f_i\ i \in J\}$. If t is the smallest supra fuzzy topology on X, then t is generated by $f_j^{-1}(u_j); u_j \in t_j$ and $j \in t_j$.

1.5.17. Definition: Let (X, t*) and (Y, s*) be two fuzzy supra topological spaces. If u_1 and u_2 are two fuzzy supra open subsets of X and Y respectively, then the Cartesian product $u_1 \times u_2$ is a fuzzy supra subset of X × Y defined by $(u_1 \times u_2)(x, y) = \min(u_1(x), u_2(y))$, for each pair $(x, y) \in X \times Y$

1.5.18. Definition: Let $\{(X_i, t_i)_{i\in J}\}$ be a collection of fuzzy supra topological spaces. Let $X = \prod_{i\in k} X_i$ be their Cartesian product and $\pi_i : X \to Xi$ be a projection map assigning to each element of its ith. coordinate, $\pi_i\ \{(x_i, t_i)_{i\in J}\} = x_i$; it is called the projection mapping associated with the index i.

1.5.19. Definition: Let (X, t^*) and (Y, s^*) be two fuzzy supra topological spaces and f: $(X, t^*) \rightarrow (Y, s^*)$ be any function, then f is called fuzzy supra homeomorphism if and only if f is fuzzy supra bijective, fuzzy supra continuous and fuzzy supra open.

1.5.20. Definition: Let (X, t^*) is a fuzzy supra topological space. A fuzzy set A is called quasi coincident with a fuzzy set B denoted by AqB if $A(x)+B(x)>1$ for some $x \in X$. A fuzzy point $x_t \leq A$ is called quasi coincident with the fuzzy set A denoted by $x_t q A$ if, $t + A(x)>1$. The negation relation is denoted by $x_t \neg q\ A$

1.5.21. Definition: Let X be set and $(X', t^{*'})$ be an fuzzy supra topological space. Let us consider a function f: $X \rightarrow (X', t^{*'})$. Suppose $t^* = \{f^{-1}(U) : U \in t^{*'}\}$. Then t^* is a fuzzy supra topology on X. We call t^*, the reciprocal supra topology on X.

1.5.22. Definition: Let (X, t^*) is a fuzzy supra topological space. Then (i) a subfamily B of t^* **is** called a base for t^* iff each member of t^* can be expressed as a supremum of member of B and (ii) a subfamily S of t^* is a subbase for t^* iff the family of all the finite infima of members of S is a base for t^*.

1.5.23. Definition: - Let f be a real valued function on a fuzzy supra topological space (X, t^*). If $\{x \in X: f(x)>\alpha\}$ is supra open and for every real α where $\alpha \in I$, the f is called lower semi continuous function.

1.5.24. Definition: Let X be non empty set and T^* be a supra topology on X, and let $t^* = \omega(T^*)$ be the set of all lower semi continuous function from (X, T^*) to I with usual topology. Thus $t^* = \omega(T^*) = \{u \in I^X: u^{-1}(\alpha, 1] \in T^*\}$ for each $\alpha \in I_1$, $t^* = \omega(T^*)$ to be a fuzzy supra topology on X.

Let P be the property of a supra topological space (X, T^*) and FSP is fuzzy supra topological analogue. Then FSP is called good extension of P "If the statement(X, T^*) has P if and only if $(X, \omega(T^*))$ has FSP" holds good for every fuzzy supra topological space (X, T^*).

CHAPTER-II

Fuzzy Supra R_0 Topological Spaces

2. Introduction:

The separation axiom R_0 was introduced and studied by Shanin, N.A., [53] in general topology. Several topologists introduced fuzzy R_0 spaces in various ways, after the introduction of fuzzy topological space by Chang, C.L., [21] 1968.In fuzzy topology, this property was introduced and studied mainly by Srivastava A.K.[57], Lowen, R., and Wuyts, P., [36]; and also by Ali D.M.,Wuyts,P.,and Srivastava A.K. [8]. In this chapter, we introduce and study some R_0 properties in fuzzy supra topological spaces and obtain their several features mainly in the sense of Chang, C.L. We symbolize, fuzzy supra R_0 topological space by FSR_0.

2.1. Definitions of FSR_0 spaces.

2.1.1. Definitions:- Let (X, t^*) be fuzzy supra topological space, R_0- properties of (X, t^*) as follows: [We recall first nine axiom from [**8**]]

FSR_0(i) : For every pair $x, y \in X$, $x \neq y$, $\overline{1_y}(x) = 0 \Rightarrow \overline{1_x}(y) = 0$.

FSR_0(ii): $\forall$ pair $x, y \in X$, $x \neq y$, $\forall\ \alpha \in I_0 : \overline{\alpha 1_x}(y) = \alpha \Leftrightarrow \forall\ \beta \in I_0 : \overline{\beta 1_y}(x) = \beta$

FSR_0(iii) : $\forall\ \lambda \in t^*$, $\forall\ x \in X$ and $\forall\ \alpha < \lambda(x)$, $\overline{\alpha 1_x} \leq \lambda$

FSR_0(iv) : $\forall\ \lambda \in t^*$, $\forall\ x \in X$ and $\forall\ \alpha \leq \lambda(x)$, $\overline{\alpha 1_x} \leq \lambda$.

FSR_0(v) : For every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) = 1 \Rightarrow \overline{1_y}(x) = 1$.

FSR_0(vi): For every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x)$.

FSR_0(vii) : For every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0, 1\}$.

FSR_0(viii) : For every pair $x, y \in X$, $x \neq y$ and $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \alpha \Rightarrow \overline{\alpha 1_y}(x) = \alpha$.

FSR_0(ix) : For every pair $x, y \in X$, $x \neq y$ and $\forall \alpha \in I$, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$.

$FSR_0(x)$: $\forall$ x, y $\in$X, x$\neq$ y, whenever $\exists$ $\lambda \in t^*$, with λ (x)= 0 and λ (y)>0, there also also exist $\mu \in t^*$ with μ (x)>0,and $\mu(y) = 0$.

$FSR_0(xi)$: $\forall$ x, y $\in$X, x$\neq$y , whenever $\exists$ $\lambda \in t^*$, with λ (y)< λ (x) there also exist $\mu \in t^*$ with μ (x)< μ (y).

2.1.1. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) $FSR_0(i)$, i.e. for every pair x, y$\in$X, x $\neq$ y, $\overline{1_y}(x) = 0 \Rightarrow \overline{1_x}(y) = 0$.

(b) For every pair x, y$\in$X, x$\neq$y, $\overline{1_x}(y) = 0 \Leftrightarrow \overline{1_y}(x) = 0$

(c) $\forall$ x, y $\in$X, x$\neq$y, whenever $\exists$ $\lambda \in t^*$, with λ(x) = 1 and λ(y) = 0 there also exist $\mu \in t^*$ with μ(x) = 0 and μ(y) = 1. **[59]**

Proof: (a)$\Rightarrow$(b):

Suppose (X, t^*) is $FSR_0(i)$. Suppose $\overline{1_x}(y) = 0$. Then since (X, t^*) is $FSR_0(i)$, so $\overline{1_y}(x) = 0$. On the other hand if $\overline{1_y}(x) = 0$, then by $FSR_0(i)$, $\overline{1_x}(y) = 0$. Thus we see that $\overline{1_x}(y) = 0 \Leftrightarrow \overline{1_y}(x) = 0$.

(b)$\Rightarrow$(c):

Suppose x, y$\in$X, x $\neq$ y and there exists $\lambda \in t^*$, such that, λ(x) = 1 and λ(y) = 0. Put m = 1$-$ λ. Then m$\in t^{*c}$, m(x) = 0 and m(y) = 1. Again we have for every x, y$\in$X, such that x $\neq$ y, m(x) = 0 and m(y) = 1. Taking m = $\overline{1_y}$ and so $\overline{1_y}$ (x) = 0. By (b) $\overline{1_x}(y) = 0$. This implies that there exists a t^*-supra closed set k such that k(x) = 1 and k(y) = 0. Put μ= 1 $-$ k. Then clearly $\mu \in t^*$, μ(x) = 0 and μ(y) = 1.

(c)$\Rightarrow$(a):

Suppose x, y$\in$X, x $\neq$ y and $\overline{1_x}(y) = 0$. This implies that there exists a t^*-supra closed set k = $\overline{1_x}$ such that k(y) = 0 and k(x) = 1. Put λ = 1$-$k. Then λ is a t^*-supra open set such that λ(x) = 0 and λ (y) = 1. By (c) there exists a t^*-supra open set μ such that μ (x) = 1 and μ (y) = 0. Put m = 1 $-$ μ. Then m is a t^*-supra closed set such that m(y) = 1 and m(x) = 0. Thus there exist a t *-supra closed set m such that m(y) = 1 and m(x) = 0. Therefore, $\overline{1_y}(x) = 0$. ■

2.1.2. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) $FSR_0(ii)$, i. e. For every pair x, y$\in$X, x$\neq$ y,

$\left(\forall \alpha \in I_0 : \overline{\alpha 1_x}(y) = \alpha \Leftrightarrow \forall \beta \in I_0 : \overline{\beta 1_y}(x) = \beta\right)$

(b) For every x, y$\in$ X, x $\neq$ y, if there exists $\alpha \in I_0$ such that $\overline{\alpha 1_x}$ (y) $<\alpha$, then there exists $\beta \in I_0$ such that $\overline{\beta 1_y}(x) < \beta$.

(c) $FSR_0(xi)$.

Proof: (a)⇒(b):

Suppose x, y $\in$X, x $\neq$ y and there exists $\alpha \in I_0$ such that $\overline{\alpha 1_x}$ (y) $<\alpha$...............(1) Suppose for every $\beta \in I_0$, $\overline{\beta 1_x}(y) = \beta$. Then by (a) for every $\alpha \in I_0$, $\overline{\alpha 1_x}$ (y) $=\alpha$, which contradicts (1). Therefore there exists $\beta \in I_0$ such that $\overline{\beta 1_y}(x) < \beta$.

(b)⇒(c):

Suppose for every x, y$\in$ X, x $\neq$ y, there exists a t *-supra open set λ such that λ (y) < λ (x). Let $\beta = \lambda$ (y), then $\overline{\beta 1_y}(x) < \beta$. Hence by (b), there exist $\alpha_0 \in I_0$ such that $\overline{\alpha_0 1_x}(y) < \alpha_0$.This implies that there exists a t *-supra closed set, say η such that η (y) $\leq \alpha_0 < \eta$ (x) and so η (y) < η(x). Put $\mu = 1 - \eta$ Then μ is a t^*-supra open set and μ (x) < μ (y).

(c)⇒(a):

Suppose for every pair x, y$\in$X. x $\neq$ y and for every $\alpha \in I_0$, $\overline{\alpha 1_x}$ (y) $=\alpha$,. Let $\beta \in I_0$. Then $\overline{\beta 1_x}$ (y) = β. We have to show that $\overline{\beta 1_y}$ (x) $=\beta$. Suppose $\overline{\beta 1_y}$ (x)$\neq \beta$ Thus $\overline{\beta 1_y}$ (x) $<\beta$. This implies that there exists a t *-supra closed set, say η such that η (x) < η (y). Put μ = $1 - \eta$. Thus μ is supra open and μ (y) < μ (x). Hence by (c), there exists a t *-supra open set say λ such that λ (x) < λ (y). Therefore, $\overline{\lambda(y)1_y}$(x)< λ (x), and so $\overline{\lambda(y)1_y}$ (x) $<\lambda(y)$, which is a contradiction, so $\overline{\beta 1_y}$(x) $=\beta$.■

2.1.3. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) FSR_0(iii).

(b) For every $\lambda \in t^*$, there exists $M \subset t^{*c}$ such that $\lambda = \text{Sup}\,\mu$, $\mu \in M$

Proof: (a)⇒(b):

Let $u \in t^*$. Put M = { $\overline{\alpha 1_x}$: $x \in X, \alpha < \lambda(x)\}$. By FSR_0(iii), for every $\alpha < \lambda$ (x), $\overline{\alpha 1_x} \leq \lambda$.

Clearly, $\lambda = \text{Sup}\,\mu$, $\mu \in M$.■

(b)⇒(a):

Let $x \in X$ and λ is a t^*- supra open set such that $\alpha < \lambda$ (x). By (b) there exists $M \subset t^{*c}$, such that $\lambda = \text{Sup}\ \mu$, $\mu \in M$. Thus there exists a $\mu \in M$. such that $\alpha < \mu$ (x). That is $\overline{\alpha 1_x} \leq \mu$, so. $\overline{\alpha 1_x} \leq \mu \leq \lambda$ .Thus (X, t^*), is FSR_0(iii).■

2.1.4. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent **[8]**.

(a) FSR_0(iv), i.e $\forall \lambda \in t^*$, $x \in X$ and $\forall \alpha \leq \lambda$ (x), $\overline{\alpha 1_x} \leq \lambda$

(b) For every $\lambda \in t^*$, and for every $x \in X$, $\overline{\lambda(x)1_x} \leq \lambda$.

(c) For every $\lambda \in t^*$, $\lambda = \text{Sup}\,\overline{\lambda(x)1_x}$, $x \in X$.

(d) For every pair x, y ∈ X, x ≠ y and for every $\lambda \in t^*$, there exists $\mu \in t^{*c}$ such that μ (x) = λ (x) and μ (y) = λ (y).

(e) For every pair x, y ∈ X, x ≠ y, the subspace ({x, y}, $t^*|$ {x, y}) is self duel, i.e. ({x, y}, $t^*|$ {x, y})= ({x, y}, $t^{*c}|$ {x, y})

(f) For every pair x, y ∈ X, x ≠ y and for every pair α, β∈I, α ≠ β, $\overline{\alpha 1_x}(y) \leq \beta \Rightarrow \overline{(1-\beta)1_y}(x) \leq 1-\alpha$.

Proof: (a)⇒(b):

Let $\lambda \in t^*$, and $x \in X$, Put $\alpha = \lambda$ (x). By FSR_0(iv), $\overline{\alpha 1_x} \leq \lambda$,Thus $\overline{\lambda(x)1_x} \leq \lambda$■

(b)⇒(c):

Suppose $\lambda \in t^*$, If (b) is satisfied, then for every $x \in X$, $\overline{\lambda(x)1_x} \leq \lambda$. Therefore

$$\text{Sup } \overline{\lambda(x)1_x} \leq \lambda,\ x \in X \tag{1}$$

Now if $y \in X$, we also have $\lambda(y) = \overline{\lambda(y)1_y}(y) \leq \text{Sup } \overline{\lambda(x)1_x}(y)$, $x \in X$. Thus

$$\lambda \leq Sup\, \overline{\lambda(x)1_x} \tag{2}$$

From (1) and (2) $\lambda = \overline{\lambda(x)1_x}$, $x \in X$.

(c)⇒(d):

Let $\lambda \in t^*$, and $x, y \in X$ such that $x \neq y$. Without loss of generality suppose $\alpha = \lambda(x) \leq \lambda(y) = \beta$. Then $\overline{\beta 1_y} \leq \lambda(x) = \alpha$. Put $\mu_1 = \overline{\beta 1_y}$. Now μ_1 is a t^*-supra closed set such that $\mu_1(y) = \lambda(y) = \beta$ and $\mu_1(x) \leq \alpha = \lambda(x)$. Put $\mu = \mu_1 \vee \alpha$. Now $\mu(x) = \alpha = \lambda(x)$, and $\mu(y) = \beta = \lambda(y)$. Thus we see that there exists a $\mu \in t^{*c}$ such that $\mu(x) = \lambda(x)$, and $\mu(y) = \lambda(y)$. ■

(d)⇔ (e) :

Suppose (d) is satisfied. Therefore with the notations of (d) we have $\lambda|\{x, y\} = \mu|\{x, y\}$, thus $(\{x, y\}, t^*|\{x, y\}) = (\{x, y\}, t^{*c}|\{x, y\})$.

On the other hand, suppose (e) is satisfied, i.e. $(\{x, y\}, t^*|\{x, y\}) = (\{x, y\}, t^{*c}|\{x, y\})$, then for every pair $x, y \in X$, $x \neq y$ and for every $\lambda \in t^*$ there exists $\mu \in t^{*c}$ such that

$\mu(x) = \lambda(x)$ and $\mu(y) = \lambda(y)$.■

(d)⇒(f):

Suppose $x, y \in X$, $x \neq y$, and $\alpha, \beta \in I$, $\alpha \neq \beta$ such that $\overline{\alpha 1_x}(y) \leq \beta$. If $\beta < \alpha$ there is a $\mu \in t^*c$ such that $\mu(x) = \alpha$ and $\mu(y) \leq \beta$. Let $\mu_1 = \mu \vee \beta$. Then $\mu_1(x) = \alpha$ and $\mu_1(y) = \beta$. If (d) is satisfied, there is a $\lambda \in t^*$, such that $\lambda(x) = \alpha$ and $\lambda(y) = \beta$. Let $\eta = 1 - \lambda$, Then $\eta \in t^*c$. Now $\eta(x) = 1 - \alpha, \eta(y) = 1 - \beta$. Therefore, $\overline{(1-\beta)1_y}(x) = \text{Inf}\{\eta(x): \eta \in t^*c \text{ and } (1-\beta)1_y \leq \eta\} \leq \eta(x) = 1 - \alpha$. Therefore $\overline{(1-\beta)1_y}(x) \leq 1-\alpha$.

(f)⇒(a):

Suppose (f) is satisfied, $\lambda \in t^*$, and $\alpha \leq \lambda(x)$. We have to show that $\overline{\alpha 1_x} \leq \lambda$.

Let $y \in X - \{x\}$ and $\lambda(y) = \beta$. If $\beta > \alpha$, then it is clear that $\overline{\alpha 1_x} \leq \lambda$.

Suppose $\beta < \alpha$, let $\mu = 1 - \lambda$, Then $\mu \in t^{*c}$ such that $\mu(y) = 1 - \beta > 1 - \alpha \geq 1 - \lambda(x) = \mu(x)$.

Thus we have, $\mu(x) < \mu(y)$. Therefore, $\overline{\mu(y) 1_y}(x) \leq \mu(x)$.

Applying (f), $\overline{(1 - \mu(x)) 1_x}(y) \leq 1 - \mu(y)$

$\Rightarrow \overline{\lambda(x) 1_x}(y) \leq \lambda(y)$

$\Rightarrow \overline{\alpha 1_x}(y) \leq \lambda(y)$, [Since $\alpha \leq \lambda(x)$]

Therefore, $\overline{\alpha 1_x} \leq \lambda$. (Proved) ■

2.1.5. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) FSR_0 (v).

(b) For every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) = 1 \Leftrightarrow \overline{1_y}(x) = 1$.

(c) For every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) < 1 \Leftrightarrow \overline{1_y}(x) < 1$

(d) $\forall$ Pair $x, y \in X$, $x \neq y$, if there exists a t^*-supra closed set λ such that $\lambda(y) < 1 = \lambda(x)$, then there exists a t^*-supra closed set μ such that $\mu(x) < 1 = \mu(y)$.

Proof:

(a)⇒(b): Trivial.

(b)⇒(c):

Suppose, $\overline{1_x}(y) < 1$. We have to show that, $\overline{1_y}(x) < 1$. If $\overline{1_y}(x)$ is not less than 1, then $\overline{1_y}(x) = 1$, by (b) $\overline{1_x}(y) = 1$ which is a contradiction .Therefore, $\overline{1_y}(x) < 1$. Thus we see that $\overline{1_x}(y) < 1 \Rightarrow \overline{1_y}(x) < 1$. Similarly we can show that $\overline{1_y}(x) < 1 \Rightarrow \overline{1_x}(y) < 1$.

(c)⇒(d):

Suppose there exists a t^*- supra closed set μ such that $\mu(y) < 1 = \mu(x)$. Then $\overline{1_x}(y) < 1$. By (c) $\overline{1_y}(x) < 1$, Put $\lambda = \overline{1_y}$. Then clearly $\lambda(y) = 1$ and $\lambda(x) < 1$. Thus we see that there exists a t^*- supra closed set, say λ such that $\lambda(x) < 1 = \lambda(y)$.

(d)⇒(a):

Suppose $\overline{1_y}(x) = 1$. We have to show that $\overline{1_x}(y) = 1$.

Suppose $\overline{1_x}(y) < 1$ and $\overline{1_x} = \mu$. Thus μ is a t^*-supra closed set such that $\mu(y) < 1 = \mu(x)$. By (d) there exists a t^*-supra closed set λ such that $\lambda(x) < 1 = \lambda(y)$. This implies that $\overline{1_y}(x) < 1$, which is a contradiction. Therefore, $\overline{1_x}(y) = 1$. Thus we see that, for every pair x, y$\in$X, x $\neq$ y, $\overline{1_x}(y) = 1 \Rightarrow \overline{1_y}(x) = 1$. Thus (a) is satisfied.■

2.1.6. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) $FSR_0(vi)$. i.e For every pair x, y$\in$X, x $\neq$ y, $\overline{1_x}(y) = \overline{1_y}(x)$

(b) For every pair x, y $\in$ X, x $\neq$ y and for every $\alpha \in I_1$, $\overline{1_x}(y) \le \alpha \Rightarrow \overline{1_y}(x) \le \alpha$

(c) For every pair x, y $\in$ X, x $\neq$ y and for every $\alpha \in I_0$, if there exists a t^*-supra open set λ such that $\lambda(y) = 0 < \alpha = \lambda(x)$, then there exists a t^*-supra open set μ such that $\mu(x) = 0 < \alpha = \mu(y)$, i.e $FSR_0(x)$.

Proof:

(a)$\Rightarrow$(b):

Suppose, x, y $\in$ X, x $\neq$ y and $\alpha \in I_1$ such that $\overline{1_x}(y) \le \alpha$. By (a), $\overline{1_y}(x) = \overline{1_x}(y)$ Therefore, $\overline{1_y}(x) \le \alpha$.

(b)$\Rightarrow$(c):

Suppose, x, y $\in$ X, x $\neq$ y, $\alpha \in I_0$, and $\exists$ a t^*-supra open set λ such that $\lambda(y) < \alpha = \lambda(x)$. Put $\eta = 1 - \lambda$. Then η is a t^*-supra closed set such that, $\eta(y) > \eta(x) = 1 - \alpha$. Therefore $\overline{1_y}(x) \le 1 - \alpha$. Hence by (b) $\overline{1_x}(y) \le 1 - \alpha$. This implies that there exists a t^*-supra closed set ν such that $\nu(x) > \nu(y) = 1 - \alpha$. Put $\mu = 1 - \nu$. Then μ is a t^*-supra open set such that $\mu(x) > \mu(y)$

(c)$\Rightarrow$(a):

Suppose, $\overline{1_y}(x) < \overline{1_x}(y)$. Let $\eta = \overline{1_y}$ and $\alpha = \eta(x) \neq 1$. Then η is t^*-supra closed set such that $\eta(y) = 1$, $\eta(x) = \alpha < \overline{1_x}(y)$. Let $\lambda = 1 - \eta$. Then λ is a t^*-supra open set such that $\lambda(y) = 0$ and $\lambda(x) = 1 - \alpha > 0$. By (c), there exists a t^*-supra open set μ such that, $\mu(x) =$

0 and $\mu(y) = 1 - \alpha$. Put $\nu = 1 - \mu$. Then ν is a t^*-supra closed such that $\nu(x) = 1$ and $\nu(y) = \alpha$. This implies that $\overline{1_x}(y) < \alpha = \overline{1_y}(x)$, a contradiction. ■

2.1.7. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent **[8]**.

(a) $FSR_0(vii)$, i.e for every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0, 1\}$.

(b) $\{\overline{1_x} : x \in X\}$ defines a partition of 1, i.e. there is a partition $\mathcal{A}$ of X such that for every $x \in A \in \mathcal{A}$. $\overline{1_x} = 1_A$.

Proof: (a)⇒(b):

We have $\overline{1_x}(y) = \overline{1_y}(x) \in \{0,1\}$. Therefore, $\overline{1_x}(X) \subset \{0,1\}$, and so there exists, for each $x \in X$, an $A(x) \subset X$ such that $\overline{1_x} = 1_{A(x)}$. Now if $y \in A(x)$, then $\overline{1_x}(y) = 1$ i.e. $1_y \leq 1_{A(x)}$. It follows that $1_{A(y)} \leq 1_{A(x)}$, so $A(y) \subset A(x)$. Now $\overline{1_x}(y) = \overline{1_y}(x) = 1$. Therefore, $x \in A(y)$, hence $A(x) \subset A(y)$. Therefore $A(x) = A(y)$. Hence $\{A(x): x \in X\}$ is a partition of X.
(b)⇒(a):

Given $\{\overline{1_x} : x \in X\}$ is a partition of X. This implies that, either $\overline{1_x} = \overline{1_y}$ or $\overline{1_x} \wedge \overline{1_y} = 0$. If $\overline{1_x} = \overline{1_y}$, then clearly $\overline{1_x}(y) = \overline{1_y}(x) = 1$. On the other hand, if $\overline{1_x} \wedge \overline{1_y} = 0$, then $(\overline{1_x} \wedge \overline{1_y})(x) = 0$ and $(\overline{1_x} \wedge \overline{1_y})(y) = 0$. Therefore, $\overline{1_x}(y) = 0 = \overline{1_y}(x)$. Thus
$\overline{1_x}(y) = \overline{1_y}(x) \in \{0, 1\}$.

2.1.8. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) $FSR_0(viii)$, i.e For every pair $x, y \in X$, $x \neq y$ and $\forall \alpha \in I_0$,

$$\overline{\alpha 1_x}(y) = \alpha, \Rightarrow \overline{\alpha 1_y}(x) = \alpha.$$

(b) For every pair, $x, y \in X$, $x \neq y$ and for every $\alpha \in I_0$, $\overline{\alpha 1_x}(y) < \alpha \Rightarrow \overline{\alpha 1_y}(x) < \alpha$.

Proof: (a)⇒(b):

Suppose x, y $\in$ X, x $\neq$ y and $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) < \alpha$. Suppose $\overline{\alpha 1_y}(x) = \alpha$. Then by (a) $\overline{\alpha 1_x}(y) = \alpha$, which is a contradiction .Therefore $\overline{\alpha 1_y}(x) < \alpha$.

(b)$\Rightarrow$(a):

Suppose x, y $\in$ X and $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) = \alpha$. Suppose $\overline{\alpha 1_y}(x) \neq \alpha$. Therefore,

$\overline{\alpha 1_y}(x) < \alpha$, Then by (b), $\overline{\alpha 1_x}(y) < \alpha$, which is a contradiction .Therefore $\overline{\alpha 1_y}(x) = \alpha$

2.1.9. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) $FSR_0(ix)$, i.e For every pair x, y$\in$X, x $\neq$ y and $\forall \alpha \in I_0, \overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$

(b) For every pair, x, y $\in$ X, x $\neq$ y and for every t^*- Supra closed set, μ there exists a t^*- Supra closed set, ν such that $\nu(x) = \mu(y)$, $\nu(y) = \mu(x)$.

Proof: **(a)$\Rightarrow$(b):**

Let x, y $\in$X, x $\neq$ y and μ is a t^*- Supra closed set. Let $\alpha = \mu(x)$ and $\beta = \mu(y)$. This implies that $\overline{\alpha 1_x}(y) \leq \beta$. Therefore, $\overline{\alpha 1_y}(x) \leq \beta$. Hence there exists a t^*- Supra closed set ν such that $\nu(y) = \alpha$ and $\nu(x) = \beta$. Thus $\mu(x) = \nu(y)$ and $\mu(y) = \nu(x)$.

(b)$\Rightarrow$(a):

Without loss of generality suppose, $\overline{\alpha 1_x}(y) < \overline{\alpha 1_y}(x)$ (3)

Let $\mu = \overline{\alpha 1_x}$. Then $\alpha = \mu(x)$. Let $\beta = \mu(y)$. Then by (3) $\beta < \overline{\alpha 1_y}(x)$ (4)

By (b) there exists t^*- Supra closed set ν such that $\nu(x) = \mu(y) = \beta$ and $\nu(y) = \mu(x) = \alpha$.

Wehave, $\overline{\nu(y) 1_y}(x) \leq \nu(x)$

$$\Rightarrow \overline{\alpha 1_y}(x) \leq \beta$$

Using (3), $\overline{\alpha 1_x}(y) < \overline{\alpha 1_y}(x) \leq \beta$.

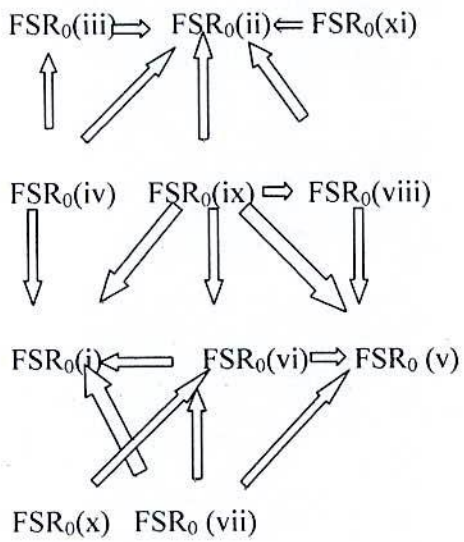
or, $\overline{\alpha 1_x}(y) < \beta$, or $\mu(y) < \beta$ which is a contradiction.

Therefore, $\overline{\alpha 1_x}(y) < \overline{\alpha 1_y}(x)$ is not true. Similarly we can show that

$\overline{\alpha 1_y}(x) < \overline{\alpha 1_x}(y)$ is also not true, hence $\overline{\alpha 1_y}(x) = \overline{\alpha 1_x}(y)$.

2.2. Relationships among FSR_0 Spaces.

2.2.1 Theorem: The following implications are true:



Thus we have

(a): $FSR_0(iv) \Rightarrow FSR_0(iii) \Rightarrow FSR_0(ii)$

(b): $FSR_0(iv) \Rightarrow FSR_0(i)$

(c): $FSR_0(vii) \Rightarrow FSR_0(vi) \Rightarrow FSR_0(i)$

(d): $FSR_0(vi) \Rightarrow FSR_0(v)$

(e): $FSR_0(ix) \Rightarrow FSR_0(viii) \Rightarrow FSR_0(ii)$

(f): $FSR_0(viii) \Rightarrow FSR_0(v)$

(g): $FSR_0(ix) \Rightarrow FSR_0(vi) \Leftarrow FSR_0(x)$

Proof:

(a): Suppose (X, t^*) is $FSR_0(iv)$ Let $\lambda \in t^*$, $x \in X$ and $\alpha < \lambda(x)$. Then since (X, t^*), is

FSR_0(iv) hence $\overline{\alpha 1_x} \le \lambda$. So $\forall\ \lambda \in t^*$, $\forall$ x$\in$X and $\forall\ \alpha < \lambda(x)$, $\overline{\alpha 1_x} \le \lambda$.Therefore (X, t^*) is FSR_0(iii).

Suppose, there exists $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) = \beta < \alpha$. Take $\beta < \gamma < \alpha$. Let $\lambda = 1 - \overline{\alpha 1_x}$.Then $\lambda(x) = 1 - \alpha$, $\lambda(y) = 1 - \beta > 1 - \gamma$. Since (X, t^*) is FSR_0(iii), $\overline{(1-\gamma)1_y} \le \lambda$. Now $\overline{(1-\gamma)1_y}(x) \le \lambda(x) = 1 - \alpha < 1 - \gamma$. Thus we see that, if $\overline{\alpha 1_x}(y) < \alpha$.then there exists $\delta \in I_0$ such that $\overline{\delta 1_y}(x) < \delta$. So by lemma, 2.1.3. (X, t^*), is FSR_0(ii).

(b): Suppose (X, t^*) is FSR_0(iv) , by lemma-2.1.5,we have for every pair x, y $\in$ X , x $\neq$ y and for every pair α, $\beta \in$ I, $\alpha \neq \beta$, $\overline{\alpha 1_x}(y) \le \beta \Rightarrow \overline{(1-\beta)1_y}(x) \le 1 - \alpha$. Taking α = 1 and $\beta = 0$, $\overline{1_x}(y) \le 0 \Rightarrow \overline{1_y}(x) \le 0$. Or $\overline{1_x}(y) = 0 \Rightarrow \overline{1_y}(x) = 0$. Which is FSR_0(i).

(c) Suppose (X, t^*) is FSR_0(vii). Then clearly, for every x, y $\in$X, x $\neq$ y, $\overline{1_x}(y) = \overline{1_y}(x)$. Therefore (X, t^*) is FSR_0(vi).

Let, $\overline{1_x}(y)$ =0. As (X, t^*) is FSR_0(vi), $\overline{1_x}(y) = \overline{1_y}(x)$ and so $\overline{1_y}(x) = 0$.Therefore, (X, t^*) is FSR_0(i).

(d): Suppose (X, t^*) is FSR_0(vi).Then $\overline{1_x}(y) = \overline{1_y}(x)$. Therefore if $\overline{1_x}(y)$ =1, then $\overline{1_y}(x) = 1$. Therefore (X, t^*) is FSR_0(v).

(e): Suppose (X, t^*) is FSR_0(ix). Therefore for every pair x, y$\in$X, x $\neq$ y and $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$. Therefore, if $\overline{\alpha 1_x}(y) = \alpha$ then $\overline{\alpha 1_y}(x) = \alpha$. Hence (X, t^*) is FSR_0(viii).■

Again, suppose for every $\alpha \in I_0$, $\overline{\alpha 1_x}(y) = \alpha$. Then clearly for every $\beta \in I_0$, $\overline{\beta 1_x}(y) = \beta$. Since, (X, t^*) is FSR_0(viii), $\overline{\beta 1_x}(y) = \beta \Rightarrow \overline{\beta 1_y}(x) = \beta$. Therefore, we see that, for every pair x, y$\in$X, x $\neq$ y, $\forall\ \alpha \in I_0 : \overline{\alpha 1_x}(y) = \alpha \Rightarrow \forall\ \beta \in I_0 : \overline{\beta 1_y}(x) = \beta$ .Similarly we can show that, for every pair x, y$\in$X, x $\neq$ y, $\forall\ \beta \in I_0$; $\overline{\beta 1_y}(x) = \beta \Rightarrow \forall\ \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \alpha$. Thus (X, t^*) is FSR_0(ii).

(f): Suppose(X, t^*) is FSR_0(viii), x, y ∈ X and $\overline{1_x}$ (y)=1. Since (X, t^*) is FSR_0 (viii), if α = 1, $\overline{1_y}(x) = 1$. Therefore (X, t^*) is FSR_0(v).

(g): Suppose (X, t^*) is FSR_0 (ix), then for every pair x, y ∈ X, x ≠ y and for every α ∈ I_0, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$. In particular, if α = 1, $\overline{1_x}(y) = \overline{1_y}(x)$. Therefore, (X, t^*) is FSR_0(vi).Also from lemma 2.1.7. ; it is clear that if (X, t^*) is FSR_0(x),then it satisfies FSR_0(vi).

Now we give some examples:

2.2.1. Example: Let the fuzzy supra topological space (X, t^*) where X={x, y}, and t^*= {0,u={(x, 0),(y, 1)}; v={(x, 1),(y, 1)}} ,$u'(x)=1$, $u'(y)=0$; $v'(x)=0$, $v'(y)=0$.Let $\overline{1_x} = u'$, $\overline{1_y} = v'$, Here $\overline{1_x}(y) = 0$, $\overline{1_y}(x)=0$, let α=.4 ≮u(x) and $\overline{\alpha 1_x}(y) = \alpha \vee \overline{1_x}(y)$ =.4∨0=.4 ≤u(y). So, FSR_0(i) ⇏ FSR_0(iii). Similarly we shall prove that, FSR_0 (1)⇏ FSR_0 (iv).

2.2.2 . Example: Let (X, t^*) be a fuzzy supra topological space, where X={x, y} and t^*= {0,u={(x, 0),(y, 1)}; v={(x, 1),(y, 1)}} ,$u'(x)=1$, $u'(y)=0$; $v'(x)=0$, $v'(y)=0$ $\overline{1_x} = u'$, $\overline{1_y} = v'$ then $\overline{1_x}(y) = \overline{1_y}(x)$, hence (X, t^*) is FSR_0(vi) ;but u(x)=0, u(y)=1>0; and there does not exist v(x)=0, v(y)>0. Hence FSR_0 (vi)⇏ FSR_0(x).

2.2.3. Example: Let (X, t^*) be a fuzzy supra topological space, where X={x, y} and t^*= {0,u={(x, 0),(y, 1)}; v={(x, 1),(y, 1)}}, $u'(x)=1$, $u'(y)=0$; $v'(x)=0$, $v'(y)=0$.Let $\overline{1_x} = u'$, $\overline{1_y} = v'$, Here $\overline{1_x}(y) = 0$, $\overline{1_y}(x)=0$, then $\overline{1_x}(y) = \overline{1_y}(x)$, hence (X, t^*) is FSR_0(vi) ;here u(x)<u(y), but v(x)=v(y).Hence FSR_0(vi)⇏ FSR_0(x).

2.2.4. Example: Let (X, t^*) be a fuzzy supra topological space, where X={x, y} and t^*= {0, u={(x, 0),(y, 1)}; v={(x, 1), (y, 1)}} ,$u'(x)=1$, $u'(y)=0$; $v'(x)=0$, $v'(y)=0$ Here u(x)<u(y), and Let $\overline{1_x} = u'$, $\overline{1_y} = v'$, then $\overline{1_x}(y)=u'(y)=0$, then $\overline{\alpha 1_x}(y) = \alpha \vee 1_x(y) = \alpha \vee 0 = \alpha$ .Now $\overline{\beta 1_y}(x) = \beta \vee 1_y(x) = \beta \vee 0 = \beta$ Hence (X, t^*)is FSR_0(ii), Here u(x)<u(y), but v(y)≮v(x) , So FSR_0(ii)⇏ FSR_0(xi).

2.2.5. Example: Let (X, t^*) be a fuzzy supra topological space, where $X=\{x, y\}$ and $t^*=\{1, u=\{(x, 0),(y, 0)\}; v=\{(x, 0),(y, 1)\}\}$, $u'(x)=1$, $u'(y)=1$; $v'(x)=1$, $v'(y)=0$; $\overline{1_x} = u'$, $\overline{1_y} = v'$ then $\overline{1_x}(y) = \overline{1_y}(x) = 1$ hence (X, t^*) is $FSR_0(v)$ but for all $\alpha \in I_0$ may not $\overline{1_x}(y) = \overline{1_y}(x)$, so $FSR_0(vi) \not\Rightarrow FSR_0(ix)$.

2.3 . Good extension property:

In this section we show that all $FSR_0(k)(i \le k \le xi)$ properties are good extensions of their supra topological counter parts; all of them are also found to be hereditary.

2.3.1. Definition: Let (X, T^*) be a Supra topological space, the space (X, T^*) is called Supra R_0 topological space, if $\forall$ x, y $\in$ X, $x \neq y$, then if $x \in \{\overline{y}\}$ their also exist $y \in \{\overline{x}\}$.

2.3.2. Definition Let (X, T^*) be a Supra topological space, the space (X, T^*) is called Supra R_0 topological space, if $\forall$ x, y $\in$ X, $x \neq y$, then if $\exists$ supra open set $u \in T^*$ with $x \in u$ and $y \notin u$ then their also $\exists$ an supra open set $v \in T^*$ such that $y \in v$ and $x \notin v$. We denote Supra R_0 topological space, by SR_0 -space.

2.3.1. Theorem: All $FSR_0(k)$ $(i \le k \le xi)$ properties are good extensions of the topological SR_0-property. That is,

(a) If (X, T^*) is an SR_0-space, then $(X, \omega(T^*))$ are also $FSR_0(k)$ $(i \le k \le xi)$ spaces.

(b) If $(X, \omega(T^*))$ satisfies $FSR_0(k)$ $(i \le k \le ix)$ then (X, T^*) is an SR_0-space.

Proof (a): Suppose (X, T^*) is an SR_0-space. Let $\lambda \in \omega(T^*) = \{u \in I^X : u^{-1}(\alpha, 1] \in T^*, \alpha \in I_1\}$, $\lambda(x) = \alpha < \lambda(y) = \beta$. Let $F = \lambda^{-1}(0, \alpha]$, then F is closed in (X, T^*). We have $y \notin F$. Therefore, $F \cap \overline{\{y\}} = \varnothing$, also $\overline{\{x\}} \subset F$. Put $\mu = \alpha 1_{\overline{\{x\}}} \vee \beta 1_{\overline{\{y\}}}$. Then μ is closed in $\omega(T^*)$. Now, $\mu(x) = \alpha$ and $\mu(y) = \beta$. Thus $\mu(x) = \lambda(x)$ and $\mu(y) = \lambda(y)$. Therefore $(X, \omega(T^*))$ is $FSR_0(iv)$. We know $FSR_0(iv) \Rightarrow FSR_0(iii) \Rightarrow FSR_0(ii)$ and $FSR_0(iv) \Rightarrow FSR_0(i)$.

Again, $\overline{\alpha 1_x} = \alpha 1_{\{\bar{x}\}}$, $\overline{\alpha 1_y} = \alpha 1_{(\bar{y})}$, we have $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x) = \alpha$ if and only if $\overline{\{x\}} = \overline{\{y\}}$ and $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x) = 0$ if and only if $\overline{\{x\}} \cap \overline{\{y\}} = \varphi$, So $(X, \omega(T^*))$ is $FSR_0(ix)$. We know $FSR_0(i) \Rightarrow FSR_0(vi)$, $FSR_0(ix) \Rightarrow FSR_0(viii) \Rightarrow FSR_0(ii)$, $FSR_0(vi) \Rightarrow FSR_0(v)$, $FSR_0(vii) \Rightarrow FSR_0(vi) \Rightarrow FSR_0(i)$. Thus (X, $\omega(T^*)$) is $FSR_0(k)$ $(i \le k \le xi)$.

Proof (b): (1) Suppose (X, $\omega(T^*)$) is an $FSR_0(i)$ space and $x \in \overline{\{y\}}$, then $\overline{1_y}(x) = 1_{\overline{\{y\}}}(x) = 1 \neq 0$, and so $1_{\overline{\{x\}}}(y) = \overline{1_x}(y) \neq 0$. Therefore, $y \in \overline{\{x\}}$ which proves that (X, T^*) is an SR_0-space.

(2) Suppose (X, $\omega(T^*)$) is an FSR_0 (ii) space and $x \in \overline{\{y\}}$,

then $\overline{\alpha 1_y}(x) = \alpha 1_{\overline{\{y\}}}(x) = \alpha$ for all $\alpha \in I_0$. Therefore, $\overline{\beta 1_x}(y) = \beta$, for every $\beta \in I_0$. So in particular $\overline{1_x}(y) = 1_{\overline{\{x\}}}(y) = 1$. Hence $y \in \overline{\{x\}}$, which proves that (X, T^*)) is an SR_0-space.

(3) Suppose (X, ω (T)) is an $FSR_0(v)$ space and $x \in \overline{\{y\}}$, then $\overline{1_y}(x) = 1_{\overline{\{y\}}}(x) = 1$. By $FSR_0(v)$, $\overline{1_x}(y) = 1_{\overline{\{x\}}}(y) = 1$. Therefore, $y \in \overline{\{x\}}$ which proves that (X, T^*) is an SR_0-space. Thus we see that, if (X, $\omega(T^*)$) satisfies FSR_0 (k) (k = i, ii, v) then (X, T^*) is an SR_0-space. Also we know that, $FSR_0(iv) \Rightarrow FSR_0(iii) \Rightarrow FSR_0(ii)$, $FSR_0(vii) \Rightarrow FSR_0(vi) \Rightarrow FSR_0(i)$, $FSR_0(ix) \Rightarrow FSR_0(viii) \Rightarrow FSR_0(ii)$.

(4) Let (X, $\omega(T^*)$) is FSR_0 (x), we shall prove that (X, T^*) is supra R_0. Let x, y $\in$X; with $x \neq y$, $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Suppose $x \in U$ and $y \notin U$ and $U \in T^*$, by the definition of lsc $1_U \in \omega(T^*)$ and with $1_U(x) = 1 > 0$, $1_U(y) = 0$, since (X, $\omega(T^*)$)is $FSR_0(x)$, then $\exists\ v \in \omega(T^*)$ $\ni v(x)=0$, $v(y)>0$. Choose α such that $v(y)>\alpha>0$. Then $v^{-1}(\alpha,1] \in t^*$, $y \in v^{-1}(\alpha,1]$ and $x \notin v^{-1}(\alpha,1]$. Hence it is clear that (X, T^*) is supra R_0 spaces.

Conversely suppose that (X, T^*) is supra R_0 space. We shall prove that(X, $\omega(T^*)$) is FSR_0 (x). Let x, y $\in$X; with $x \neq y$ and there exist $\lambda \in \omega(T^*)$ such that $\lambda(x)>0$, $\lambda(y) = 0$, Choose $\alpha \in I_0$ such that $\lambda(x) > \alpha > 0$. Then $\lambda^{-1}(\alpha,1] \in T^*$, hence $x \in \lambda^{-1}(\alpha,1]$, $y \notin \lambda^{-1}(\alpha, 1]$ as $\lambda(x)>\alpha$, $\lambda(y)=0$; Since (X, T^*) is supra R_0 space then $\exists\ \mu \in T^*$ such that $x \notin \mu$ and $y \in \mu$, but $1\mu \in \omega(T^*)$and $1\mu(x) = 0$, $1\mu(y) = 1 > 0$. Hence it is clear that (X, $\omega(T^*)$) is FSR_0 (x). This completes the proof.

Similarly we can prove this theorem for $FSR_0(xi)$.

Therefore, if (X, $\omega(T^*)$) satisfies $FSR_0(k)$ $(i \le k \le xi)$, then (X, T^*) is an SR_0-space.

2.4. Reciprocal properties of FSR_0 spaces.

2.4.1. Definition: Let X be set and $(X', t^{*'})$ be an fuzzy supra topological space. Consider a function f: $X \to (X', t^{*'})$. Let $t^* = \{ f^{-1}(u) : u \in t^{*'} \}$. Then t^* is a fuzzy supra topology on X. We call t^*, the reciprocal topology on X.

2.4.1.Theorem : Let X be a set, $(X', t^{*'})$ be a fuzzy supra topological space having the property $FSR_0(k)$ (k=I, ii, iii, ...xi), then the reciprocal fuzzy supra topology t^* on X for f: $X \to (X', t^{*'})$ also has $FSR_0(k)$, (k=I, ii, iii, ...xi).

Proof: Suppose $(X', t^{*'})$ a fuzzy supra topological space having the property $FSR_0(K)$ $(1 \le k \le 11)$. Suppose, $t^* = \{ f^{-1}(u) : u \in t^{*'} \}$. Now (X, t^*) is a fuzzy supra topological space. We have to show that (X, t^*) has $FSR_0(k)$ $(i \le k \le xi)$.

We have, $\overline{\alpha 1_x} = f^{-1}(\overline{f(\alpha 1_x)}) = f^{-1}\left(\overline{\alpha 1_{f(x)}}\right)$ i.e. $\forall$ y $\in$ X, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_{f(x)}}(f(y))$ ....(**).

1. Suppose x, y $\in$ X, x $\neq$ y, $\overline{1_x}(y)=0$, then $\overline{1_{f(x)}}(f(y)) = 0$, and since $(X', t^{*'})$ has $FSR_0(i)$, $\overline{1_{f(y)}}(f(x)) = 0$. Using (**), $f^{-1}(\overline{1_{f(y)}})(x) = 0$, and so $\overline{1_y}(x) = 0$. Therefore, (X, t^*) has $FSR_0(i)$.

2. Suppose x, y $\in$ X, x $\neq$ y, $\alpha \in I_0$ and $\overline{\alpha 1_x}(y) = 0$. Then $\overline{\alpha 1_{f(x)}}(f(y)) = 0$. and since $(X', t^{*'})$ has $FSR_0(ii)$, $\overline{\beta 1_{f(y)}}(f(x)) = 0$, for every $\beta \in I_0$. Using (**), $\overline{\beta 1_y}(x) = 0$ for every $\beta \in I_0$. This implies that (X, t') has $FSR_0(ii)$.■

3. Suppose $x \in X$, $\lambda \in t^*$ and $\alpha < \lambda(x)$. There is a $\lambda' \in t^{*'}$ such that $\lambda = f^{-1}(\lambda') = \lambda' \circ f$. Now, $\alpha < \lambda(x) = \lambda'(f(x))$. Since $(X', t^{*'})$ has $FSR_0(iii)$., $\overline{\alpha 1_{f(x)}} \le \lambda'$.Now, Using (**), $\overline{\alpha 1_x} = f^{-1}\left(\overline{\alpha 1_{f(x)}}\right) \le f^{-1}(\lambda') = \lambda$. Therefore (X, t^*) has $FSR_0(iii)$.

4. Suppose $x \in X$, $\lambda \in t^*$ and $\alpha \le \lambda(x)$. There is a $\lambda' \in t^{*'}$ such that $\lambda = f^{-1}(\lambda') = \lambda' \circ f$. Now, $\alpha \le \lambda(x) = \lambda'(f(x))$. Since $(X', t^{*'})$ has $FSR_0(iv)$, $\overline{\alpha 1_{f(x)}} \le \lambda'$.Now

$\overline{\alpha 1_x} = f^{-1}\left(\overline{\alpha 1_{f(x)}}\right) \leq f^{-1}(\lambda') = \lambda$. Therefore (X, t *) has FSR_0(iv).

5. Suppose x, y $\in$ X, x $\neq$ y, $\overline{1_x}(y) = 1$. Then $\overline{1_{f(x)}}(f(y)) = 1$ and since (X$'$, t $^{*/}$) has FSR_0(v), $\overline{1_{f(y)}}(f(x)) = 1$ and so $\overline{1_y}(x) = 1$. Therefore, (X, t *) has FSR_0(v).

6. Suppose x, y $\in$ X, x $\neq$ y. If (X$'$, t $^{*/}$) has FSR_0(vi), then $\overline{1_{f(x)}}(f(y)) = \overline{1_{f(y)}}(f(x))$.

Therefore, $\overline{1_x}(y) = \overline{1_y}(x)$. Therefore, (X, t) has FSR_0(vi).

7. Suppose x, y $\in$ X, x $\neq$ y. If (X$'$, t $^{*/}$) has FSR_0(vii), then $\overline{1_{f(x)}}(f(y)) = \overline{1_{f(y)}}(f(x))$ $\subset\{0, 1\}$. Therefore, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0,1\}$. Therefore, (X, t *) has FSR_0(vii).

8. Suppose x, y $\in$ X, x $\neq$ y, $\alpha \in I_0$. Suppose, $\overline{\alpha 1_x}(y) = \alpha$. Using (**), $\overline{\alpha 1_{f(x)}}(f(y)) = \alpha$ If (X$'$, t$'$) has FSR_0(viii), then $\overline{\alpha 1_{f(y)}}(f(x)) = \alpha$. Using (**), $\overline{\alpha 1_y}(x) = \alpha$. Therefore, (X, t *) has FSR_0(viii).

9. Suppose x, y $\in$ X, x $\neq$ y, $\alpha \in I_0$. If (X$'$, t$^{*/}$) has FSR_0(ix), then $\overline{\alpha 1_{f(x)}}(f(y)) = \overline{\alpha 1_{f(y)}}(f(x))$.

Using (**), $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$. We have, (X, t *) has FSR_0(ix).

It is easy to prove the conditions for FSR_0(x) and FSR_0(xi).

2.5. Initial, Productivity and Hereditary Properties.

2.5.1. Theorem: The properties FSR_0(k) , k $\in$ {i, ii, iii, v, ...xi} are initial, i.e. if $f : X \to (X_i, t_i^*)_{i\in J}$ is a source of fsts. where all (X_i, t_i) are FSR_0(k) then initial fuzzy supra topological spaces is also FSR_0(k) spaces.

Proof: Let $\{(X_i, t_i^*)_{i\in J}\}$ be a family of FSR_0(iii) , and $\{ f : X \to (X_i, t_i^*)_{i\in J} \}$ be a family of functions and t^* be the initial fuzzy supra topology on X induced by the family $\{ f_i : i\in J\}$. Let $\alpha \in I_{0,1}$ x$\in$ X, and $\lambda \in t^*$ such that $\alpha 1_x < \lambda$. Since $\lambda \in t^*$, we can find basic t^*-supra open sets λ_i, i$\in$J such that $\lambda = \text{Sup}\{ \lambda_i , i\in J \}$. Also λ_i must be expressible as $\lambda_i = \text{Inf}\{ f_{ik}^{-1}(\lambda_{ik}) : 1 \leq k \leq n\}$ where $\lambda_{ik} \in t_{ik}^*$ and ik$\in$J. Now we can find some k, ($1 \leq k \leq n$), say k_1 such that $\alpha 1_x < f^{-1}_{ik_1}(\lambda_{ik_1})$ that is $\alpha < f^{-1}_{ik_1}(\lambda_{ik_1})(x)$ or $\alpha < \lambda_{ik_1} f_{ik_1}(x)$. Since $(X_{ik_1}, t^*_{ik_1})$ is FSR_0(iii), $\overline{\alpha 1_{f_{jk_1}}} < \lambda_{ik_1}$ Since f is continuous $f_{ik_1}(\overline{\alpha 1_x}) < \overline{\alpha 1_{f_{jk_1}}}(x)$, thus $f_{ik_1}(\overline{\alpha 1_x}) < \lambda_{ik_1} \Rightarrow \overline{\alpha 1_x} \leq f^{-1}_{ik_1}(\lambda_{ik_1})$. But each $f^{-1}_{ik_1}(\lambda_{ik_1}) \leq \lambda$, therefore $\overline{\alpha 1_x} \leq \lambda$ and hence (X, t^*) is FSR_0(iii). So the properties FSR_0(iii) is initial

(b) Let $\{(X_i, t^*_i)_{i\in J}\}$ be a family of FSR_0(v) , and $\{ f : X \to (X_i, t^*_i)_{i\in J} \}$ be a family of functions and t^* be the initial fuzzy supra topology on X induced by the family $\{ f_i : i\in J\}$. Let x, y$\in$ X, x$\neq$y and $\exists$ $\mu \in t^{*c}$ such that $\mu(y) < 1 = \mu(x)$, Put $\lambda = 1 - \mu$ then $\lambda \in t^*$, such that $\lambda(x)=0$ and $\lambda(y)>0$, since $\lambda \in t^*$, we can find basic t^*- supra open sets λ_i, i$\in$J such that λ=Sup$\{ \lambda_i , i\in J \}$. Also each λ_i must be expressible as, λ_i = Inf $\{ f^{-1}_{ik}(\lambda_{ik}) : 1 \leq k \leq n\}$ where $\lambda_{ik} \in t^*_{ik}$ and ik$\in$J .Since $\lambda(x)=0$ and $\lambda(y)>0$ Now we can find some k, $(1 \leq k \leq n)$, say k_1 such that $f^{-1}_{ik_1}(\lambda_{ik_1})$ (x)=0 and that is $f^{-1}_{ik_1}(\lambda_{ik_1})(y) > 0$. $\Rightarrow \lambda_{ik_1} f_{ik_1}(x) = 0$ and $\Rightarrow \lambda_{ik_1} f_{ik_1}(y) > 0$. Since $(X_{ik_1}, t^*_{ik_1})$ is FSR_0(v), and hence (X, t^*) is FSR_0(v).$\exists$ $V_{i_{k_1}} \in t^*_{ik_1}$ such that $f^{-1}_{ik_1}(V_{i_{k_1}})(y)=0$ and $f^{-1}_{ik_1}(V_{i_{k_1}})(x)>0$. Now let $v = 1 - f^{-1}_{ik_1}(V_{i_{k_1}})$. Then v$\in t^{*c}$ such that $v(x) < 1 = v(y)$. So by lemma 2.1.6 implies that (X, t^*) is FSR_0(v).

(c) Let $\{(X_i, t^*_i)_{i\in J}\}$ be a family of FSR_0(x) , and $\{ f : X \to (X_i, t^*_i)_{i\in J} \}$ be a family of functions and t^* be the initial fuzzy supra topology on X induced by the family $\{ f_i : i\in J\}$. Let x, y$\in$ X, x$\neq$y .Let $\alpha \in I_{0,1}$ and $\lambda \in t^*$ such that $\lambda(y)=0 < \alpha = \lambda(x)$, since $\lambda \in t^*$, we can find basic t^*- supra open sets λ_i, i$\in$J such that λ=Sup$\{ \lambda_i , i\in J \}$. Also each λ_i must be expressible as, λ_i = Inf $\{ f^{-1}_{ik}(\lambda_{ik}) : 1 \leq k \leq n\}$ where $\lambda_{ik} \in t^*_{ik}$ and ik$\in$J .Since $\lambda(y)=0 < \alpha = \lambda(x)$ Now we can find some k, $(1 \leq k \leq n)$, say k_1 such that $f^{-1}_{ik_1}(\lambda_{ik_1})$ (y)=0 $< \alpha = f^{-1}_{ik_1}(\lambda_{ik_1})(x) > 0$. $\Rightarrow \lambda_{ik_1} f_{ik_1}(y) = 0 < \alpha = \lambda_{ik_1} f_{ik_1}(x)$. Since $(X_{ik_1}, t^*_{ik_1})$ is FSR_0(x), hence there exist $\mu_{i_{k_1}} \in t^*_{ik_1}$ such that $\mu_{ik_1} f_{ik_1}(x) = 0 < \alpha = \mu(y) = \mu_{ik_1} f_{ik_1}(y)$.

Now let $f_{ik_l}^{-1}(\mu_{i_{k_1}}) = \mu \in t^*$ .Then $\mu(x)=0 < \alpha = \mu(y)$. Hence that (X, t^*) is $FSR_0(x)$. Since $FSR_0(x) \Rightarrow FSR_0(vi)$. So (X, t^*) is also $FSR_0(vi)$ and $FSR_0(i)$

(d) Let $\{(X_i, t_i^*)_{i\in J}\}$ be a family of $FSR_0(xi)$, and $\{ f: X \rightarrow (X_i, t_i^*)_{i\in J} \}$ be a family of functions and t^* be the initial fuzzy supra topology on X induced by the family $\{ f_i : i\in J\}$. Let $x, y \in X$, $x \neq y$ and $\exists\ \lambda \in t^*$ such that $\lambda(y) < \lambda(x)$. We can find basic t^*- supra open sets λ_i, $i\in J$ such that $\lambda = \text{Sup}\ \{ \lambda_i, i\in J \}$. Also λ_i must be expressible as $\lambda_i = \text{Inf}\{ f_{ik}^{-1}(\lambda_{ik}) : \ 1 \leq k \leq n\}$ where $\lambda_{ik} \in t_{ik}^*$ and $ik \in J$. Now we can find some k, $(1 \leq k \leq n)$, sayk$_1$ such that $f_{ik_l}^{-1}(\lambda_{ik_l})\ (y) < f_{ik_l}^{-1}(\lambda_{ik_l})(x) \Rightarrow \lambda_{ik_l} f_{ik_l}(y) < \lambda_{ik_1} f_{ik_1}\ (x)$. Since $(X_{ik_l}, t_{ik_l}^*)$ is $FSR_0(xi)$, there exists $V_{i_{k_l}} \in t_{ik_1}^*$ such that $V_{i_{k_1}} f_{ik_l}(x) < V_{i_{k_1}} f_{ik_l}(y) \Rightarrow f_{ik_l}^{-1}(V_{i_{k_l}})(x) < f_{ik_l}^{-1}(V_{i_{k_1}})(y)$. Put $V = f_{ik_l}^{-1}\ (V_{i_{k_1}}) \in t^*$.Thus $v(x) < v(y)$. Hence (X, t^*) is $FSR_0(xi)$. Since $FSR_0(xi) \Rightarrow FSR_0(ii)$. So the properties $FSR_0(ii)$ is also initial. Similarly we can show that initial properties hold for $FSR_0(vii)$ and $FSR_0(viii)$. The initiality of FSR_0 (iv) is not yet done. We hope to do this in a latter work.

2.5.2. Theorem: The properties $FSR_0(k)$, $k \in \{ii, iii, iv, ...xi\}$ are productive, i.e. if $(X_i, t_i^*)_{i\in J}$ is a family of fuzzy supra topological spaces, each of them having the property $FSR_0(k)$, the product space $\left(X = \prod_{i\in J} X_i, t^* \right)$ also has $FSR_0(k)$.

Proof:

(a) Suppose each of $(X_i, t_i^*)_{i\in J}$ has the property $FSR_0(ii)$. Suppose $x, y \in X$, $x \neq y$, where $x = (x_i)_{i\in J}$ and $y = (y_i)_{i\in J}$, $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) = \alpha$. Let $\beta \in I_0$. We have to show that $\overline{\beta 1_y}(x) = \beta$. We have $\overline{\alpha 1_x}(y) = \inf_{i\in J} \overline{\alpha 1_{x_i}}(y_i)$. Thus $\inf_{i\in J} \overline{\alpha 1_{x_i}}(y_i) = \alpha$. Since each of $(X_i, t_i)_{i\in J}$ has the property $FSR_0(ii)$, therefore $\inf_{i\in J} \overline{\beta 1_{y_i}}(x_i) = \beta$.Thus $\overline{\beta 1_y}(x) = \beta$ and so (X, t^*) has $FSR_0(ii)$.

(b) Suppose each of $(X_i, t_i)_{i\in J}$ has the property $FSR_0(iii)$. Suppose $x \in X$ where x =

$(x_i)_{i\in J}$ and $\lambda\in t^*$ such that $\alpha<\lambda(x)$. We have to show that, $\overline{\alpha 1_x} \le \lambda$. It follows from the definition of t^* that there exist $K \subset J$ and a family $(\lambda_i)_{i\in K}$ with $\lambda_i\in t^*_i$ such that for every $y = (y_i)_{i\in J}\in X$, $\inf_{i\in K} \lambda_i(x_i) \le \lambda(y)$ and more over $\alpha < \inf_{i\in K} \lambda_i(x_i) \le \lambda(x)$. From this it follows that $\alpha < \lambda_i(x_i)$, hence $\alpha 1_{x_i}(y_i) \le \lambda(y_i)$, for $i\in K$, and therefore $\overline{\alpha 1_x}(y) = \inf_{i\in J} \overline{\alpha 1_{x_i}}(y_i) \le \inf_{i\in K} \overline{\alpha 1_{x_i}}(y_i) \le \inf_{i\in K} \lambda_i(y_i) \le \lambda(y)$. so $\overline{\alpha 1_x} \le \lambda$. Hence (X, t^*) has FSR_0(iii).

(c) Suppose each of $(X_i, t^*_i)_{i\in J}$ has the property FSR_0(v). Suppose $x, y\in X$, $x \ne y$, where $x = (x_i)_{i\in J}$ and $y = (y_i)_{i\in J}$. Let $\overline{1_x}(y) = 1$. We have to show that $\overline{1_y}(x) = 1$. Now $\overline{1_x}(y) = 1$ implies that $\inf_{i\in J} \overline{1_{x_i}}(y_i) = 1$. Since each of $(X_i, t^*_i)_{i\in J}$ has the property FSR_0(v), $\inf_{i\in J} \overline{1_{y_i}}(x_i) = 1$ and so $\overline{1_y}(x) = 1$. Hence (X, t^*) has FSR_0(v)

(d) Suppose $x, y\in X$, $x \ne y$, where $x = (x_i)_{i\in J}$ and $y = (y_i)_{i\in J}$. If each of $(X_i, t^*_i)_{i\in J}$ has the property FSR_0(vi), then $\inf_{i\in J} \overline{1_{x_i}}(y_i) = \inf_{i\in J} \overline{1_{y_i}}(x_i)$. Therefore, $\overline{1_x}(y) = \overline{1_y}(x)$. Hence (X, t) has FSR_0(vi).

(e) Suppose $x, y\in X$, $x \ne y$, where $x = (x_i)_{i\in J}$ and $y = (y_i)_{i\in J}$. If each of $(X_i, t^*_i)_{i\in J}$ has the property FSR_0(vii), then $\inf_{i\in J} \overline{1_{x_i}}(y_i) = \inf_{i\in J} \overline{1_{y_i}}(x_i) \in \{0, 1\}$. Therefore, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0, 1\}$. Hence (X, t^*) has FSR_0(vii).

(f) Suppose $\alpha\in I_0$ and $x, y \in X$, $x \ne y$, where $x = (x_i)_{i\in J}$ and $y = (y_i)_{i\in J}$ such that $\overline{\alpha 1_x}(y) = \alpha$ Thus $\inf_{i\in J} \overline{1_{x_i}}(y_i) = \alpha$. If each of $(X_i, t^*_i)_{i\in J}$ has the property FSR_0(viii), then $\inf_{i\in J} \overline{\alpha 1_{y_i}}(x_i) = \alpha$. Therefore, $\overline{\alpha 1_y}(x) = \alpha$. Hence (X, t^*) has FSR_0(viii).

(g) Suppose $\alpha\in I_0$ and $x, y \in X$, $x \ne y$, where $x = (x_i)_{i\in J}$ and $y = (y_i)_{i\in J}$. If each of $(X_i,t^*_i)_{i\in J}$ has the property FSR_0(ix), then $\inf_{i\in J} \overline{\alpha 1_{x_i}}(y_i) = \inf_{i\in J} \overline{\alpha 1_{y_i}}(x_i)$.Therefore. $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$. Hence (X, t^*) has FSR_0(ix).

(h) Let $(X_i, t_i^*)_{i\in J}$ be $FSR_0(x)$. We shall prove that (X, t^*) is $FSR_0(x)$. Let x, y ∈ X, with x≠y , and ∃ $\lambda \in t^*$, with λ (x)=0 and λ(y)>0. But we have λ (x)= min{ $\lambda_i(x_i)$: i∈J}and λ(y)= min{(λ_i (y_i) : i∈J} . Since $(X_i, t_i^*)_{i\in J}$ be $FSR_0(x)$, so exist $\mu_i \in t_i^*$ with μ_i (x)>0 and μ_i (y)=0. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$ and hence μ_i (π_i(x))>0 and μ_i(π_i(y)) =0. It follows that ∃ ($\mu_i \circ \pi_i$) $\in t^*$, such that ($\mu_i \circ \pi_i$) (x)>0, and ($\mu_i \circ \pi_i$)(y)=0. Hence it is clear that (X, t^*) is $FSR_0(x)$. Similarly we can proof that FSR_0(xi) is productive.

2.5.3. Corollary: The product space of a non-void family $(X_i, t_i^*)_{i\in J}$ of fuzzy supra topological space is $FSR_0(k)$ k ∈ {ii, iii, ...ix} if and only if each factor is $FSR_0(k)$.

Proof: Suppose, $(X_i, t^*_i)_{i\in J}$ is a family of fuzzy topological spaces, each of them having the property $FSR_0(k)$. k ∈ {ii, iii, ...ix}. Then by the theorem 2.5.2, the product space $\left(X = \prod_{i\in J} X_i, t^* \right)$ also has $FSR_0(k)$.

Converse:

(a). Let (X, t^*) is FSR_0(viii), Suppose i∈J, $x_i, y_i \in X_i$, $x_i \neq y_i$ and $\alpha \in I_0$ such that $\overline{\alpha 1_{y_i}}(x_i) = \alpha$. Since (X, t^*) is FSR_0(viii) $\overline{\alpha 1_{y_i}}(x_i) = \alpha \Rightarrow \overline{1_{x_i}}(y_i) = \alpha$, for each i. Let for some i∈J, a_i be a fixed element of X_i, suppose that $A_i = \{x \in X = \prod_{i\in J} X_i \,/\, x_j = a_j$ for some i≠j}. So that A_i is the subset of X, and this implies that $(A_i, t^*_{A_i})$ is also subspace of (X, t^*) then$(A_i, t^*_{A_i})$ is also FSR_0(viii) and A_i is a homeomorphic image of X_i. Hence it is clear that (X_i, t^*_i) is FSR_0(viii).Similarly we can prove this theorem for FSR_0 (k) where k ∈ {ii, iii, ...ix}.

2.5.4. Theorem: All the properties $FSR_0(k)$, i≤k≤ xi are hereditary.

Proof: Let the fsts (X, t^*), and A⊂X. where $t^*_A = \{u \wedge A : u \in t^*\}$, we have to show that, the subspace (A, t^*_A) has $FSR_0(k)$ i≤k≤ xi, if (X, t^*) has $FSR_0(k)$ i≤k≤ xi)

We have, $t^*\text{-cl}(1_x) \cap 1_A = t_A^*\text{-cl}(1_x)$.

(1) Let $x, y \in A$, $x \neq y$ and $(t_A^*\text{-cl}(1_x))(y) = 0$. Therefore, $(t^*\text{-cl}(1_x)) \cap 1_A)(y) = 0$. $\Rightarrow$ $(t^*\text{-cl}(1_x))(y) \wedge 1_A(y) = 0 \Rightarrow (t^*\text{-cl}(1_x))(y) = 0$. [Since, $y \in A$]. Now, $x, y \in X$, $x \neq y$, and $(t^*\text{-cl}(1_x))(y) = 0$. So if (X, t^*) has FSR_0(i) then $(t^*\text{-cl}(1_y))(x) = 0$. Now $(t_A^*\text{-cl}(1_y))(x) = (t^*\text{-cl}(1_y) \cap 1_A)(x) = (t^*\text{- cl}(1_y))(x) \wedge 1_A(x) = 0$. This implies that, (A, t_A^*) has FSR_0(i).

(2) Let $x, y \in A$, $x \neq y$, $\alpha \in I_0$ and $(t_A^*\text{-cl}(\alpha 1_x))(y) = \alpha$.
Therefore, $(t^*\text{-cl}(\alpha 1_x) \cap 1_A)(y) = \alpha \Rightarrow (t^*\text{-cl}(\alpha 1_x))(y) = \alpha$, [since $y \in A$.]
Now, $x, y \in X$, $x \neq y$, $\alpha \in I_0$ and $(t^*\text{-cl}(\alpha 1_x))(y) = \alpha$. So if (X, t^*) has FSR_0(ii), $(t^*\text{-cl}(\alpha 1_y))(x) = \alpha$. Again, $(t_A^*\text{-cl}(\alpha 1_y))(x) = (t^*\text{-cl}(\alpha 1_y) \cap 1_A)(x) = (t^*\text{- cl}(\alpha 1_y)(x) \wedge 1_A(x)$ $= \alpha \wedge 1 = \alpha$. Therefore, (A, t_A^*) has FSR_0(ii).■

(3) Let $x \in A$, $\lambda \in t_A^*$ such that $\alpha < \lambda(x)$. There exist $\lambda' \in t^*$ such that $1_A \cap \lambda' = \lambda$. Since $x \in A$, $\lambda(x) = \lambda'(x)$. Now $\lambda' \in t^*$ and $\alpha < \lambda'(x)$. So if (X, t^*) has FSR_0(iii), then $t^*\text{-cl}(\alpha 1_x) \leq \lambda'$. Now, $t_A^*\text{-cl}(\alpha 1_x) = 1_A \cap (t^*\text{-cl}(\alpha 1_x)) \leq 1_A \cap \lambda' = \lambda$. Therefore, (A, t_A^*) has FSR_0(iii).

(4) Let $x \in A$, $\lambda \in t_A^*$ such that $\alpha \leq \lambda(x)$. There exist $\lambda' \in t^*$ such that $1_A \cap \lambda' = \lambda$. Since $x \in A$, $\lambda(x) = \lambda'(x)$. Now $\lambda' \in t^*$ and $\alpha \leq \lambda'(x)$. So if (X, t^*) has FSR_0(iv), then $t^*\text{-cl}(\alpha 1_x) \leq \lambda'$. Now, $t_A^*\text{-cl}(\alpha 1_x) = 1_A \cap (t^*\text{- cl}(\alpha 1_x)) \leq 1_A \cap \lambda' = \lambda$. Therefore, (A, t_A^*) has FSR_0(iv).

(5) Let $x, y \in A$, $x \neq y$ and $(t_A^*\text{-cl}(1_y))(x) = 1$. Therefore, $(1_A \cap t^*\text{-cl}(1_y))(x) = 1 \Rightarrow (t^*\text{-cl}(1_y))(x) = 1$. If (X, t^*) has FSR_0(v), $(t\text{-cl}(1_x))(y) = 1$. Now $(t_A^*\text{-cl}(1_x))(y) = (1_A \cap t^*\text{-cl}(1_x))(y) = 1_A(y) \wedge (t^*\text{-cl}(1_x))(y) = 1 \wedge 1 = 1$. Therefore, (A, t_A^*) has FSR_0(v).

(6) Let $x, y \in A$, and $x \neq y$. If (X, t^*) has FSR_0(vi) then $(t^*\text{-cl}(1_x))(y) = (t^*\text{-cl}(1_y))(x)$.
Now, $(t_A^*\text{-cl}(1_x))(y) = (t^*\text{-cl}(1_x))(y) = (t^*\text{-cl}(1_y))(x) = (t_A^*\text{-cl}(1_y))(x)$. Therefore, (A, t_A^*) has FSR_0(vi).

(7) Let x, y $\in$ A, x $\neq$ y .Now x, y $\in$ X, x $\neq$ y. So if (X, t^*) has FSR_0(vii) then (t^*-cl(1_x))(y) = (t^*-cl(1_y))(x) $\in$ {0, 1}. We have, (t_A^*-cl(1_x))(y) = (t^*-cl(1_x))(y) = (t^*-cl(1_y))(x) = (t_A^*-cl(1_y))(x). Therefore, (t_A^*-cl(1_x))(y) = (t_A^*-cl(1_y))(x) $\in$ {0, 1}. This implies that (A, t_A^*) has FSR_0(vii) .

(8) Let x, y $\in$A, x $\neq$ y, $\alpha\in I_0$ and t^*_A-cl($\alpha 1_x$)(y) = α. Therefore, ($1_A \cap$t-cl($\alpha 1_x$))(y) = $\alpha \Rightarrow$ t^*-cl($\alpha 1_x$)(y) = α, since y $\in$ A. If (X, t^*) has FSR_0(viii), then t^*-cl($\alpha 1_y$)(x) = α. Now, t_A^*-cl($\alpha 1_y$)(x) = ($1_A \cap t^*$-cl($\alpha 1_y$))(x) = 1 $\wedge$ α = α. Therefore, (A, t_A^*) has FSR_0(viii).

(9) Let x, y$\in$ A, x $\neq$ y, $\alpha\in I_0$. Now x, y$\in$X, x $\neq$ y, $\alpha\in I_0$. So if (X, t^*) has FSR_0(ix) then t^*-cl($\alpha 1_x$)(y) = t^*-cl($\alpha 1_y$)(x). We have, (t_A^*-cl(1_x))(y) = (t^*-cl(1_x))(y) = (t^*-cl(1_y))(x) = (t_A^*-cl(1_y))(x). Therefore, (A, t_A^*) has FSR_0(ix).

(10). Let (X, t^*) is FSR_0(x) space, we shall prove that (A,t_A^*) is FSR_0(x) space. Let x, y $\in$A, with x$\neq$y, so that x, y $\in$X, as A$\subseteq$X. Since (X, t^*) is FSR_0(x) space,$\exists$ u$\in t^*$ with u(x) = 0 and u(y)>0.For A$\subseteq$X,we have u$\wedge$A$\in t_A^*$ and (u$\wedge$A)(x)=0, (u$\wedge$A)(y)>0 because u $\in t^*$ with membership function μ_u then min(μ_u(x), μ_A(x))=0,and min(μ_u(y), μ_A(y))>0 as x, y $\in$A with x$\neq$y .Hence it is clear that (A, t_A^*)is FSR_0(x) space.

(11). Let (X, t^*) is FSR_0(xi) space, we shall prove that (A,t_A^*) is FSR_0(xi) space. Let x, y $\in$A, with x$\neq$y, so that x, y $\in$X, as A$\subseteq$X. Since (X, t^*) is FSR_0(xi) space,$\exists$ u$\in t^*$ with u(x) < u(y) .For A$\subseteq$X,we have $\mu\wedge$A$\in t_A^*$ and (u$\wedge$A)(x)< (u$\wedge$A)(y), because u $\in t^*$ with membership function μ_u then min(μ_u(x), μ_A(x))< min(μ_u(y), μ_A(y)) as x, y $\in$A with x$\neq$y .Hence it is clear that (A, t_A^*)is FSR_0(xi) space.

2.6. Homeomorphism in FSR_0 spaces.

2.6.1 Definition: Let (X, t^*) and (Y, s^*) be two fuzzy supra topological spaces and f: (X, t^*)$\rightarrow$(Y, s^*) be any function, then f is called fuzzy supra homeomorphism if and only if f is fuzzy supra bijective, fuzzy supra continuous and fuzzy supra open.

2.6.1. Theorem: Every homeomorphic image of FSR_0(k) is also FSR_0(k), (i<k<ix)

Proof:(a). Let f:(X, t_1^*)$\rightarrow$(Y, t_2^*) be a homeomorphism between fsts, where (X, t_1^*) has

FSR$_0$(i), then $\overline{1_{f(x_1)}}(f(x_2)) = \overline{1_{x_1}}(x_2)$, for every pair, x_1, x_2 ∈X. Let y_1, y_2 ∈Y, $y_1 \neq y_2$ such that $\overline{1_{y_1}}(y_2)$=0, Let $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$.Then $x_1 \neq x_2$.Again since (X, t_1^*) is FSR$_0$(i) so $\overline{1_{x_1}}(x_2)$ =0 ⇒ $\overline{1_{x_2}}(x_1)$=0, and therefore , $\overline{1_{f(x_2)}}(f(x_1))=0=\overline{1_{y_2}}(y_1)$. This implies that (Y, t_2^*) is an FSR$_0$(i)

(b). Let f:(X, t_1^*)→(Y, t_2^*) be a homeomorphism between fsts, where (X, t_1^*) has FSR$_0$(ii), then $\overline{\alpha 1_{f(x_1)}}(f(x_2)) = \overline{1_{x_1}}(x_2)$ for every pair, x_1, x_2 ∈X and for every α ∈I$_0$. Let y_1, y_2 ∈Y, $y_1 \neq y_2$ and α ∈I$_0$ such that $\overline{\alpha 1_{y_1}}(y_2) = \alpha$, Let $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. Then $x_1 \neq x_2$.Since $\overline{\alpha 1_{y_1}}(y_2) = \alpha$, $\overline{\alpha 1_{x_1}}(x_2) = \alpha$, Again since (X, t_1^*) is FSR$_0$(ii) so $\overline{\beta 1_{x_1}}(x_2) = \beta$ ∀ β ∈I$_0$. Therefore, $\overline{\beta_{f(x_1)}}(f(x_2)) = \beta \Rightarrow \overline{\beta 1_{y_1}}(y_2) = \beta$. This implies that (Y, t_2^*) is an FSR$_0$(ii) .

(c). Let f:(X, t_1^*)→(Y, t_2^*) be a homeomorphism between fsts, where (X, t_1^*) has FSR$_0$(iii), then $\overline{\alpha 1_{f(x)}} = f(\overline{\alpha 1_x})$ for every x ∈X and for every α ∈I$_0$. Let y ∈Y, $\lambda \in t_2^*$ and α ∈I$_0$ such that $\alpha < \lambda(y)$. Let $f^{-1}(y) = x$ and $f^{-1}(\lambda) = \mu$.Then x∈X and $\mu \in t_1^*$. such that $\alpha < \mu(x)$. Since (X, t_1^*) is FSR$_0$(iii), $\overline{\alpha 1_x} \leq \mu$. Now $\overline{\alpha 1_y}$ =

$\overline{\alpha 1_{f(x)}} = f(\overline{\alpha 1_x}) \leq f(\mu) = \lambda$. This implies that (Y, t_2^*) is an FSR$_0$(iii).

(d). Let f: (X,t_1^*)→(Y, t_2^*) be a homeomorphism between fsts, where (X, t_1^*) has FSR$_0$(iv), then $\overline{\alpha 1_{f(x)}} = f(\overline{\alpha 1_x})$ for every x ∈X and for every α ∈I$_0$. Let y ∈Y, $\lambda \in t_2^*$ and α ∈I$_0$ such that $\alpha \leq \lambda(y)$. Let $f^{-1}(y) = x$ and $f^{-1}(\lambda) = \mu$.Then x ∈X and $\mu \in t_1^*$.such that $\alpha < \mu(x)$. Since (X, t_1^*) is FSR$_0$(iv), $\overline{\alpha 1_x} \leq \mu$. Now

$\overline{\alpha 1_y} = \overline{\alpha 1_{f(x)}} = f(\overline{\alpha 1_x}) \leq f(\mu) = \lambda$. This implies that (Y, t_2^*) is an FSR$_0$(iv).

(e). Let f: (X, t_1^*) → (Y, t_2^*) be a homeomorphism between fsts, where (X, t_1^*) is FSR$_0$(v). Let y_1, y_2 ∈Y, $y_1 \neq y_2$, $\mu \in t_2^*$ such that $\overline{1_{y_1}}(y_2)$=1. Let $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$, Since f is a homeomorphism, $\overline{1_{x_1}}(x_2) = \overline{1_{f(x_1)}}(f(x_2)) = \overline{1_{y_1}}(y_2)$=1.By the

$FSR_0(v)$ property of (X, t_1^*) we have $\overline{1_{x_2}}(x_1)=1$. Now, $\overline{1_{y_2}}(y_1) = \overline{1_{f(x_2)}}(f(x_1)) = \overline{1_{x_2}}(x_1)=1$.This implies that (Y, t_2^*) is an $FSR_0(v)$.

(f). Let f: $(X, t_1^*) \to (Y, t_2^*)$ be a homeomorphism between fsts, where (X, t_1^*) is $FSR_0(vi)$. Let y_1, $y_2 \in Y$, $y_1 \neq y_2$, $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$,then $x_1 \neq x_2$. By the $FSR_0(vi)$ property of (X, t_1^*)we have $\overline{1_{x_1}}(x_2)= \overline{1_{x_2}}(x_1)$. Since f is a homeomorphism, $\overline{1_{f(x_1)}}(f(x_2)=\overline{1_{x_1}}(x_2)$ for every x_1, $x_2 \in X$ which together with $\overline{1_{x_1}}(x_2)= \overline{1_{x_2}}(x_1)$imply that $\overline{1_{y_1}}(y_2)=\overline{1_{y_2}}(y_1)$. This implies that (Y, t_2^*) is an $FSR_0(vi)$.

(g). Let (X, t_1^*), (Y, t_2^*) be two fuzzy supra topological spaces, where (X, t_1^*) is $FSR_0(vii)$.Let f: $(X, t_1^*) \to (Y, t_2^*)$ be a homeomorphism.). Let y_1, $y_2 \in Y$, $y_1 \neq y_2$, Let $\overline{1_{y_1}}(y_2) \notin \{0,1\}$. This implies that there exists $\lambda \in t_2^{*c}$ such that $\lambda(y_1) =1$ but $0< \lambda(y_2)<1$. Since f is a homeomorphism we have $f^{-1}(y_1)$, $f^{-1}(y_2) \in X$ and $f^{-1}(\lambda) \in t_1^{*c}$ such that $(f^{-1}(\lambda))(f^{-1}(y_1)) =1$ and $0<(f^{-1}(\lambda))(f^{-1}(y_2))<1$ .This implies that $\overline{1_{f^{-1}(y_1)}}(f^{-1}(y_2)) \notin \{0, 1\}$, which is a contradiction . Since (X, t_1^*) is $FSR_0(vii)$. Again let $\overline{1_{y_2}}(y_1) \neq \overline{1_{y_1}}(y_2)$.Without any loss of generality we can assume that $0= \overline{1_{y_1}}(y_2) < \overline{1_{y_2}}(y_1)=1$. This implies that there exist $\eta, \lambda \in t_2^{*c}$ such that $\eta(y_1)=1$, $\eta(y_2)=0$; $\lambda(y_1)=0, \lambda(y_2)=1$. Now since f is a homeomorphism, we have $f^{-1}(\eta)$, $f^{-1}(\lambda) \in t_1^{*c}$, $(f^{-1}(\eta))(f^{-1}(y_1)) =1$, $(f^{-1}(\eta))(f^{-1}(y_2)) =0$; $(f^{-1}(\lambda))(f^{-1}(y_1)) =1$, $(f^{-1}(\lambda))(f^{-1}(y_2)) =0$; This implies that $\overline{1_{f^{-1}(y_1)}}(f^{-1}(y_2)=0$ and $\overline{1_{f^{-1}(y_2)}}(f^{-1}(y_1)=1$. Therefore, $\overline{1_{f^{-1}(y_1)}}(f^{-1}(y_2) \neq \overline{1_{f^{-1}(y_2)}}(f^{-1}(y_1)$, which is also contradiction. Therefore, that $\overline{1_{y_1}}(y_2) = \overline{1_{y_2}}(y_1) \in \{0,1\}$, and so (Y, t_2^*) is $FSR_0(vii)$.

(h). Let f:$(X, t_1^*) \to (Y, t_2^*)$ be a homeomorphism between fsts, where (X, t_1^*) has $FSR_0(viii)$. Let y_1, $y_2 \in Y$, $y_1 \neq y_2$ and $\alpha \in I_0$ such that $\overline{\alpha 1_{y_1}}(y_2)=\alpha$. Again Let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. $\overline{\alpha 1_{f(x_1)}}(f(x_2) = \overline{1_{x_1}}(x_2), \forall$ x_1, $x_2 \in X$. Since f is a homeomorphism, Now $\alpha=\overline{\alpha 1_{y_1}}(y_2)=\overline{\alpha 1_{f(x_1)}}(f(x_2) = \overline{1_{x_1}}(x_2)$.By the $FSR_0(viii)$ property of (X, t_1^*), $\overline{1_{x_1}}(x_2)=\alpha$.Now $\overline{\alpha 1_{y_2}}(y_1)$=$\overline{\alpha 1_{f(x_2)}}$(f(x$_1$))= $\overline{\alpha 1_{x_2}}$(x$_1$)= α. Therefore (Y, t_2^*) is FSR$_0$(viii).

(i). Let f:(X, t_1^*)→(Y, t_2^*) be a homeomorphism between fsts, where (X, t_1^*) has FSR$_0$(ix). Let y_1, y_2∈Y, $y_1 \neq y_2$ and α ∈I$_0$ such that $\overline{\alpha 1_{y_1}}(y_2)$=$\alpha$. Again Let x$_1$ $=f^{-1}(y_1)$ and x$_2$=f^{-1}(y$_2$). By the FSR$_0$(ix) property of (X, t_1^*), $\overline{\alpha 1_{x_1}}(x_2)$ = $\overline{\alpha 1_{x_2}}$(x$_1$), we have $\overline{\alpha 1_{f(x_1)}}$(f($x_2$) =$\overline{\alpha 1_{x_1}}(x_2)$,∀ x$_1$, x$_2$ ∈X. Since f is a homeomorphism, Now $\overline{\alpha 1_{y_1}}(y_2)$ = $\overline{\alpha 1_{f(x_1)}}$(f($x_2$) =$\overline{\alpha 1_{x_1}}(x_2)$. Similarly $\overline{\alpha 1_{y_2}}(y_1)$ = $\overline{\alpha 1_{x_2}}$(x$_1$). Therefore $\overline{\alpha 1_{y_1}}(y_2)$ = $\overline{\alpha 1_{y_2}}(y_1)$. Therefore (Y, t_2^*) is FSR$_0$(ix).

The proof for FSR$_0$ (x) and FSR$_0$ (xi) are similar.

2.6.2 .Definition: Let x_t be a point in fsts (X, t^*).Then the fuzzy supra Ker{ x_t}=∨ {λ ∈ t^* : x_t∈ λ, t ∈(0,1] and t≤ λ (x)}

2.6.3. Definition: Let (X, t^*)is a fuzzy supra topological space, then (X, t^*)is said to be FSR$_0$ if for every fuzzy supra open set λ ∈ t^*,: x_t∈ λ ⟹ cl{ x_t }≤ λ .This is the alternative definition of FSR$_0$(iv).

2.6.4 Definition: Let (X, t^*) is a fuzzy supra topological space. A fuzzy set A is called quasi coincident with a fuzzy set B denoted by AqB if A(x)+B(x)>1 for some x∈X. A fuzzy point x_t≤A is called quasi coincident with the fuzzy set A denoted by x_tqA if t+A(x)>1. The negation relation is denoted by x_t ¬q A. **[47].**

2.6.2 Theorem: A fuzzy supra topological space (X, t^*) is fuzzy supra R$_0$- space iff for every of pair of fuzzy point x_t and y_r in X with x≠y and cl(x_t)≠cl(y_r), cl(x_t) ¬q cl(y_r).

Proof: Let a A fuzzy supra topological space (X, t^*) is fuzzy supra R$_0$- space. Let x_t and y_r be a pair of fuzzy point in X with x ≠y and cl(x_t)≠cl(y_r), then ∃ a fuzzy point z_μ in X ∋ z_μ≤ cl(x_t) and $z_\mu \nleq$ cl(y_r), If x_t ≤cl(y_r), then cl(x_t)≤ cl(y_r), Hence z_μ≤ cl(y_r), but this is a contradiction. Then $x_t \nleq$ cl(y_r),so that x_t≤ 1- cl(y_r),Then since 1- cl(y_r),is fuzzy supra open and (X, t^*) is fuzzy supra R$_0$- space, cl(x_t) < 1- cl(y_r). Hence cl(x_t) ¬q

$cl(y_r)$.

Conversely let $\lambda \in t^*$ and $x_t \leq \lambda$, we will prove that $cl(x_t) \leq \lambda$. Let $y_r \nleq \lambda$, then $y_r \leq 1-\lambda$ and $x \neq y \Rightarrow cl(y_r) \leq cl(1-\lambda) = 1-\lambda$. Since $x_t \leq \lambda$, then $x_t \nleq cl(y_r)$, i.e $cl(x_t) \neq cl(y_r)$. Then by assumption $cl(x_t) \neg q\ cl(y_r)$. i.e $cl(x_t) \leq 1 - cl(y_r). \leq \lambda$. This completes the theorem.

2.6.3. Theorem : Let (X, t^*) is a fuzzy supra topological space. Then the following conditions are equivalent:

(a) (X, t^*) is FSR_0.

(b) $cl\{x_t\} \leq$ fuzzy supra $Ker\{x_t\}$.

(c) For all, $\lambda \in t^{*c}$ $\lambda = \wedge\{\mu : \lambda < \mu,\ \mu \in t^*\}$

(d) For all $\gamma \in t^*$, $\gamma = \vee\{\lambda \text{ closed } \lambda < \gamma\}$.

(e) For every fuzzy set $\delta \neq 0$ and for each $\gamma \in t^*$ such that $\delta \wedge \gamma \neq 0$ there exists a supra closed set λ such that $\lambda < \gamma$ and $\lambda \wedge \delta \neq 0$.

Proof: (a)⇔(b).

From the definition 2.6.3 and fuzzy supra $Ker\{x_t\}$ (definition 2.6.2), it is clear that (a)⇔(b).

(a)⇒(c):

Let $\beta = \wedge\{\mu : \lambda < \mu, \mu \in t^*\}$.Then clearly $\lambda < \beta$, where $\lambda \in t^{*c}$, We need to prove that $\beta < \lambda$. Let $x_{t \in \lambda \in t^{*c}}$, then $x_t' \in \lambda^c \in t^*$.Since X is FSR_0, $cl\{x_t'\} \leq \lambda^c = \gamma$. Let $cl\{x_{t'}\}^c = \mu \in t^*$.Hence $x_t \in \mu$ and $\lambda < \mu$,therefore $x_t \notin \beta$.

With the help of Demorgan law it is clear that (c)⇒(d),

(d)→(e):

Let $\delta \neq 0$ and $\gamma \in t^*$ and $x_t \in \delta \wedge \gamma$ by (d) there exist a closed set λ, $x_t \in \lambda < \gamma$ therefore $\lambda \wedge \delta \neq 0$.

(e)⇒(a):

Let $\{x_t\} \in \mu \in t^* \Rightarrow \{x_t\} \wedge \mu \neq 0$.by (e)there exist a supra closed set λ such that $\{x_t\} \in \lambda < \mu$.Hence $cl\{x_t\} < \lambda < \mu$. Hence (X, t^*) is FSR_0.

2.6.4 Theorem: Let (X, t^*) be a fuzzy supra topological space, x_t, $y_r \in X$, Then fuzzy supra Ker{ x_t } ≠ fuzzy supra Ker{ y_r } if and only if cl{ x_t } ≠ cl { y_r }.**[51]**

Proof : Firstly suppose that cl{ x_t } ≠ cl { y_r } then there is a fuzzy point z_p such that $z_p \leq$ cl{ x_t } and $z_p \nleq$ cl { y_r }. If $x_t \leq$ cl { y_r } then cl{ x_t } ≤ cl { y_r } and hence $z_p \leq$ cl{ x_t } ≤ cl { y_r } ⇒ $z_p \leq$ cl { y_r } which is a contradiction . Hence $x_t \nleq$ cl { y_r }, hence $x_t \leq 1-$ cl { y_r } , since $1-$ cl { y_r } is a fuzzy supra open set containing x_t , not containing y_r, so from definition of kernel $y_r \leq$ fuzzy supra Ker{ y_r } and $y_r \nleq$ fuzzy supra Ker{ x_t } and hence fuzzy supra Ker{ x_t } ≠ fuzzy supra Ker{ y_r}.

Conversely suppose that fuzzy supra Ker{ x_t } ≠ fuzzy supra Ker{ y_r } then then there is a fuzzy point z_p such that $z_p \leq$ fuzzy supra Ker{ x_t }and $z_p \nleq$ fuzzy supra Ker{ y_r } . If $z_p \leq$ fuzzy supra Ker{ x_t } then $x_t \leq$ cl{ z_p } and so then cl{ x_t } ≤ cl { z_p } and similarly $z_p \nleq$ fuzzy supra Ker{ y_r } ⇒ $y_r \nleq$ cl { z_p } and slnce cl{ x_t } ≤ cl { z_p } then $y_r \nleq$ cl{ x_t } and hence cl{ x_t } ≠ cl { y_r }.

2.6.5 Theorem: A fuzzy supra topological space (X, t^*) is fuzzy supra R_0- space implies that for every of pair of fuzzy point x_t and y_r in X with $x \neq y$ and fuzzy supra ker $\{x_t\}$ ≠ fuzzy supra ker $\{y_r\}$, then fuzzy supra ker $\{x_t\}$ ¬ q fuzzy supra ker $\{y_r\}$.

Proof: Firstly suppose that (X, t^*) is fuzzy supra R_0- space with for every of pair of fuzzy point x_t and y_r in X , $x \neq y$ and fuzzy supra ker $\{x_t\}$ ≠ fuzzy supra ker $\{y_r\}$, we have to be prove that fuzzy supra ker $\{x_t\}$ ¬ q fuzzy supra ker $\{y_r\}$. Since fuzzy supra ker $\{x_t\}$ ≠ fuzzy supra ker $\{y_r\}$, so we have by previous theorem cl{ x_t } ≠ cl { y_r } .Next suppose that fuzzy supra ker $\{x_t\}$ q fuzzy supra ker $\{y_r\}$. Then for some $z \in X$, Let $\mu =$ fuzzy supra ker $\{x_t\}(z) \vee$ fuzzy supra ker $\{y_r\}(z) \in (0,1]$. Hence $z_\mu \leq$ fuzzy supra ker $\{x_t\}$ and $z_\mu \leq$fuzzy supra ker { y_r }. Hence when $z_\mu \leq$ fuzzy supra ker $\{x_t\}$ then $x_t \leq$cl { z_μ} and. Hence cl(x_t) ≤cl (z_μ). Similarly we prove that cl { z_μ } ≤cl (y_r). Hence we conclude that cl{ x_t } = cl { y_r }. Hence fuzzy supra ker $\{x_t\}$ ¬ q fuzzy supra ker $\{y_r\}$. ■

CHAPTER-III

Fuzzy Supra R_1 Topological Spaces

3. Introduction:

In this chapter, we introduce and study several fuzzy supra R_1 topological spaces (FSR_1 in short). We obtain their properties in the following sections. First we give definitions of FSR_1 spaces and then study implications and non-implications among them. Also, we study Good extension, reciprocal and hereditary properties of FSR_1 spaces. Moreover, some other properties such as $I_\alpha(t^*)$, homeomorphism among FSR_1 spaces, initiality and productivity of FSR_1 spaces are studied..

3.1. Definitions of FSR_1 spaces.

3.1.1. Definition: Let (X, t*) be a fuzzy supra topological space, we define fuzzy supra R_1-properties as follows.

We recall the twelve definitions (FSR_1(i)-FSR_1(vi) and FSR_1(xii)- FSR_1(xviii)) of **[9, 10]** to show it can significantly used in FSR_1 space.

FSR_1(i) If $\forall$ x, y$\in$X, x$\neq$y, $\exists$ w$\in t^*$ such that w(x)$\neq$ w(y), then $\exists\ \mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu = 0$.

FSR_1(ii) If $\forall$ x, y$\in$X, x$\neq$y, $\exists$ w$\in t^*$ such that w(x)$\neq$ w(y), then $\exists\ \mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \leq 1-\nu$.

FSR_1(iii) If $\forall$ x, y$\in$X, x$\neq$y, $\exists$ w$\in t^*$ such that w(x)$\neq$ w(y), then $\exists\ \mu, \nu \in t^*$ such that μ(x)=1 = ν (y) and $\mu \wedge \nu$=0.

FSR_1(iv) If $\forall$ x, y$\in$X, x$\neq$y, $\exists$ w$\in t^*$ such that w(x)$\neq$ w(y), then $\exists\ \mu, \nu \in t^*$ such that μ(x)= 1= ν (y) and $\mu \leq 1-\nu$.

FSR_1(v) If $\forall$ x, y$\in$X, x$\neq$y, $\exists$ w$\in t^*$ such that w(x)$\neq$ w(y), then $\forall\ \beta, \delta \in I_{0,1}$, $\exists$ $\mu, \nu \in t^*$ such that μ(x)>β, ν (y)>δ and $\mu \wedge \nu$=0.

FSR_1(vi) If $\forall$ x, y$\in$X, x$\neq$y, $\exists$ w$\in t^*$ such that w(x)$\neq$ w(y), then $\exists\ \mu, \nu \in t^*$ such that μ(x)>0, ν (y)>0 and $\mu \wedge \nu$=0.

FSR_1(vii) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) > \alpha \in I_0$, and $w(y) = 0$ or $w(y) > \alpha \in I_0$, and $w(x) = 0$, then $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu = 0$.

FSR_1(viii) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) > \alpha \in I_0$, and $w(y) = 0$ or $w(y) > \alpha \in I_0$, and $w(x) = 0$, then $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \leq 1 - \nu$.

FSR_1(ix) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) > \alpha \in I_0$, and $w(y) = 0$ or $w(y) > \alpha \in I_0$, and $w(x) = 0$, then $\exists$ $\mu, \nu \in t^*$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \wedge \nu = 0$.

FSR_1(x) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) > \alpha \in I_0$, and $w(y) = 0$ or $w(y) > \alpha \in I_0$, and $w(x) = 0$, then $\exists$ $\mu, \nu \in t^*$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \leq 1 - \nu$.

FSR_1(xi) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) > \alpha \in I_0$, and $w(y) = 0$ or $w(y) > \alpha \in I_0$, and $w(x) = 0$, then $\forall$ $\beta, \delta \in I_{0,1}$, $\exists$ $\mu, \nu \in t^*$ such that $\mu(x) > \beta$, $\nu(y) > \delta$ and $\mu \wedge \nu = 0$.

FSR_1(xii) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) > \alpha \in I_{0,1}$, and $w(y) = 0$ or $w(y) > \alpha \in I_0$, and $w(x) = 0$, then $\exists$ $\mu, \nu \in t^*$ such that $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$.

FSR_1(xiii) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) = 1$, and $w(y) = 0$ or $w(y) = 1$, $w(x) = 0$; then $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu = 0$.

FSR_1(xiv) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) = 1$, and $w(y) = 0$ or $w(y) = 1$, $w(x) = 0$; then $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \leq 1 - \nu$.

FSR_1(xv) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) = 1$, and $w(y) = 0$ or $w(y) = 1$, $w(x) = 0$; then $\exists$ μ, $\nu \in t^*$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \wedge \nu = 0$.

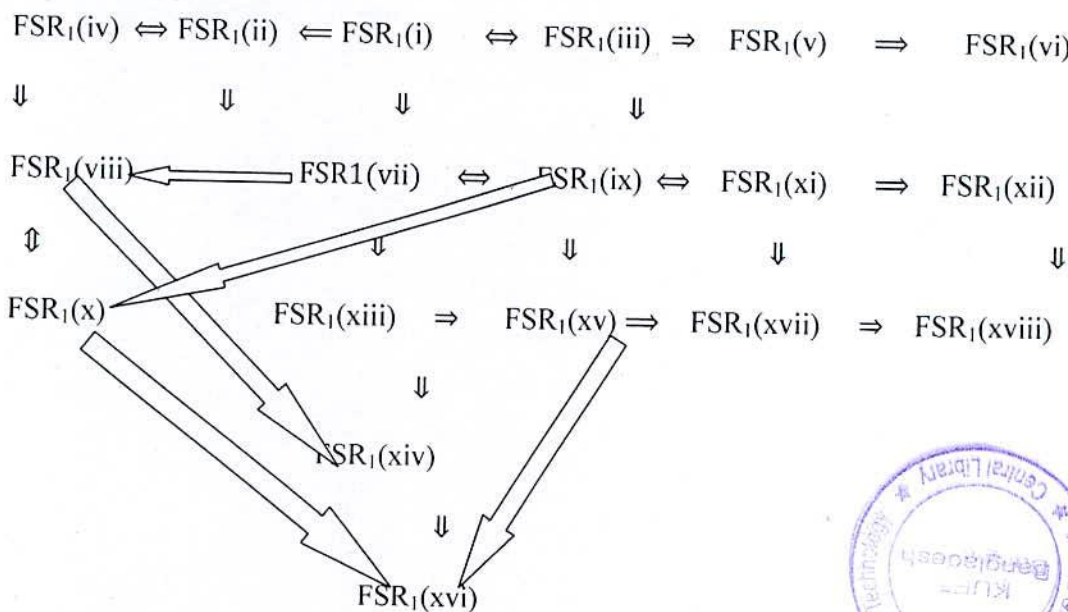
FSR_1(xvi) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) = 1$, and $w(y) = 0$ or $w(y) = 1$, $w(x) = 0$; then $\exists$ $\mu, \nu \in t^*$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \leq 1 - \nu$.

FSR_1(xvii) If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) = 1$, and $w(y) = 0$ or $w(y) = 1$, $w(x) = 0$;, then $\forall$ $\beta, \delta \in I_{0,1}$, $\exists$ $\mu, \nu \in t^*$ such that $\mu(x) > \beta$, $\nu(y) > \delta$ and $\mu \wedge \nu = 0$.

FSR_1(xviii) If $\forall$ x, y $\in$ X, x$\neq$y, $\exists$ w$\in t^*$ such that either w(x)=1, and w(y)=0 or w(y) = 1, w(x)=0; then $\exists$ $\mu, \nu \in t^*$ such that μ(x)>0, ν (y)>0 and $\mu \wedge \nu$=0.

3.2: Implications among FSR_1(k) , i≤k≤xviii

3.2.1: Theorem [12]: The following implications hold among the FSR_1(k) (i≤k≤xviii) properties in the above section



Proof:

FSR_1(vii) ⇒FSR_1(ix).

Let (X, t^*) be an fsts. Which has the property FSR_1(vii) .Suppose that, x, y$\in$X, x$\neq$y, and w$\in t^*$ such that w(x)> α $\in I_{0,1}$ and w(y)=0. Then, by the FSR_1(vii)-property of (X, t^*), $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu$=0. Clearly, μ(x) =1 = ν(y) and $\mu \wedge \nu$=0.Hence (X, t^*) has the property FSR_1(ix).

Thus FSR_1(vii) ⇒FSR_1(ix). Similarly we can show that FSR_1(i) ⇒ FSR_1(iii).

FSR_1(vii) ⇒FSR_1(viii)

Let (X, t^*) be an fsts. Which has the property FSR_1(vii).Suppose that, x, y$\in$X, x$\neq$y, and w$\in t^*$ such that w(x)> α $\in I_0$, and w(y) =0. Then, by the FSR_1(vii)- property of (X, t^*), $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu$=0. Clearly, μ ≤1-ν . Hence (X, t^*) has the property FSR_1(viii). Thus FSR_1(vii) $\Rightarrow FSR_1$(viii) Similarly we can show that. FSR_1(i) $\Rightarrow$ FSR_1(ii).

FSR_1(i) $\Rightarrow$ FSR_1(xii)

Let (X, t^*) be an fsts. Which has the property FSR_1(i).Suppose that, x, $y \in X$, $x \neq y$, and $w \in t^*$ such that $w(x) \neq w(y)$. Hence we can treat $w(x)=\alpha$ and $w(y)=0$.where $\alpha \in I_0$.Again by FSR_1(i).- property of (X, t^*), $\exists\ \mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu=0$. Hence (X, t^*) has the property FSR_1(vii)

FSR1(vii) $\Rightarrow$ FSR_1(xiii)

Let (X, t^*) be an fsts. Which has the property FSR_1(vii).Let x, $y \in X$, and $w \in t^*$ such that $w(x) > \alpha \in I_0$, and $w(y) = 0$, Then clearly $w(x)= 1$, and $w(y)= 0$. Again by FSR_1(vii)-property of (X, t^*), $\exists\ \mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu=0$. Hence (X, t^*) has the property FSR_1(xiii).

FSR_1(viii) $\Rightarrow FSR_1$(x).

Let (X, t^*) be an fsts. Which has the property FSR_1(viii) .Suppose that, x, $y \in X$, $x \neq y$, and $w \in t^*$ such that $w(x) > \alpha \in I_0$, and $w(y)=0$. Then, by the FSR_1(viii) - property of (X, t^*), $\exists\ \mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$, and $\mu \leq 1-\nu$ .Clearly $\mu(x)=1=\nu(y)$ and $\mu \leq 1-\nu$. Hence (X, t^*) has the property FSR_1(x). Thus FSR_1(viii) $\Rightarrow FSR_1$(x). Similarly we can show that FSR_1(ii) $\Rightarrow$ FSR_1(iv).

FSR_1(x) $\Rightarrow$ FSR_1(viii).

Let (X, t^*) be an fsts. This has the property FSR_1(x). Let x, $y \in X$, $x \neq y$, $\alpha \in I_0$, and $w \in t^*$ such that $w(x) =\alpha$ and $w(y) = 0$. Then by FSR_1(x), $\exists\ u, \nu \in t^*$ such that $u(x) =1= \nu(y)$ and $u \leq 1-\nu$.Let $z \in X$ and $\beta \in I_{0,1}$ such that $\beta 1_z \nleq u$. This implies $\beta > u(z)$. Now let $u(z)=\delta \in I_{0,1}$, Then $u(z)=\delta \in I_{0,1}$ and $u(y)=0$ together imply that $\exists\ \eta,\ \lambda \in t^*$ such that $\eta(y)= 1 =\lambda(z)$ and $\lambda \leq 1-\eta$. Now $1-\lambda(y) = 1$, therefore $\overline{1_y} \leq 1-\lambda$, again $\overline{1_y}(z) \leq 1-\lambda(z)$ $= 0$, and $\beta 1_z \nleq \overline{1_y}$, Therefore $\overline{1_y} \leq u$, which is a contradiction as $u(y) \neq 1$. Therefore, $u(z)=0$. Now $\beta \wedge u \in t^*$ such that $\beta \wedge u(z)=0$, $\beta \wedge u(x)=\beta$. Therefore $\exists\ \eta, \lambda \in t^*$ such that $\eta(y) = 1 = \lambda(z)$ and $\lambda \leq 1-\eta$. Now $1-\lambda(x) = 1$, Therefore $\overline{1_x} \leq 1-\lambda$ but $\overline{1_x}(z) \leq 1-\lambda(z)$ $=0$. Therefore $\beta 1_z \nleq \overline{1_x}$ Thus we see that if $\beta 1_z \nleq u$ then $\beta 1_z \nleq \overline{1_x}$, hence $\overline{1_x} \leq u$. Similarly we can show that $\overline{1_y} \leq u$ .Therefore(X, t^*) is FSR_1(viii). Thus

$FSR_1(x) \Longrightarrow FSR_1(viii)$. Similarly we can show that $FSR_1(iii) \Longrightarrow FSR_1(i)$, $FSR_1(ix) \Longrightarrow FSR_1(vii)$, $FSR_1(x) \Longrightarrow FSR_1(viii)$.

$FSR_1(ix) \Rightarrow FSR_1(xi)$.

Let (X, t^*) be an fsts. Which has the property $FSR_1(ix)$. Suppose that, $x, y \in X$, $x \neq y$, and $w \in t^*$ such that $w(x) > \alpha \in I_0$, and $w(y) = 0$. Then, by the $FSR_1(ix)$- property of (X, t^*), $\exists\ \mu, \nu \in t^*$ such that, $u(x)=1=\nu(y)$ and $u \wedge \nu=0$. Clearly $u(x) > \alpha$, $v(y) > \beta$, $\alpha, \beta \in I_{0,1}$, $u \wedge \nu=0$. Hence (X, t^*) has the property $FSR_1(xi)$, Thus $FSR_1(ix) \Rightarrow FSR_1(xi)$. Similarly we can show that $FSR_1(iii) \Longrightarrow FSR_1(v)$.

$FSR_1(xi) \Rightarrow FSR_1(xii)$.

Let (X, t^*) be an fsts. Which has the property $FSR_1(xi)$ Suppose that, $x, y \in X$, $x \neq y$, and $w \in t^*$ such that $w(x) > \alpha \in I_{0,1}$ and $w(y)=0$. Then, by the $FSR_1(xi)$-property of (X, t^*), $\exists\ u, \nu \in t^*$ such that, $u(x) > \alpha$, $v(y) > \beta, \alpha, \beta \in I_{0,1}$, and $\mu \wedge \nu=0$, clearly $u(x) > 0$, $v(y) > 0$, and $\mu \wedge \nu=0$. Hence (X, t^*) has the property $FSR_1(xii)$. Thus $FSR_1(xi) \Rightarrow FSR_1(xii)$.Similarly we can show that $FSR_1(v) \Rightarrow FSR_1(vi)$.

$FSR_1(ix) \Rightarrow FSR_1(x)$.

Let (X, t^*) be an fsts. Which has the property $FSR_1(ix)$. Suppose that, $x, y \in X$, $x \neq y$, and $w \in t^*$ such that $w(x) > \alpha \in I_0$, and $w(y)=0$. Then, by the $FSR_1(ix)$ property of (X, t^*), $\exists\ u, \nu \in t^*$ such that, $u(x)=1=\nu(y)$ and $\mu \wedge \nu=0$.clearly $u \leq 1-\nu$, Hence (X, t^*) has the property $FSR_1(x)$, Thus $FSR_1(ix) \Rightarrow FSR_1(x)$ .Similarly we can show that $FSR_1(iii) \Longrightarrow FSR_1(iv)$

$FSR_1(vii) \Longrightarrow FSR_1(xiii)$.

Let (X, t^*) be an fsts. Which has the property $FSR_1(vii)$. So $\forall\ x, y \in X$, $x \neq y$, $\exists\ w \in t^*$ such that $w(x) > \alpha \in I_0$, and $w(y) = 0$. Define $w' = w \vee \alpha$. Clearly $w' \in t^*$ such that $w'(x) = 1$ and $w'(y) = 0$. Then by $FSR_1(vii)$ property of (X, t^*) $\exists\ \mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu=0$. and therefore (X, t^*) has the property $FSR_1(xiii)$,

All others proofs are similar.

We now give some examples below:

3.2.1.Example:- Let (X, t^*) be a fuzzy supra topological space, and $X=\{x, y\}$ and $t^*=<\{u, v\} \cup$ constant$>$ then (X, t^*) be a fuzzy supra topological space on X, where $u(x)=0.6$, $u(y)=0$ and $v(x)=v(y)=0.4$, for $\alpha = 0.6$, (X, t^*) vacuously satisfies the $FSR_1(vii)$ property. Now $u(x) = 0.6 = \alpha$ and $u(y) = 0$. But there does not exist any $u, v \in t^*$

with $u(x) = 1 = v(y)$ and $u \wedge v = 0$ .and hence (X, t^*) is not $FSR_1(iv)$ thus we see that $FSR_1(vii) \neq> FSR_1(i)$. This example also shows that $FSR_1(i) \neq> FSR_1(xii)$. Thus, $FSR_1(q) \neq> FSR_1(p)$; (p= i, ii, iii, ..vi and q= vii,........xii).

3.2.2.Example:- Let (X, t^*) be a fuzzy supra topological space, and $X=\{x, y, z\}$ and $t^*=\{1, 0, w=\{(x, 1), (y, 0),(z, 0.4)\}, v=\{(x, 0), (y, 1), (z, 1)\}\}$, then (X, t^*) be a fuzzy supra topological space on X, for $\alpha = 1$, (X, t^*) vacuously satisfies the $FSR_1(i)$ property. But (X, t^*) is not $FSR_1(ix)$ Here $w(y)= 0$, $w(z)=.4$, but there does not exist any $u, v \in t^*$ with $u(y)=1= v(z)$ and $u \wedge v=0$ .and hence (X, t^*) is not $FSR_1(ix)$ thus we see that $FSR_1(i) \neq> FSR_1(ix)$. This example also shows that $FSR_1(i) \neq> FSR_1(xii)$. Thus, $FSR_1(q) \neq> FSR_1(p)$ (p= vii,........xii and q= i, ii, iii, ..vi).

3.2.3. Example:- Let X be an infinite set and for any $x, y \in X$, we define u_{xy}, a fuzzy set in X, as follows : $u_{xy}(x)=1$, $u_{xy}(y)=0$ and $u_{xy}(z)=0.5$ $\forall$ $z \in X$, $z \neq x, y$. Now consider the fuzzy supra topology t^* on X generated by $\{u_{xy}: x, y \in X, x \neq y\} \cup \{constants\}$. It is clear that $\overline{1_x} < u_{xy}$, $\overline{1_y} < u_{yx}$ and $u_{xy} \leq 1 - u_{yx}$, thus(X, t^*) is $FSR_1(viii)$, not $FSR_1(xii)$ as $u_{xy} \wedge u_{yx} \neq 0$ thus $FSR_1(viii) \nRightarrow FSR_1(xii)$, and so $FSR_1(x) \nRightarrow FSR_1(xii)$. Hence $FSR_1(p) \nRightarrow FSR_1(q)$,$p \in \{viii, X\}$; $q \in \{vii, ix, xi, xii\}$**[11]**.

3.2.4. Example: Let (X, t^*) be a fuzzy supra topological space, and $X=\{x, y\}$ where $t^*= < \{\beta 1_x, \alpha 1_y\} \cup \{constants\}>$, and $\beta > \alpha$. $\alpha, \beta \in I_{0,1}$. Then it is clear that (X, t^*) is $FSR_1(xi)$. But (X, t^*) is not $FSR_1(x)$, since there exist no $u, v \in t^*$ such that, $u(x)=1= v(y)$ and $u \leq 1-v$.Thus we see that $FSR_1(xi) \neq> FSR_1(x)$. Thus, $FSR_1(p) \neq> FSR_1(q)$ (p=xi, xii and q=vii, viii, ix, x).

3.2.5. Example: Let (X, t^*) be a fuzzy supra topological space, where $X=\{x, y\}$ and $t^*= < \{ \frac{1}{2} 1_x \ \frac{1}{2} 1_y\} \cup \{constants\}>$, But (X, t^*) is not $FSR_1(iv)\} \cup \{constants>$, then it is clear that (X, t^*) is $FSR_1(vi)$. But (X, t^*) is not $FSR_1(v)$. For if we take $\beta, \delta \in I_{0,1}$ such that $\beta>0.5$ and $\delta>0.5$ there exist no $u, v \in t^*$ such that, $u(x)> \beta$, $v(y)> \delta$ and $u \wedge v=0$. Thus we have $FSR_1(xii) \neq> FSR_1(xi)$. This example also shows that $FSR_1(vi) \neq> FSR_1(v)$.

3.2.6.Example:- Let (X, t^*) be a fuzzy supra topological space, and X={x, y, z} and t^*={ 1, 0, u={(x, 1), (y, 0),(z, 0.5)}, v={(x, 0), (y, 1), (z, .4)}, w = {(x, 1), (y, 1), (z, 0.5)} } then (X, t^*) be a fuzzy supra topological space on X, (X, t^*) vacuously satisfies the FSR_I(ii) property. As $\overline{1_y} \leq u$, $\overline{1_z} \leq w$ and $u \leq$ 1- w . But(X, t^*) is not FSR_I(vi) Here v={(x, 0), (y, 1) (z, .4)},Therefore v(y) ≠ v(z), where w(x)=1,w(y)=1,w(z)=.5, but there dose not exist any u, w∈ t^* with u(y)>0, w(z)>0, u ∧w =0, hence (X, t^*) is not FSR_I(vi) thus we see that FSR_I(ii) ≠> FSR_I(vi). . Thus, FSR_I(p) ≠> FSR_I(q) (p=ii, iv and q=i, iii, v, vi).

3.2.7. Example: Let X={x, y} and u, v, w∈I^X where t^*= {0, u= {(x, 1), (y, 0)}, v= {(x, 0), (y, 1), 1={(x,1), (y, 1)}},be a fuzzy supra topology on X, is generated by {0, u, v, p}, now u(x)= 1,u(y)= 0 and v(x) =0, v(y)=1 . p ={(x,1), (y,1)}.Suppose u=w and u, v∈ t^* with u(x)>0, v(y)>0 and u ∧v =0, Hence it is clear that (X, t^*) is FSR_I (xviii) but (X, t^*) is not FSR_I (xv).

3.2.8. Example: Example 3.2.6 Also shows that FSR1(ii) ⇏ FSR_I(xviii)

3.2.9. Example: Let (X, t^*) be a fuzzy supra topological space, and X={x, y} where t^*= < {w}∪{Constant}>; w is defined as w(x)=1 and w(y)= 0, vacuously(X, t^*) satisfies the FSR_I(Xiii) property. We see that;

■ (X, t^*) doesn't satisfy the property FSR_I(x). For if we take α=0.4 .Then w(x)>α and w(y) =0, but there doesn't exist μ, $\nu \in t^*$ such that μ(x)=1= ν(y) and $\mu \wedge \nu$=0.

■(X, t^*) doesn't satisfy the property FSR_I(xii). For if we take α=0.4 .Then w(x)>α and w(y) =0, but there don't exist μ, $\nu \in t^*$ such that μ(x)>0, ν(y)>0 and $\mu \wedge \nu$=0.

■ (X, t^*) doesn't satisfy the property FSR_I(iv). For if we take α=0.5 .Then w(x) =α and w(y) =0, but there don't exist μ, $\nu \in t^*$ such that μ(x)=1= ν(y) and $\mu \wedge \nu$=0.

■ (X, t^*) doesn't satisfy the property FSR_I(vi). For if we take α=0.5 .Then w(x) = α and w(y) =0, but there don't exist μ, $\nu \in t^*$ such that μ(x)>0, ν(y)>0 and $\mu \wedge \nu$=0.

3.3 Goodness and permanency properties:

In this section we show that all $FSR_1(k)$(i≤k≤xviii) properties are good extensions of their supra topological counter parts.

3.3.1. Definition: A space (X, T*) is said to be a Supra R_1-space if for x, y ∈ X such that x ∉ scl{y}, there exist supra open sets U, V such that x ∈ U, y ∈ V and U ∩ V = φ. We denote it by SR_1-space.or (X, T^*) be a supra R_1 space,∃ U, V∈ T^* such that x∈U, y∈V and U∩V=ϕ.

3.3.1 Theorem: All $FSR_1(k)$ (i≤k≤xviii) are good extensions of the topological SR_0-property. That is,

(a) If (X, T^*) is an SR_1-space, then (X, $\omega(T^*)$) satisfies $FSR_1(k)$ (i≤k≤xviii).

(b) If (X, ω (T^*)) satisfies $FSR_1(k)$ (i≤k≤xviii) then (X, T^*) is an SR_1 topological -space.

Note: For if part we only prove FSR_1(i), FSR_1(vii), FSR_1(xiii) because We know that $FSR_1(i) \Rightarrow FSR_1(ii)$, $FSR_1(i) \Rightarrow FSR_1(iii)$, $FSR_1(ii) \Rightarrow FSR_1(iv)$, $FSR_1(iii) \Rightarrow FSR_1(v)$, $FSR_1(v) \Rightarrow FSR_1(vi)$, and $FSR_1(vii) \Rightarrow FSR_1(ix)$, $FSR_1(vii) \Rightarrow FSR_1(viii)$, $FSR_1(viii) \Rightarrow FSR_1(x)$, $FSR_1(ix) \Rightarrow FSR_1(xi)$, $FSR_1(xi) \Rightarrow FSR_1(xii)$, $FSR_1(vii) \Rightarrow FSR_1(xiii)$, $FSR_1(viii) \Rightarrow FSR_1(xiv)$, $FSR_1(ix) \Rightarrow FSR_1(xv)$, $FSR_1(x) \Rightarrow FSR_1(xvi)$, $FSR_1(xi) \Rightarrow FSR_1(xvii)$, $FSR_1(xii) \Rightarrow FSR_1(xviii)$ and only if part If (X, ω (T^*)) satisfies $FSR_1(k)$ (k∈ {iv, v, vi, viii, x, xii}) then (X, T^*) is an SR_1-space.

Proof (a): (X, T^*) is an $SR_1 \Rightarrow$ (X, $\omega(T^*)$) be an FSR_1(i) space.

Suppose (X, T^*) is an SR_1- topological space. Let x, y∈X, x≠y, and $\alpha \in I_{0,1}$ ∃ w∈t^* such that w(x)> $\alpha \in I_{0,1}$, and w(y)=0 . Let ω (T^*) = {w ∈ I^X: $w^{-1}(\alpha, 1] \in T^*$, $\alpha \in I_1$}, we shall prove that (X, $\omega(T^*)$) be an FSR_1(i)space. Let w∈ $\omega(T^*)$such that w(x)≠ w(y) i.e either w(x)< w(y) or w(x)> w(y).Suppose w(y)< α < w(x). Then it is clear that $w^{-1}(\alpha, 1] \in T^*$ as w∈ $\omega(T^*)$and y∉ $w^{-1}(\alpha, 1]$, x∈ $w^{-1}(\alpha, 1]$. Since (X, T^*) be a supra R_1 space,∃ U, V∈ T^* such that x∈U, y∈V and U∩V=ϕ.Since an R_1 –topological space is R_0 –topological space,$\overline{\{x\}} \subseteq$ U and $\overline{\{y\}} \subseteq$V. Also we know that $1_{\overline{\{x\}}} = \overline{1_x}$ and $\overline{1_y} = 1_{\overline{\{y\}}}$, therefore $\overline{1_x} \leq 1_U$ and $\overline{1_y} \leq 1_v$ and since 1_U and 1_V are lower semi continuous functions from (X, T^*) into I, then 1_U, $1_V \in \omega(T^*)$ and $1_U(x)=1$,$1_V(y)=1$,and $1_U \wedge 1_V=0$,i.e (X, $\omega(T^*)$) be an FSR_1(i) space.

(X, T^*) is an $SR_1 \Longrightarrow$ (X, $\omega(T^*)$) be an FSR_1(vii) space.

Suppose (X, T^*) is an SR_1- topological space. Let x, y$\in$X, x$\neq$y, and $\alpha \in I_{0,1}$ $\exists$ w$\in t^*$ such that w(x)= $\alpha \in I_{0,1}$, and w(y)=0 .Take $\beta \in I_{0,1}$ such that $\alpha > \beta$ .Let ω (T^*) ={w $\in$ I^X: $w^{-1}(\beta, 1] \in T^*$, $\beta \in I_1$}, we shall prove that (X, $\omega(T^*)$) be an FSR_1(Vii)space. Let w $\in \omega(T^*)$such that w(x)$\neq$ w(y) i.e either w(x)< w(y) 0r w(x)> w(y). Suppose w(y)< β < w(x). Then it is clear that $w^{-1}(,\beta\ 1] \in T^*$ as w$\in \omega(T^*)$and y$\notin$ $w^{-1}(,\beta\ 1]$, x$\in$ $w^{-1}(\beta, 1]$. Since (X, T^*) be a supra R_1 space,$\exists$ U, V$\in T^*$ such that x$\in$U, y$\in$V and U$\cap$V=ϕ.Since an R_1 –topological space is R_0 –topological space,$\overline{\{x\}} \subseteq$ U and $\overline{\{y\}} \subseteq$V. Also we know that $1_{\overline{\{x\}}} = \overline{1_x}$ and $\overline{1_y} = 1_{\overline{\{y\}}}$, therefore $\overline{1_x} \leq 1_U$ and $\overline{1_y} \leq 1_V$and Since 1_U and 1_V are lower semi continuous functions from (X, T^*) into I, then 1_U, $1_V \in \omega(T^*)$ and $1_U(x)=1$, $1_V(y)=1$, and $1_U \wedge 1_V=0$,i.e (X, $\omega(T^*)$) be an FSR_1(vii) space.

(X, T^*) is an $SR_1 \Longrightarrow$ (X, $\omega(T^*)$) be an FSR_1(xiii) space.

Let (X, T^*) is an SR_1- topological space. Let x, y$\in$X, x$\neq$y, and $\alpha \in I_{0,1}$ $\exists$ w$\in t^*$ such that w(x)= $\alpha \in I_{0,1}$, and w(y)=0 .Again let α=1, and take $\beta \in I_{0,1}$ therefore $\alpha > \beta$ .Let ω (T^*) = {w $\in$ I^X: $w^{-1}(\beta, 1] \in T^*$, $\beta \in I_1$}, we shall prove that (X, $\omega(T^*)$) be an FSR_1(xiii)space. Let w $\in \omega(T^*)$ such that w(x)$\neq$ w(y) i.e either w(x)<w(y) 0r w(x)> w(y).Suppose w(y)< β < w(x). Then it is clear that $w^{-1}(,\beta\ 1] \in T^*$ as w$\in \omega(T^*)$and y$\notin$ $w^{-1}(\beta\ 1]$, x$\in$ $w^{-1}(\beta, 1]$. Since (X, T^*) be a supra R_1 space,$\exists$ U, V$\in T^*$ such that x$\in$U, y$\in$V and U$\cap$V=ϕ.Since an R_1 –topological space is R_0 –topological space,$\overline{\{x\}} \subseteq$ U and $\overline{\{y\}} \subseteq$V. Also we know that $1_{\overline{\{x\}}} = \overline{1_x}$ and $\overline{1_y} = 1_{\overline{\{y\}}}$, therefore $\overline{1_x} \leq 1_U$ and $\overline{1_y} \leq 1_V$ and since 1_U and 1_V are lower semi continuous functions from (X, T^*) into I, then 1_U, $1_V \in \omega(T^*)$ and $1_U(x)=1$, $1_V(y)=1$, and $1_U \wedge 1_V=0$, i. e (X, $\omega(T^*)$) be an FSR_1(xiii) space. Similarly we can show other conditions.

(b) (X, $\omega(T^*)$) be an FSR_1(iv) space$\Rightarrow$ (X, T^*) is an SR_1

Suppose (X, ω (T^*)) satisfies FSR_1(iv).Let x, y$\in$X such that x$\notin$ $\overline{\{y\}}$ in T^*. Then $\exists$ w$\in$

T^* such that x∈w and y∉w. Now $1_w \in \omega(T^*)$ such that $1_w(y) = 0$ and $1_w(x) = 1 > \alpha$ $\forall \alpha \in I_{0,1}$ .Therefore $\exists$ $\mu, \nu \in \omega$ (T^*) such that $\mu(x)=1=\nu(y)$ and $\mu \leq 1-\nu$. Taking $\mu^{-1}(\frac{1}{2}, 1]$ and V= $\nu^{-1}(\frac{1}{2}, 1]$. Clearly U, V∈ T^* such that x∈U, y∈V and U∩V=ϕ. Therefore, (X, T^*) be an SR_1- topological space.

(X, $\omega(T^*)$) be an FSR_1(vi) space⇒ (X, T^*) is an SR_1

Suppose (X, ω (T^*)) satisfies FSR_1(vi). Let x, y∈X, such that x∉ $\overline{\{y\}}$ in T^* . Then $\exists$ w∈ T^* such that x∈w and y∉w. Now $1_w \in \omega(T^*)$ such that $1_w(y) = 0$ and $1_w(x) = 1 > \alpha$ $\forall \alpha \in I_{0,1}$. Therefore $\exists$ $\mu, \nu \in \omega$ (T^*) $\mu(x)>0$, $\nu(y)>0$ and $\mu \wedge \nu=0$.Now x∈ $\mu^{-1}(0, 1]$∈ T^*, y∈ $\nu^{-1}(0, 1]$∈ T^* such that $\mu^{-1}(0, 1] \cap \nu^{-1}(0, 1] = \emptyset$. Therefore, (X, T^*) be an SR_1- topological space.

(X, $\omega(T^*)$) be an FSR_1(x) space⇒ (X, T^*) is an SR_1

Again suppose (X, ω (T^*)) satisfies FSR_1(x). In T^* let x, y∈X such that x∉ $\overline{\{y\}}$, Then $\exists$ w∈ T^* such that x∈w and y∉w.Let $\alpha \in I_{0,1}$ Now $\alpha 1_w \in \omega(T^*)$ implies $\alpha 1_w = 0$ and $\alpha 1_w = \alpha$ $\forall \alpha \in I_{0,1}$. Then $\exists$ $\mu, \nu \in \omega$ (T^*) such that $\mu(x) = 1 = \nu(y)$ and $\mu \leq 1-\nu$. Take U $= \mu^{-1}(\frac{1}{2}, 1]$ and V$= \nu^{-1}(\frac{1}{2}, 1]$. Clearly U, V∈ T^* such that x∈U, y∈V and U∩V=ϕ.For if z∈ U∩V then $\frac{1}{2} < \mu(z) \leq 1-\nu(z) < \frac{1}{2}$ a contradiction. Therefore (X, T^*) is an SR_1- topological space. Suppose (X, ω (T^*)) satisfies FSR_1(xii). In T^* let x, y∈X such that x∉ $\overline{\{y\}}$ .Then $\exists$ w∈ T^* such that x∈w and y∉w. Let $\alpha \in I_{0,1}$ Now $\alpha 1_w \in \omega(T^*)$ implies $\alpha 1_w = 0$ and $\alpha 1_w = \alpha$ $\forall \alpha \in I_{0,1}$ .Then $\exists$ $\mu, \nu \in \omega$ (T^*) such that $\mu(x)>0$, $\nu(y)>0$ and $\mu \wedge \nu=0$, x∈ $\mu^{-1}(0, 1]$∈ T^*, y∈ $\nu^{-1}(0, 1]$∈ T^* such that $\mu^{-1}(0, 1] \cap \nu^{-1}(0, 1]= \emptyset$. Therefore, (X, T^*) be an SR_1- topological space.

The proofs for the other properties are similar.

3.4. Reciprocal properties of FSR_1 spaces.

3.4.1. Theorem: If X is a set, $(X', t^{*'})$ be an fuzzy supra topological space having the property FSR_1(k) (i≤k≤xviii), then the reciprocal topology t^* on X for f: X → $(X', t^{*'})$ also has FSR_1(k).

Proof: Suppose $(X', t^{*'})$ be an fuzzy supra topological space having the property $FSR_1(k)$ $(i \le k \le xviii)$.Suppose, $t^* = \{f^{-1}(u) : u \in t^{*'}\}$. Now (X, t^*) is a fuzzy supra topological space. We have to show that (X, t^*) has $FSR_1(k)$ $(i \le k \le xviii)$.

(a): Let $(X', t^{*'})$ be an $FSR_1(i)$, t^* be the reciprocal topology on X for $f: X \to (X', t^{*'})$.Let $x, y \in X$, $x \ne y$, $w \in t^*$ such that $w(x) \ne w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t^*$, there exist $w' \in t^{*'}$ such that $w = f^{-1}(w')$. Now $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$, similarly $w'(y') = w(y)$.So $w'(x') \ne w'(y')$. Therefore there exists $\mu, \nu \in t^{*'}$ such that $\overline{1_{x'}} \le \mu$, $\bar{1}_y \le \nu$ and $\mu \wedge \nu = 0$, we have , $f(\overline{1_z}) \le \overline{1_{f(z)}}$ for every $z \in X$, since f is continuous. Thus, $f(\overline{1_x}) \le \overline{1_{f(x)}} = \overline{1_{x'}} \le \mu$, and $f(\overline{1_y}) \le \overline{1_{f(y)}} = \overline{1_{y'}} \le \nu$. So $\overline{1_x} \le f^{-1}(\mu)$, and $\overline{1_y} \le f^{-1}(\nu)$.Moreover, $f^{-1}(\mu) \wedge f^{-1}(\nu) = 0$, Clearly $f^{-1}(\mu), f^{-1}(\nu) \in t^*$. Hence (X, t^*) is an $FSR_1(i)$ space.

(b): Let $(X', t^{*'})$ be an $FSR_1(ii)$, t^* be the reciprocal topology on X for $f: X \to (X', t^{*'})$.Let $x, y \in X$, $x \ne y$, $w \in t^*$ such that $w(x) \ne w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t^*$, there exist $w' \in t^{*'}$ such that $w = f^{-1}(w')$. Now $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly $w'(y') = w(y)$. So $w'(x') \ne w'(y')$. Therefore there exists $\mu, \nu \in t^{*'}$ such that $\overline{1_x} \le \mu$, $\bar{1}_y \le \nu$ and $\mu \le 1 - \nu$, we have , $f(\overline{1_z}) \le \overline{1_{f(z)}}$ for every $z \in X$, since f is continuous .Thus, $f(\overline{1_x}) \le \overline{1_{f(x)}} = \overline{1_x} \le \mu$, and $f(\overline{1_y}) \le \overline{1_{f(y)}} = \overline{1_{y'}} \le \nu$, hence $\overline{1_x} \le f^{-1}(\mu)$, and $\overline{1_y} \le f^{-1}(\nu)$.Moreover $f^{-1}(\mu) \le 1 - f^{-1}(\nu)$, Clearly $f^{-1}(\mu)$, $f^{-1}(\nu) \in t^*$. Hence (X, t^*) is an $FSR_1(ii)$ space.

(c): Suppose $(X', t^{*'})$ be an fuzzy supra topological space having the property $FSR_1(iii)$ Suppose, $t^* = \{f^{-1}(u) : u \in t^{*'}\}$. Now (X, t^*) is a fuzzy supra topological space. We have to show that (X, t^*) has $FSR_1(iii)$

Since $(X', t^{*'})$ be an $FSR_1(iii)$, t^* be the reciprocal topology on X for $f: X \to (X', t^{*'})$.Let $x, y \in X$, $x \ne y$, $w \in t^*$ such that $w(x) \ne w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t^*$, there exist $w' \in t^{*'}$ such that $w = f^{-1}(w')$. Now $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$, similarly $w'(y') = w(y)$. So $w'(x') \ne w'(y')$.. Therefore there exists $\mu, \nu \in t^{*'}$ such that $\mu(x')$

$= \nu(y') = 1$ and $\mu \wedge \nu = 0$, Now $f^{-1}(\mu(x)) = \mu f(x) = \mu(x') = 1$. Similarly $f^{-1}(\nu(y)) = 1$ Moreover, $f^{-1}(\mu) \wedge f^{-1}(\nu)) = 0$, Clearly $f^{-1}(\mu), f^{-1}(\nu) \in t^*$. Hence (X, t^*) is an FSR_1(iii).

(d): Let $(X', t^{*'})$ be an FSR_1(iv), t^* be the reciprocal topology on X for f: $X \to (X', t^{*'})$.Let x, $y \in X$, $x \neq y$, $w \in t^*$ such that $w(x) \neq w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t^*$, there exist $w' \in t^{*'}$ such that $w = f^{-1}(w')$. Now $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$, similarly $w'(y') = w(y)$. So $w'(x') \neq w'(y')$.. Therefore there exists $\mu, \nu \in t^{*'}$ such that $\mu(x') = \nu(y') = 1$ and $\mu \leq 1-\nu$. Now $f^{-1}(\mu(x)) = \mu f(x) = \mu(x') = 1$. Similarly $f^{-1}(\nu(y)) = 1$ .Moreover $f^{-1}(\mu) \leq 1 - f^{-1}(\nu)$, Clearly $f^{-1}(\mu), f^{-1}(\nu) \in t^*$.Hence (X, t^*) is an FSR_1(iv).

(e): Let $(X', t^{*'})$ be an FSR_1(v), t^* be the reciprocal topology on X for f: $X \to (X', t^{*'})$.Let x, $y \in X$, $x \neq y$, $w \in t^*$ such that $w(x) \neq w(y)$.. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t^*$, there exist $w' \in t^{*'}$ such that $w = f^{-1}(w')$. Now $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$, similarly $w'(y') = w(y)$. So $w'(x') \neq w'(y')$. Therefore there exists $\mu, \nu \in t^{*'}$ such that $\mu(x) > \alpha$, $\nu(y) > \beta$ and $\mu \wedge \nu = 0$. Now $f^{-1}(\mu(x)) = \mu f(x) = \mu(x') > \alpha$.Similarly $f^{-1}(\nu(y)) > \beta$ Moreover, $f^{-1}(\mu) \wedge f^{-1}(\nu)) = 0$, Clearly $f^{-1}(\mu), f^{-1}(\nu) \in t^*$, hence (X, t^*) is an FSR_1(v).

(f): Let $(X', t^{*'})$ be an FSR_1(vi), t^* be the reciprocal topology on X for f: $X \to (X', t^{*'})$.Let x, $y \in X$, $x \neq y$, $w \in t^*$ such that $w(x) = \alpha \in I_{0,1}$, and $w(y) = \beta \in I_{0,1}$ where $\alpha \neq \beta$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t^*$, there exist $w' \in t^{*'}$ such that $w = f^{-1}(w')$. Now $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) > \alpha$, similarly $w'(y') = w(y)$. So $w'(x') > \alpha$, and $w'(y') = 0$. Therefore there exists $\mu, \nu \in t^{*'}$ such that $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$. Now $f^{-1}(\mu(x)) = \mu f(x) = \mu(x') > 0$.Similarly we can show that $f^{-1}(\nu(y)) > 0$.Moreover, $f^{-1}(\mu) \wedge f^{-1}(\nu)) = 0$, clearly $f^{-1}(\mu), f^{-1}(\nu) \in t^*$. Hence (X, t^*) is an FSR_1(vi).

(g) : Let $(X', t^{*'})$ be an FSR_1(vii), t^* be the reciprocal topology on X for f: $X \to (X', t^{*'})$.Let x, $y \in X$, $x \neq y$, $w \in t^*$ such that $w(x) > \alpha \in I_{0,1}$, and $w(y) = 0 \in I_{0,1}$, . Let $f(x) = x'$ and f(y) = $y^{/}$. As w∈t^*, there exist $w^{/}$∈ $t^{*/}$ such that w = $f^{-1}(w^{/})$. Now $w^{/}(x^{/})$ = $w^{/}$(f(x))=($f^{-1}(w^{/})$)(x)= w(x)> α, similarly $w^{/}(y^{/})$=w(y). So $w^{/}(x^{/})>\alpha$, and $w^{/}(y^{/})$ =0. Therefore there exists $\mu, \nu \in t^{*/}$ such that $\overline{1_x} \leq \mu$, $\bar{1}_y \leq \nu$ and $\mu \wedge \nu$=0. We have, f($\bar{1}_z$)≤ $\overline{1_{f(z)}}$ for every z∈X, since f is continuous . Now f($\bar{1}_x$)≤ $\overline{1_{f(x)}} = \overline{1_x} \leq \mu$, and f($\bar{1}_y$)≤ $\overline{1_{f(y)}}$ = $\overline{1_y} \leq \nu$ .So $\bar{1}_x \leq f^{-1}(\mu)$, and $\bar{1}_y \leq f^{-1}(\nu)$. Moreover, $f^{-1}(\mu) \wedge f^{-1}(\nu)$ =0, Clearly $f^{-1}(\mu), f^{-1}(\gamma) \in t^*$. Hence (X, t^*) is an FSR_1(vii) space.

(h) : Let ($X^{/}$, $t^{*/}$) be an FSR_1(viii), t^* be the reciprocal topology on X for f: X → ($X^{/}$, $t^{*/}$).Let x, y∈X, x≠y, w∈t^* such that w(x)> α ∈$I_{0,1}$, and w(y)=0 ∈$I_{0,1}$,. Let f(x) =$x^{/}$ and f(y) = $y^{/}$. As w∈t^*, there exist $w^{/}$∈ $t^{*/}$ such that w = $f^{-1}(w^{/})$. Now $w^{/}(x^{/})$ = $w^{/}$(f(x))=($f^{-1}(w^{/})$)(x)= w(x)> α, similarly $w^{/}(y^{/})$=w(y). So $w^{/}(x^{/})>\alpha$, and $w^{/}(y^{/})$ =0. Therefore there exists $\mu, \nu \in t^{*/}$ such that $\overline{1_x} \leq \mu$, $\bar{1}_y \leq \nu$ and $\mu \wedge \nu$=0. we have , f($\bar{1}_z$)≤ $\overline{1_{f(z)}}$ for every z∈X, since f is continuous .Now f($\bar{1}_x$)≤ $\overline{1_{f(x)}} = \overline{1_x} \leq \mu$, and f($\bar{1}_y$)≤ $\overline{1_{f(y)}} = \overline{1_y}$ $\leq \nu$. So $\bar{1}_x \leq f^{-1}(\mu)$, and $\bar{1}_y \leq f^{-1}(\nu)$ Moreover $f^{-1}(\mu) \leq 1 - f^{-1}(\nu)$, Clearly $f^{-1}(\mu), f^{-1}(\nu) \in t^*$. Hence (X, t^*) is an FSR_1(viii) space.

(i): Let ($X^{/}$, $t^{*/}$) be an FSR_1(ix), t^* be the reciprocal topology on X for f: X → ($X^{/}$, $t^{*/}$).Let x, y∈X, x≠y, w∈t^* such that w(x)> α ∈$I_{0,1}$, and w(y)=0 ∈$I_{0,1}$. Let f(x) =$x^{/}$ and f(y) = $y^{/}$. As w∈t^*, there exist $w^{/}$∈ $t^{*/}$ such that w = $f^{-1}(w^{/})$. Now $w^{/}(x^{/})$ = $w^{/}$(f(x)) ($f^{-1}(w^{/})$)(x)= w(x)> α, similarly $w^{/}(y^{/})$=w(y). So $w^{/}(x^{/})>\alpha$, and $w^{/}(y^{/})$ =0. Therefore there exists $\mu, \nu \in t^{*/}$ such that $\mu(x^{/}) = \nu(y^{/})$ =1 and $\mu \wedge \nu$=0. Now $f^{-1}(\mu(x)) = \mu$ f(x) = $\mu(x^{/})$ =1 .Similarly $f^{-1}(\nu(y))$=1. Moreover, $f^{-1}(\mu) \wedge f^{-1}(\nu)$ =0, Clearly $f^{-1}(\mu), f^{-1}(\gamma) \in t^*$. Hence (X, t^*) is an FSR_1(ix) space.

Proofs are same for the remaining properties.

3.5. Hereditary Properties of FSR_1 spaces.

3.5.1. Theorem: All the properties FSR_1(k) of subspace topology where (i≤k≤xviii) are hereditary.

(a) Proof : FSR_1(vii) : If $\forall$ x, $y \in X$, $x \neq y$, $\exists$ $w \in t^*$ such that either $w(x) > \alpha \in I_{0,1}$, and $w(y)=0$, or $w(y) > \alpha \in I_{0,1}$, and $w(x)=0$, then $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu = 0$.Let (X, t^*) is FSR_1(i), we shall prove that (A, t^*_A) is FSR_1(vii) .Let x, $y \in A$, with $x \neq y$ then x, $y \in X$, and $x \neq y$. Let $w \in t^*$ such that either $w(x) > \alpha \in I_{0,1}$, and $w(y)=0$, or $w(y) > \alpha \in I_{0,1}$, and $w(x)=0$, Let $w \in t^*$ such that $w(x) > \alpha \in I_{0,1}$, and $w(y)=0$ since (X, t^*) is FSR_1(vii), then $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu = 0$. We have, t^*-cl(1_x) $\cap 1_A = t^*_A$-cl(1_x). Let $x \in A$, $m \in t^*_A$ then there exist $\mu \in t^*$ such that $1_A \wedge \mu = m$. Since $x \in A$ then $m(x) = \mu(x)$, So, t^*-cl(1_x)$\leq \mu$. Similarly we can prove that, t^*-cl(1_y)$\leq \nu$. Now, t^*_A-cl(1_x) = $1_A \cap$(t^*-cl(1_x)) $\leq 1_A \wedge \mu = m$, $\forall$ $\mu \in t^*$ and so on. We observed that $\overline{1_x} \leq m$, $\overline{1_y} \leq n$, where $n \in t^*_A$ and $m \wedge n = (1_A \wedge \mu) \wedge (1_A \wedge \nu) = 1_A \wedge (\mu \wedge \nu) = 1_A \wedge 0 = 0$. This implies that, (A, t^*_A) has FSR_1(vii).

(b) Let (X, t^*) is FSR_1(ii), we shall prove that (A, t^*_A) is FSR_1(viii).Let x, $y \in A$, with $x \neq y$ then x, $y \in X$, and $x \neq y$. Let $w \in t^*$ such that either $w(x) > \alpha \in I_{0,1}$, and $w(y)=0$, or $w(y) > \alpha \in I_{0,1}$, and $w(x)=0$, Let $w \in t^*$ such that $w(x) > \alpha \in I_{0,1}$, and $w(y)=0$ since (X, t^*) is FSR_1(viii), then $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \leq 1 - \nu$. We have, t^*-cl(1_x) $\cap 1_A = t^*_A$-cl(1_x). Let $x \in A$, $m \in t^*_A$ then there exist $\mu \in t^*$ such that $1_A \wedge \mu = m$. Since $x \in A$ then $m(x) = \mu(x)$, So, t^*-cl(1_x)$\leq \mu$. Similarly we can prove that, t^*-cl(1_y)$\leq \nu$. Now, t^*_A -cl(1_x) = $1_A \cap$(t^*-cl(1_x)) $\leq 1_A \wedge \mu = m$, $\forall$ $\mu \in t^*$ and so on. We observed that $\overline{1_x} \leq m$, $\overline{1_y} \leq n$, where $n \in t^*_A$. Now $1 - n = 1 - 1_A \wedge \nu \geq 1 - \nu \geq \mu \geq 1_A \wedge \mu = m$ .Hence, (A, t^*_A) has FSR_1(viii).

All other proofs are similar and so omitted.

3.6. $I_\alpha(t^*)$ Properties of FSR_1 spaces.

3.6.1. Theorem: Let (X, t^*) be a fuzzy supra topological space and $I_\alpha(t^*) = \{u^{-1}(\alpha, 1]: u \in t^*, \alpha \in I_1\}$ then

(a) (X, t^*) is FSR_1(iii) $\Rightarrow$(X, $I_\alpha(t^*)$) is supra R_1

(b) (X, t^*) is FSR_1 (iv) $\Rightarrow$ (X, $I_\alpha(t^*)$) is supra R_1

(c) (X, t^*) is $FSR_1(v) \Rightarrow (X, I_\alpha(t^*))$ is supra R_1

(d) (X, t^*) is $FSR_1(vi) \Rightarrow (X, I_\alpha(t^*))$ is supra R_1

Proof :-(a) Let (X, t^*) is $FSR_1(iii)$ we shall prove that $(X, I_\alpha(t^*))$ is supra R_1. Let x, y $\in$X; x$\neq$y, and M$\in I_\alpha(t^*)$with x$\in$M and y$\notin$M or x$\notin$M and y$\in$M. Suppose x$\in$M and y$\notin$M. We can write M= $w^{-1}(\alpha, 1]$, where w$\in t^*$, then we can have $w(x)>\alpha$, $w(y)\leq\alpha$, therefore w(x)$\neq$w(y). Since(X, t^*) is $FSR_1(iii)$. Then $\exists$ u, v$\in t^*$ such that u(x)= 1, v(y)=1, and u$\wedge$v =0 .It follows that $\exists$ $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1]\in$, $I_\alpha(t^*)$ and x$\in$ $u^{-1}(\alpha, 1]$ and y$\in v^{-1}(\alpha, 1$],and $u^{-1}(\alpha, 1]\cap v^{-1}(\alpha, 1]=\phi$, as u$\wedge$v =0. Hence it is clear that $(X, I(t^*))$ is supra R_1.

Proof :-(b) Let (X, t^*) is $FSR_1(iv)$ we shall prove that $(X, I_\alpha(t^*))$ is supra R_1. Let x, y $\in$X; x$\neq$y, and M$\in$I$(t^*)$with x$\in$M and y$\notin$M or x$\notin$M and y$\in$M. Suppose x$\in$M and y$\notin$M. We can write M= $w^{-1}(\alpha, 1]$, where w$\in t^*$, then we can have $w(x)>\alpha$, $w(y)\leq\alpha$, therefore w(x)$\neq$w(y). Since(X, t^*) is $FSR_1(iv)$. Then $\exists$ u, v$\in t^*$ such that u(x) =1, v(y) =1, and u<1-v. It follows that $\exists$ $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1]\in$, $I_\alpha(t^*)$, and x$\in$ $u^{-1}(\alpha, 1]$ and y$\in$ $v^{-1}(\alpha, 1]$,and $u^{-1}(\alpha, 1]<1-v^{-1}(\alpha, 1]$, as u<1-v . Hence it is clear that $(X, \in, I_\alpha(t^*))$ is supra R_1.

Proof : (c) Let (X, t^*) is $FSR_1(v)$ we shall prove that $(X, \in, I_\alpha(t^*))$ is supra R_1. Let x, y $\in$X; x$\neq$y, and M$\in\in$, $I_\alpha(t^*)$ with x$\in$M and y$\notin$M or x$\notin$M and y$\in$M. Suppose x$\in$M and y$\notin$M. We can write M= $w^{-1}(\alpha, 1]$, where w$\in t^*$, then we can have $w(x)>\alpha$, $w(y)\leq\alpha$, therefore w(x)$\neq$w(y). Since(X, t^*) is $FSR_1(v)$ $\forall\ \beta,\delta\in I_{0,1}$, $\exists\ \mu,\nu\in t^*$ such that $u(x)>\beta, \nu(y)>\delta$ and $u\wedge\nu=0$. It follows that $\exists$ $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1]\in I(t^*)$ where $\beta>\alpha$, $\delta>\alpha$ and x$\in$ $u^{-1}(\alpha, 1]$and y$\in v^{-1}(\alpha, 1)$ as $u(x)>\beta, \nu(y)>\delta$ and $u^{-1}(\alpha, 1]\cap v^{-1}(\alpha, 1]=\phi$ as u$\wedge$v =0 and $\beta,\delta\in I_{0,1}$. Hence it is clear that$(X, I_\alpha(t^*))$ is supra R_1.

Proof: (d) Let (X, t^*) is $FSR_1(vi)$ we shall prove that $(X, I_\alpha(t^*))$ is supra R_1. Let x, y $\in$X; x$\neq$y, and M$\in$I$(t^*)$with x$\in$M and y$\notin$M or x$\notin$M and y$\in$M. Suppose x$\in$M and y$\notin$M.So we can write M= $w^{-1}(\alpha, 1]$, where w$\in t^*$, then we can have $w(x)>\alpha$, $w(y)\leq\alpha$, with w(x)$\neq$w(y). Since(X, t^*) is $FSR_1(vi)$. Then $\exists$ u, v$\in t^*$ such that u(x)> 0, v(y)>0, and u$\wedge$v =0 .It follows that $\exists$ $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1]\in I(t^*)$and x$\in$ $u^{-1}(\alpha, 1]$and y$\in v^{-1}(\alpha, 1]$,as u(x)>0, v(y)>0 and $u^{-1}(\alpha, 1]\cap v^{-1}(\alpha, 1]=\phi$, as u$\wedge$v =0. Hence it is clear that$(X, I_\alpha(t^*))$ is supra R_1.

Now we give some examples to show the following.

(a) (X, I_α (t^*)) is supra $R_1 \nRightarrow$ (X, t^*) is FSR_1(iii),

(b) (X, I_α (t^*)) is supra $R_1 \nRightarrow$ (X, t^*) is FSR_1(iv),

(c) (X, I_α (t^*)) is supra $R_1 \nRightarrow$ (X, t^*) is FSR_1(v).

(d) (X, I_α (t^*)) is supra $R_1 \nRightarrow$ (X, t^*) is FSR_1(vi).

3.6.1 Example : Let I_α (t^*)= {X, ϕ, {x}, {y}}.Then clearly I_α (t^*)is a supra topology on X and (X, I_α (t^*)) is supra R_1 space.

Now let X={x, y}and u, v, w∈I^X where t^*={1, 0, u={(x, .8), (y, .2)}, v={(x, .1), (y, .7)}, w={(x, .8),(y, .7)}} be a fuzzy supra topology on X, is generated by {0, u, v, w, 1},Here w∈t^* with w(x)≠w(y), since w(x)=.8, w(y)=.7, now u(x)=.8,u(y)=.2 and v(x)=.1, v(y) =.7.So u, v∈ t^* with u(x)>0, v(y)>0 and u ∧v≠0, Hence it is clear that (X, t^*) is not FSR_1 (vi), Also (X, t^*) is not FSR_1(iii), since u(x) ≠1 and v(y) ≠1 and u∧v≠0, Again for $\alpha = 0.6$, (X, t^*) is not FSR_1(iv), since u(x) ≠1 and v(y) ≠1. Hence (X, I_α (t^*)) is supra $R_1 \nRightarrow$ (X, t^*) is FSR_1(iii), (X, I_α (t^*)) is supra $R_1 \nRightarrow$ (X, t^*) is FSR_1(iv), (X, I_α (t^*)) is supra $R_1 \nRightarrow$ (X, t^*) is FSR_1(vi).

3.6.2 Example: Let I_α (t^*) = {X, ϕ}, then clearly (X, I(t^*))is supra R_1 space.

Let X={x, y}and u, v∈I^X where t^*={1, 0, u={(x, .8), (y, .2)}, v={(x, .1), (y, .7)}} be a fuzzy topology on X, is generated by {0, u, v, 1}, now u(x) =.8,u(y)=.2 and v(x)=.1, v(y)=.7.So u, v∈ t^* with $\alpha = .9$, we have (X, t^*) is not FSR_1 (v) space. Hence (X, I_α (t^*)) is supra $R_1 \nRightarrow$ (X, t^*) is FSR_1(v). **[65]**

This completes the proof.

3.7. Homeomorphisms among FSR_1 Spaces.

3.7.1 Theorem: Let (X, T^*) be an FSR_1 (K) (i≤k≤xviii) .Prove that every homeomorphic image of FSR_1 (K) is also FSR_1 (K).

(a) Proof: Let (X, T^*) be an FSR_1 (vii). Let f:(X, t_1^*)→(Y, t_2^*) be a homeomorphism between fsts, where (X, t_1^*) has FSR_1(vii), Let y_1, y_2∈Y, $y_1 \neq y_2$,α ∈$I_{0,1}$ and w_1∈ t_2^* such that $w_1(y_1)$ >α and $w_1(y_2)$= 0. Now $f^{-1}(y_1)$, $f^{-1}(y_2)$∈X and $f^{-1}(w_2) \in t_1^*$) such that that $(f^{-1}(w_2))(f^{-1}(y_1))>\alpha$ and $(f^{-1}(w_2))(f^{-1}(y_2))=0$. Since (X, T^*) be an FSR_1 (1), there exist u, v $\in t_1^*$ such that $\overline{1_{f^{-1}(y_1)}} \leq u$, $\overline{1_{f^{-1}(y_2)}} \leq v$ and u∧v=0 .Since f is a homeomorphism, $\overline{1_{f^{-1}(y_1)}} = f^1(\overline{1_{y_1}})$ ∀ y∈Y, f(u), f(v) $\in t_2^*$ such that $\overline{1_{y_1}} \leq f(u)$, $\overline{1_{y_2}} \leq f(v)$ and f(u)∧ f(v)=0. Therefore (Y, t_2^*) is FSR_1 (vii).

(b) Let (X, T^*) be an FSR_1(viii), Let f:(X, t_1^*)→(Y, t_2^*) be a homeomorphism between fsts, where (X, t_1^*) has FSR_1(viii). Let y_1, y_2∈Y, $y_1 \neq y_2$, $\alpha \in I_{0,1}$ and $w_1 \in t_2^*$ such that $w_1(y_1)>\alpha$ and $w_1(y_2)=0$. Now $f^{-1}(y_1)$, $f^{-1}(y_2)$∈X and $f^{-1}(w_2) \in t_1^*$) such that $(f^{-1}(w_2))(f^{-1}(y_1))>\alpha$ and $(f^{-1}(w_2))(f^{-1}(y_2))=0$. Since (X, T^*) be an FSR_1 (viii), Therefore there exist u, v $\in t_1^*$ such that $\overline{1_{f^{-1}(y_1)}} \leq u$, $\overline{1_{f^{-1}(y_2)}} \leq v$ and $u \leq 1-v$, we have , $f(\overline{1_z}) \leq \overline{1_{f(z)}}$ for every z∈X, Since f is a homeomorphism, $\overline{1_{f^{-1}(y_1)}} = f^1(\overline{1_{y_1}})$ ∀ y∈Y, f(u), f(v) $\in t_2^*$ such that $\overline{1_{y_1}} \leq f(u)$, $\overline{1_{y_2}} \leq\in f(v)$ Moreover $f(u) \leq 1 - f(v)$, Clearly f(u), f(v)∈ t_2^*. Hence (Y, t_2^*) is an FSR_1(ii) space.

All other proofs are similar, and so are omitted.

3.8. Initial properties of FSR_1 Spaces:

3.8.1 Definition : For each i ∈ Λ, Let $f_i : X \to (Y_i, \tau^*_i)$ are the functions from a set X into fsts (Y_i, τ^*_i) then the smallest fuzzy supra topology on X for which the functions f_i, i ∈ Λ are fuzzy continuous is called initial fuzzy supra topology on X generated by the collection of functions $\{f_i : i \in \Lambda\}$.

3.8.2. Theorem: The properties FSR_1 (k) (i≤k≤xii) are initial, i.e if f_j: X→ (X_j, t_j^*) is a source in fuzzy supra topological spaces where all (X_j , t_j^*) are FSR_1 (k) then the initial fuzzy supra topology t^* on X is also FSR_1 (k).

Proof: (a) Let $\{(X_j, t_j^*) : j \in J\}$ be a family of FSR_1 (vii), and $\{ f_j: X \to (X_j, t_j^*) : j \in J\}$ be a family of functions and t^* the initial fuzzy topology on X induced by the family $\{f_j : j \in J\}$. Let x, y ∈ X, x≠y, $\alpha \in I_{0,1}$ and w∈t^* such that $w(x) > \alpha \in I_{0,1}$, and w(y)=0 . Since w∈$t^*$, there exist basic t^* -supra open set, w_p such that w = sup { w_p :p∈P}. Also each must be expressible as $w_p = \inf \{ f^{-1}_{p_k} w_{p_{k'}} : 1 \leq p \leq n\}$ as $w(x) > \alpha$ and $w(y)=0$, we can find some k ($1 \leq k \leq n$) say k/ such that $f^{-1}_{p'_k} w_{p'_k}(x) > \alpha$ and $f^{-1}_{p'_k} w_{p'_k}(y)=0$ .This implies that $w_{p'_k} f^{-1}_{p'_k}(x) > \alpha$ and $w_{p'_k} f^{-1}_{p'_k}(y) = 0$. Since $(X_{p'_k}, t^*_{p'_k})$ is FSR1(vii), there exist $u_{p_{k'}}, v_{p_{k'}} \in t^*_{p_{k'}}$ such that $\overline{1_{f_{p_{k'}}}}(x) \leq u_{p_{k'}}$, $\overline{1_{f_{p_{k'}}}}(y) \leq v_{p_{k'}}$ and $u_{p_{k'}} \wedge v_{p_{k'}} = 0$ Also since $f_{p_{k'}}$ is continuous, we have $f_{p_{k'}}(\overline{1_x}) \leq \overline{1_{f_{p_{k'}}}}(x)$. Now put $u = f^{-1}_{p'_k}(u_{p_{k'}})$ and $v = f^{-1}_{p'_k}(v_{p_{k'}})$. Then $u, v \in t^*$ such that $\overline{1_x} \leq u$, $\overline{1_y} \leq v$ and $u \wedge v = 0$. Hence (X, t^*) is an FSR_1(vii) space .

(b) Let $\{(X_j , t^*_j): j \in J\}$ be a family of FSR_1 (viii), and $\{ f_j: X \rightarrow (X_j , t^*_j) : j \in J\}$ be a family of functions and t^* the initial fuzzy topology on X induced by the family $\{f_j : j \in J \}$. Let $x, y \in X$, $x \neq y$, $\alpha \in I_{0,1}$ and $w \in t^*$ such that $w(x) > \alpha \in I_{0,1}$, and $w(y)=0$. Since $w \in t^*$, there exist basic t^* -supra open set, w_p such that $w = \sup \{w_p : p \in P\}$. Also each must be expressible as $w_p = \inf \{ f^{-1}_{p_k} w_{p_k} : 1 \leq K \leq n\}$ as $w(x) > \alpha$ and $w(y)=0$, we can find some k ($1 \leq k \leq n$) say k' such that $f^{-1}_{p_k} w_{p'_k}(x) > \alpha$ and $f^{-1}_{p'_k} w_{p'_k}(y)=0$. This implies that $w_{p_k} f^{-1}_{p_k}(x) > \alpha$ and $w_{p'_k} f^{-1}_{p_k}(y) = 0$. Since $(X_{p'_k}, t^*_{p'_k})$ is FSR_1(viii), there exist $u_{p_{k'}}, v_{p_{k'}} \in t^*_{p_{k'}}$ such that $\overline{1_{f_{p_{k'}}}}(x) \leq u_{p_{k'}}$, $\overline{1_{f_{p_{k'}}}}(y) \leq v_{p_{k'}}$ and $u_{p_{k'}} \wedge v_{p_{k'}} = 0$ Also since $f_{p_{k'}}$ is continuous , we have $f_{p_{k'}}(\overline{1_x}) \leq \overline{1_{f_{p_{k'}}}}(x)$. Now put $u = f^{-1}_{p'_k}(u_{p_{k'}})$ and $v = f^{-1}_{p'_k}(v_{p_{k'}})$. Then $u, v \in t^*$ such that $\overline{1_x} \leq u$, $\overline{1_y} \leq v$ and $u < 1 - v$. Hence (X, t^*) is an FSR_1(viii) space .

All other proofs are similar.

3.9. Productivity of FSR_1 Spaces.

3.9.1. Theorem[12]: The properties $FSR_1(k)$, $k \in \{i, ii, iii, ...xii\}$ are productive, i.e. if $(X_i, t_i^*)_{i\in J}$ is a family of fuzzy supra topological spaces, each of them having the property $FSR_1(k)$, $\Leftrightarrow$ the product space $\left(X = \prod_{i\in J} X_i, t^*\right)$ also has $FSR_1(k)$.

Proof:

Suppose each of $(X_i, t_i^*)_{i\in J}$ has the property FSR_1(vii). $\forall$ x, y$\in$X, x$\neq$y, $\exists$ w$\in t^*$ such that either w(x)> $\alpha \in I_0$ and w(y)=0 or w(y) > $\alpha \in I_0$ and w(x)=0, Suppose w(x)> $\alpha \in I_{0,1}$, and w(y)=0. Now we have x, y$\in$X, x $\neq$ y, where x = $(x_i)_{i\in J}$ and y = $(y_i)_{i\in J}$, $\alpha \in I_0$, and from definition of product topology w(x)= min{ $w_j(x_i)$ j$\in J$}, w(y) = min { $w_j(y_i)$: j$\in J$ }

Hence we can find at least one $w_i \in t_i^*$ and x_j , $y_j \in X_i$ with $x_j \neq y_j$ and $w_j(x_i)>\alpha$, $w_j(y_i)$ =0 . Since each of $(X_i, t_i^*)_{i\in J}$ has the property FSR_1(vii). , then $\exists$ $\mu_j, \nu_j \in t_i^*$ such that $\overline{1_{x_j}} \leq \mu_j$, $\overline{1_{y_j}} \leq \nu_j$ and $\mu_j \wedge \nu_j$=0 . Using projection we have $\pi_j(x)$= x_j and $\pi_i(y)$= y_j and hence μ_j $(\pi_j(x))>\alpha$ and $\nu_j(\pi_i(y))$=0 .Since each of $(X_i, t^*_i)_{i\in J}$ has the property FSR_0(vii), and so (X, t^*) has FSR_1(vii).So $\exists$ $\mu, \nu \in t^*$ such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \wedge \nu$=0.

Conversely suppose that (X, t*) is FSR_1(vii), we shall prove that $(X_i, t_i^*)_{i\in J}$ has the property FSR_1(vii). Let for some i$\in \Lambda$, a_i be a fixed element of X_i, suppose that A_i= {x$\in$X =$\prod_{i\in\Delta} X_i$ / x_j =a_j for some i$\neq$j}. So that A_i is a subset of X, and hence $(A_i, t^*_{A_i})$ is also a subspace of (X, t^*) , since(X, t^*) is FSR_1(vii), then $(A_i , t^*_{A_i})$ Is also FSR_1(vii). Now we have A_i is a homeomorphic image of X_i. Hence it is clear that $(X_i, t_i^*)_{i\in J}$ is also FSR_1(vii).

All other proofs are similar and so these are omitted.

Just for record, we introduce a definition of FSR_1-space here, without study its properties. We hope to study these in separate work.

3.9.2. Definition: A fuzzy supra topological space (X, t^*) is called FSR_1 –space if for every pair of fuzzy point x_t and y_r in X with $cl(x_t) \neq cl(y_r)$,then there exists $\lambda, \mu \in t^*$ such that $cl(x_t) \leq \lambda$, $cl(y_r) \leq \mu$ and $\lambda \wedge \mu$ =0.

3.9.1. Example: Let X = {$x_{0.4}$, $y_{0.7}$}, τ^* = {0_x, 1_x, μ, λ }, where μ= $x_{0.4}$, λ = $y_{0.7}$ is defined as μ(x) = 0. 4, μ(y) = 0; λ(x) = 0, λ(y) = 0.7; clearly, (X, τ^*) is a fsts. $\mu'(x) = 0.6$, $\mu'(y) = 1$; $\lambda'(x) = 1$, $\lambda'(y) = 0.3$.

Closed sets in (X, τ^*) are 0_x, 1_x. Here the smallest closed set containing $x_{0.4}$ is $x_{0.6}$, Clearly $cl(x_{0.4}) = x_{0.6} \leq \lambda$.

3.9.2. Theorem: For a fuzzy supra topological space (X, t^*), the following properties are equivalent.

(1) For every pair of fuzzy point x_t and y_r in X with $cl(x_t) \neq cl(y_r)$,then there exists $\lambda, \mu \in t^*$ such that $cl(x_t) \leq \lambda$, $cl(y_r) \leq \mu$ and $\lambda \wedge \mu = 0$.

(2) x, y ∈ X, with $cl(x_t) \neq cl(y_r)$ then x_t and y_r have disjoint neighborhoods. $\lambda \wedge \mu = 0$.

Proof: (1) ⇒ (2)

From definition 3.9.2 of FSR_1- space, it is clear that for every x, y∈X, with $cl(x_t) \neq cl(y_r)$ then there exists $\lambda, \mu \in t^*$ such that $cl(x_t) \leq \lambda$, $cl(y_r) \leq \mu$ and $\lambda \wedge \mu = 0$.

(2) ⇒ (1)

Suppose $x_t \in (cl(y_r))^c$ then $x_t \in cl(x_t)$ and $x_t \notin cl(y_r)$ with $cl(x_t) \neq cl(y_r)$ then x_t and y_r have disjoint neighborhoods $\lambda, \mu \in t^*$ such that $\lambda \wedge \mu = 0$.Hence clearly that $cl(x_t) \leq \lambda$, $cl(y_r) \leq \mu$ and $\lambda \wedge \mu = 0$. So (X, t^*) satisfies for every pair of fuzzy point x_t and y_r in X with $cl(x_t) \neq cl(y_r)$,then there exists $\lambda, \mu \in t^*$ such that $cl(x_t) \leq \lambda$, $cl(y_r) \leq \mu$ and $\lambda \wedge \mu = 0$.

3.9.3. Theorem : A fuzzy supra topological space (X, t^*) is fuzzy supra R_1 space if for every pair of fuzzy point x_t and y_r in X with fuzzy supra kernel (x_t) ≠ fuzzy supra kernel (y_r), there exists $\lambda, \mu \in t^*$ such that $cl(x_t) \leq \lambda$, $cl(y_r) \leq \mu$ and $\lambda \wedge \mu = 0$.

Proof: We know that Let (X, t^*) be a fuzzy supra topological spaces and x_t, y_r ∈ X. then fuzzy supra kernel (x_t) ≠ fuzzy supra kernel (y_r) if and only if $cl(x_t) \neq cl(y_r)$.(by theorem 2.6.4) Now the theorem is directly follows from the definition 3.9.2. of FSR_1 –space.

CHAPTER- IV

Fuzzy Supra T_0 topological spaces

4. Introduction:

Separation axioms are very much important in topological as well as in fuzzy topological spaces. Several topologists did a lot of works in this branch. Such as Lowen, R., and Wuyts, P., [36] in 1983 have studied and give definitions of separation axioms (T_0, T_1, T_2) in fuzzy topological spaces, fuzzy neighborhood and fuzzy uniform space. Ali, D.M., [7] introduced and made a comparative study on some separation axioms. Furthermore, Hutton, B., and Reilly, I., [30], Adnadjevic, D., [3], Ganguly and Saha[50], Sarker, M.,[52], Srivastava et.el[58]., Mashhour,A.S., and Ghanim, M.H [39], Malghan and Benchalli[19] and Rodabaugh, S.E., [48], studied these properties in different ways. In this chapter following them, we introduce and study fuzzy supra T_0 topological spaces.

This chapter contains three sections; first section is on different types of definitions, implications and non-implications among these definitions with some lemmas and examples, second section is on good extension property of supra T_0 topological spaces. The third sections are on subspace, heredity and productive properties and on homeomorphic property of fuzzy supra T_0 topological spaces. Throughout this chapter we use the symbol FST_0 space for Fuzzy Supra T_0 topological space.

4.1. Definitions of FST_0 Spaces:

4.1.1. Definitions: - Let (X, t^*) be a fuzzy supra topological space. Now (X, t^*) is a

(a) **FST_0 (i)** space $\Leftrightarrow \forall\ x, y \in X, x \neq y, \exists\ u \in t^*$ s.t. $u(x) = 0, u(y) = 1$ or $\exists\ v \in t^*$ s.t $v(x)=1, v(y)=0$.

(b) **FST_0(ii)** space $\Leftrightarrow \forall\ x, y \in X, x \neq y, \exists\ u \in t^*$ s. t. $u(x)= 0, u(y)>0$ or $\exists\ v \in t^*$ s. t $v(x)>0, v(y)=0$.

(c) **FST_0 (iii)** space $\Leftrightarrow \forall\ x, y \in X, x \neq y, \exists\ u \in t^*$ s. t. $u(x) < u(y)$ or $\exists\ v \in t^*$ s. t $v(y) < v(x)$.

(d) **FST_0 (iv)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, with $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t. u(x)=1,u(y)<α , or $\exists$ v$\in$ t^* s. t v(y)=1, v(x)<α.

(e) **FST_0 (v)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, with $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t. u(x)=0, u(y)= α, or $\exists$ v$\in$ t^* s. t v(y)=0, v(x)=α.

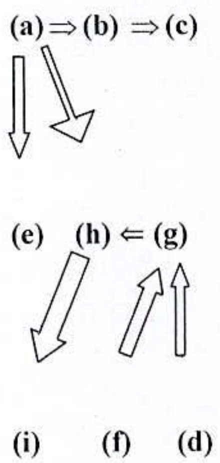
(f) **FST_0 (vi)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, with $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t. u(x)=0, u(y)>α, or $\exists$ v$\in$ t^* s. t v(y) =0, v(x)>α.

(g) **FST_0 (vii)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s. t. 0$\leq$ u(x)$\leq$ α<u(y) $\leq$ 1, or$\exists$ v $\in t^*$ s. t 0 $\leq$ v(y)$\leq$ α < v(x) $\leq$ 1

(h) **FST_0 (viii)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s. t. u (x) $\neq$u(y).

(i) **FST_0 (ix)** space $\Leftrightarrow$ $\forall$ pair of fuzzy point x_t, $y_r \in$X, x$\neq$y, cl(x_t)$\neq$cl(y_r).

4.1.1. Lemma: In view of above definitions, the following implications are true:



Proof: (a) $\Rightarrow$(b)

From definition (a) $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s.t. u(x) =0, u(y) =1 or$\exists$ v$\in$ t^* s.t v(x)=1, v(y)=0 ,then clearly u(x)=0, u(y)>0, or v(x)>0, v(y)=0, which is (b).

(b) $\Rightarrow$(c) is obvious , since from (b) $\forall$ x, y$\in$ X , x$\neq$y, $\exists$ u$\in$ t^* s. t. u(x)=0, u(y)>0 or$\exists$ v$\in$ t^* s.t v(x)>0, v(y)=0.So$\exists$ u$\in$ t^* s.t u(x) <u(y) or v(y) <v(x) , which is (c).

(a)$\Rightarrow$(e) is obvious for α=1

(a) $\Rightarrow$(h)

From (a), $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s.t. u(x) =0, u(y) =1 or $\exists$ v$\in$ t^* s.t v(x) =1, v(y) = 0, hence u(x) $\neq$ u(y).

(d)⇒(g)

Let (X, t^*) be a fuzzy supra topological spaces having properties FST_0(iv) space. We shall prove that (X, t^*) is FST_0(vii) space. Let x, y$\in$ X, x$\neq$y, since (X, t^*) is FST_0(iv) space, for $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t u(x)=1, u(y)<α, it follows that 0$\leq$ u(y) < α<u(x) $\leq$ 1, Hence it is clear that (X, t^*) is FST_0(vii).

(f)⇒(g)

Let (X, t^*) be a fuzzy supra topological spaces having properties FST_0(vi) space. We shall prove that (X, t^*) is FST_0(vii) space. Let x, y$\in$ X, x$\neq$y, since (X, t^*) is FST_0(vi) space, for $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t . u(x)=0, u(y)>α , it follows that 0$\leq$ u(x) < α<u(y) $\leq$ 1, Hence it is clear that (X, t^*) is FST_0(vii).

(g) ⇒(h)

Let (X, t^*) be a fuzzy supra topological spaces having properties FST_0(vii) space. We shall prove that (X, t^*) is FST_0(viii) space. Let x, y$\in$ X, x$\neq$y, since (X, t^*) is FST_0(vii), so for $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t. 0 $\leq$ u(y) < α<u(x) $\leq$ 1,Now we observe that u(x)$\neq$u(y). Hence (X, t^*) is FST_0(viii).

(h) ⇒ (i)

From (e), $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s. t. u(x) $\neq$u(y). Hence 1- u(x) $\neq$ 1- u(y) and since 1- u is closed put, 1- u(x) = x_t and 1- u(y) =y_r so$\exists$ u$\in$ t^* s.t. cl (x_t)$\neq$ cl (y_r). Hence (X, t^*) is FST_0(ix)

4.1. 2. Now we show the non-implications among FST_0(k), k={i, ii, iii, ……..ix} with some examples:

Example .1. Let X={x, y} and u, u_1, $u_2$$\in$ I^X, where u, u_1, u_2 are respectively defined by u(x) =0.0, u(y) =0.8; u_1(x) =0.4, u_1(y) =0.0; u_2(x) =0.4, u_2(y) =0.8; and t^*= {0, 1, u, u_1,u_2} then $t^*$is a fuzzy supra topology on X. Also (X, t^*) satisfy (b) but not (a). **[65]**

So, **(b)** $\neq$> **(a)**.

Example .2 Let X={x, y} and $u \in I^X$, where u defined by u(x) = 0, u(y) = 0.8. Consider the fuzzy supra topology t^* on X generated by and t^*= {0, 1, u}∪ {constants} for $\alpha \in I_1$ it is clear that (X, t^*) is FST_0 (v).space but (X, t^*) is not FST_0(i).

Example.3. Let X={x, y} and $u, u_1, u_2 \in I^X$, where u, u_1, u_2 are respectively defined by u(x)=0.5,u(y)=0.8; u_1(x)=0.6, u_1(y)=0.3; u_2(x)=0.6, u_2(y)=0.8; and t^*={0,1, u,u_1,u_2} then $t^*$is a fuzzy supra topology on X. Also since u(x) <u(y) 0r u_1(y) < u_1(x) so (X, t^*) satisfy (3) but not (b). So **(c)** ≠> (b).

Example .4. Let X={x, y} and u, u_1, $u_2 \in I^X$, where u, u_1, u_2 are respectively defined by u(x)= 0.5, u(y)= 0.8; u_1(x)=0.6, u_1(y)=0.5 ; u_2(x)=0.6, u_2(y)=0.8; and t^*={0,1, u, u_1, u_2} then t^* is a fuzzy supra topology on X. Also since u(x) <u(y) 0r u_1(y) < u_1(x) so (X, t^*) satisfy (3) but not (d), because for u(x) = u_1(y)≠ 1.

So **(c) ≠> (d).**

Example.5. Let X={x, y} and u, u_1, $u_2 \in I^X$, where u, u_1, u_2 are respectively defined by u(x) =0.6, u(y) = 0.6; u_1(x) = 0.3, u_1(y) = 0.4; and t^*= {0, 1, u, u_1} then $t^*$is a fuzzy supra topology on X. Also closed fuzzy sets of (X, t^*) are 1, 0, u^c, u_1^c Hence t^{*c}= {0, 1, {(x, .4), (y, .4)}, {(x, .7), (y, .6)}}

cl u(x) =.7, cl u(y) =.6; cl u(x) ≠ cl u(y), but u(x) = u(y),

Hence **(i)** ≠> **(h)** .

Example.6. .Let X={x, y} and u, u_1, $u_2 \in I^X$, where u, u_1, u_2 are respectively defined by u(x) = 0.5, u(y) =0.8; u_1(x) =0.6, u_1(y) =0.3; u_2(x) =0.6, u_2(y) =0.8; and t^*= {0, 1, u, u_1, u_2} then t^* is a fuzzy supra topology on X. We observe that u(x) ≠ u(y), so (X, t^*) satisfy (h) and clearly (X, t^*) does not satisfy (a).

Hence **(h)** ≠> **(a)** .

Example.7. .Let X={x, y} and $u \in I^X$, where u defined by u(x) = 0.6, u(y) =0.8. Consider the fuzzy supra topology t^* on X generated by and t^*= {0, 1, u}∪

{constants} for $\alpha = 0.7$ *it* is clear that (X, t^*) is FST_0(vii).space but (X, t^*) is not FST_0(iv).

Hence **(g)** $\neq>$ **(d)** .

4.1.3. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) FST_0 (i) space i.e. $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in t^*$ s. t. u(x) =0, u(y) =1 or$\exists$ v$\in t^*$ s.t v(x) =1, v(y) =0

(b) T_0: For every pair x, y$\in$X, x $\neq$ y, $\overline{1_y}(x)\wedge\overline{1_x}(y)=0$. **[36]**

(c) For every pair x, y$\in$X, x $\neq$y, $\overline{1_x}(y)=0, or \overline{1_y}(x)=0$

(a)$\Longrightarrow$(b)

Suppose (a) hold, and suppose x, y$\in$X, x $\neq$ y and $\overline{1_x}(y)=0$. This implies that there exists a, t^*-supra closed set k such that k(y) = 0 and k(x) = 1. Put u= 1–k. Then u is a t^*-supra open set such that u (x) = 0 and u (y) = 1. By (a) or there exists a t^*-supra open set v such that v (x) = 1 and v(y) = 0. Put m = 1 –v, then m is a t^*-supra closed set such that m(y) = 1 and m(x) = 0. Thus there may exist a t^*-supra closed set m such that m(y) = 1 and m(x) = 0, therefore $\overline{1_y}(x)=0$. So $\overline{1_y}(x)\wedge\overline{1_x}(y)=0$ .Hence **(a) $\Rightarrow$(b)**

(b) $\Rightarrow$(c) and **(c) $\Rightarrow$(a)** are straight forward.

4.1.4 Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) FST_0(ii) space, i.e $\forall$ x, y$\in$ X , x$\neq$y, $\exists$ u$\in t^*$ s.t. u(x)=0, u(y)>0 0r $\exists$ v$\in t^*$ s.t v(x)>0, v(y)=0.

(b) WT_0: For every pair x, y$\in$X, x $\neq$ y, $\overline{1_x}(y)\wedge\overline{1_y}(x)<1$. **[36]**

(a)$\Longrightarrow$(b)

From (a) we have $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in t^*$ s.t. u(x) = 0, u(y)>0 0r $\exists$ v$\in t^*$ s.t v(x)>0, v (y) =0. Then 1-u(x) =1 and 1-u(y) <1; or 1-v(x) <1, 1-v(y) =1; since 1-u is supra closed, so we see that $\overline{1_x}(y)<1, or\ \overline{1_y}(x)<1$. Hence $\overline{1_x}(y)\wedge\overline{1_y}(x)<1$.

(b)$\Rightarrow$(a)

Let $\overline{1_x}(y) \wedge \overline{1_y}(x) < 1. \Rightarrow$ either $\overline{1_x}(y) < 1, or\ \overline{1_y}(x) < 1. \Rightarrow$ 1- $\overline{1_x}(y) > 0$ or1- $\overline{1_y}(x) > 0$ .Let 1- $\overline{1_x}$ =u, then u$\in$ t^* and u(y)>0 also u(x) =0. Similarly from 1- $\overline{1_y}(x) > 0$, we can show that v(x)>0, and v(y) =0.■

4.1.5 Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) For every pair x, y$\in$X, x $\neq$ y, $\left(\forall(\alpha,\beta) \in I_0 \times I_0, \overline{\alpha 1_x}(y) = \alpha, or \overline{\beta 1_y}(x) = \beta\right)$

(b) T_0 : For every pair x, y$\in$X, x $\neq$ y, $\forall$ (α,β) $\in I_0 \times I_0$, $\overline{\alpha 1_x}(y) < \alpha, or$ $\overline{\beta 1_y}(x) < \beta$. [36]

(c) FST_0 (iii) space if $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s.t. u(x) <u(y) or$\exists$ v$\in$ t^* s.t v(y)< v(x). .

Proof:

(a)⟹(b)

Suppose for x, y$\in$ X, x $\neq$ y, and there exists α $\in I_0$ such that $\overline{\alpha 1_x}(y) < \alpha$. ....(1) Again let for every β $\in I_0$, $\overline{\beta 1_y}(x) = \beta$.Then by (a) for every α $\in I_0$, $\overline{\alpha 1_x}(y) = \alpha$, which contradicts (1). Therefore there exist β $\in I_0$, such that $\overline{\beta 1_y}(x) < \beta$

(b) ⟹(c)

Suppose for every x, y$\in$ X, x $\neq$ y, there exists a t^*-supra open set v such that v (y) < v (x). Let $\beta = v(y)$, then clearly $\overline{\beta 1_y}(x) < \beta$. Hence by (b), there exists α $\in I_0$ such that $\overline{\alpha_0 1_x}(y) < \alpha_0$, this implies that there exists t^*-supra closed set say w such that w(y)≤ α_0 <w(x). So, w(y) < w(x) . Let u=1-w. Then u is a t^*-supra open set and u(x) <u(y). Similarly we can show that $\exists$ v$\in$ t^* s. t v(y) <v(x).Which is (c)

(a)⇒(c)

Let For every pair x, y$\in$X, x $\neq$ y, $\overline{\alpha 1_x}(y) = \alpha$, there exists t^*-supra closed set say w such that w= $\overline{\alpha 1_x}$ so w(y) =α, w(x) =0, Hence w(y) < w(x). Let u=1-w. Then u is a t^*-supra open set and u(x) <u(y).Similarly we can show that $\exists$ v$\in$ t^* s.t v(y) < v(x). Which is (c)

4.1.6. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) FST_0 (iv) space, i.e. $\forall$ x, y$\in$ X, x$\neq$y, with $\alpha \in I_0$, $\exists$ u$\in$ t^* s. t. u(x)=1, u(y)<α ,or$\exists$ v$\in$ t^* s. t v(y)=1, v(x)<α.

(b) T_0 : For every pair x, y$\in$X, x $\neq$ y and $\forall\alpha \in I_0$, $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) < \alpha$. **[36]**

Proof:

(a)$\Longrightarrow$(b)

From (a) we have $\forall$ x, y$\in$ X, x$\neq$y, and $\alpha \in I_0$, $\exists$ u$\in$ t^* s.t. u(x)=1, u(y)<α, or$\exists$ v$\in$ t^* s. t v(y)=1, v(x)<α.Then 1-u(x) =0, and 1-u(y) >1-α ; or 1-v(x)>1-α, and 1-v(y) =0; Since 1-u(x) is closed , so we see that $\overline{\alpha 1_x}(y) < \alpha$ or $\overline{\alpha 1_y}(x) < \alpha$ thus we see that $\forall\alpha \in I_0$, $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) < \alpha$.

(b)$\Longrightarrow$(a)

Suppose, for every pair x, y$\in$X, x $\neq$ y and $\forall\alpha \in I_0$, $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) < \alpha$ this implies that $\overline{\alpha 1_x}(y) < \alpha$ or$\overline{\alpha 1_y}(x) < \alpha$ put m = $\overline{\alpha 1_x}$=$\alpha \wedge \overline{1_x}$ is t^* supra closed set such that m(y)>1 − α, m(x) = 0. Taking u=1-m, then u(x) =1, and u(y) <α . Similarly we have v(y) = 1, v(x) <α .

4.1.7. Lemma: For any fuzzy supra topological space (X, t^*), the following T_0-properties are equivalent:

(a) FST_0 (ix) space i. e $\forall$ pair of fuzzy point x_t, y_r $\in$X, x$\neq$y, $\exists$ u$\in$ t^* s.t. cl(x_t)$\neq$cl(y_r).

(b) $\forall$ pair of fuzzy point x_t, y_r $\in$X, x$\neq$y, $\exists$ u$\in$ t^* s.t. x_t qu $\leq$ 1-y_r, or y_r qu$\leq$1-x_t.

(a)$\Rightarrow$(b)

Proof: Let x_t , y_r be pair of fuzzy point in X, with x$\neq$y, and cl(x_t) $\neq$cl(y_r).$\exists$ a fuzzy point z_p in X be such that $z_p\leq$ cl(x_t) and $z_p\nleq$ cl(y_r). We claim that x_t $\nleq$ cl(y_r). For if x_t $\leq$ cl(y_r). then cl(x_t) $\nleq$ cl(y_r). This contradicts the fact that $z_p\nleq$ cl (y_r). Hence x_t $\nleq$ cl (y_r) . So x_tq (1- cl(y_r)). and Since u=1- cl(y_r)$\in t^*$ also 1-$y_r\leq$1- cl(y_r)=u, therefore x_t qu $\leq$ 1-y_r, So clear it is that $\forall$ pair of fuzzy point x_t, y_r $\in$X, x$\neq$y, $\exists$ u$\in$ t^* s.t. x_t qu $\leq$ 1-y_r, or $y_r$qu $\leq$1-x_t.

(b)$\Rightarrow$ (a)

Suppose ∀ pair of fuzzy point x_t, $y_r \in X$, $x \neq y$, $\exists$ $u \in t^*$ s.t. x_t qu $\leq 1-y_r$, or $y_r$qu$\leq 1-x_t$. If x_t qu $\leq 1-y_r$, then $x_t \nleq 1$-u and u$\leq 1-y_r$. Since 1-u is fuzzy supra closed, and $cl(y_r)$ is the smallest closed set containing y_r then $cl(y_r) \leq 1$-u , and since $x_t \nleq 1$-u and $x_t \leq cl(x_t)$ this implies that $cl(x_t) \neq cl(y_r)$. ■

4.1.8. Lemma: For any fuzzy supra topological space (X, t^*), the following T_0-properties are equivalent:

(a) T_0 : For every pair x, y∈X, $x \neq y \Rightarrow \overline{1_y}(x) \wedge \overline{1_x}(y) = 0$.

(b) T_0' : For every pair x, y∈X, $x \neq y$ and $\forall(\alpha,\beta) \in I_0 \times I_0$,

$$\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) < \alpha \wedge \beta$$

(c) T_0'' : For every pair x, y∈X, $x \neq y$,

$$\left(\forall(\alpha,\beta) \in I_0 \times I_0, \overline{\alpha 1_x}(y) < \alpha, or \overline{\beta 1_y}(x) < \beta\right)$$

(d) T_0''' : For every pair x, y∈X, $x \neq y$ and $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) < \alpha$.

(e) WT_0: For every pair x, y∈X, $x \neq y$, $\overline{1_x}(y) \wedge \overline{1_y}(x) < 1$.

Proof : **(a) ⇒ (b)**

Let (X, t^*) satisfies (a), we have to shows that it satisfies (b). Now $\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) \leq$ $\overline{1_y}(x) \wedge \overline{1_x}(y) = 0 < \alpha \wedge \beta$. Thus we see that (a) ⇒ (b).

(b) ⇒ (c)

Let (X, t^*) satisfies (b), we have to shows that it satisfies (c). Since (X, t^*) satisfies $\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) < \alpha \wedge \beta$, if $\overline{\alpha 1_x}(y) = \alpha$. Then $\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) < \alpha \wedge \beta \Rightarrow \alpha \wedge$ $\overline{\beta 1_y}(x) < \alpha \wedge \beta \Rightarrow \overline{\beta 1_y}(x) < \beta$ .Thus we see that (b) ⇒ (c)

(c) ⇒ (d)

Let (X, t^*) is T_0'' , then $\forall(\alpha,\beta) \in I_0 \times I_0$, $\overline{\alpha 1_x}(y) < \alpha$ or $\overline{\beta 1_y}(x) < \beta$. In particular, we have $\overline{\alpha 1_x}(y) < \alpha$ or $\overline{\alpha 1_y}(x) < \alpha$. Therefore $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) < \alpha$, hence (X, t^*) is T_0''' .

(d)⇒ (e)

Let (X, t^*) is T_0''' , then $\forall(\alpha,\beta)\in I_0\times I_0$, $\overline{\alpha 1_x}(y)\wedge\overline{\alpha 1_y}(x)<\alpha$. Taking α =1, then $\overline{1_x}(y)\wedge\overline{1_y}(x)<1$. Thus we see that $T_0''' \Rightarrow WT_0$.

(e) ⇒(a)

It is clear from Lemma 4.1.3 and 4.1.4.

Now we give some examples,

$T_0 \not\Rightarrow T_0$

Example: We consider a fuzzy supra topological space (X, t^*) where X={x, y} and t^*= {o, u, v, 1} ∪ {constants}; u(x)= 0= v(y) , u(y) = 0.8 =v(x) Let α =0.6 , β = 0.7, then $\overline{\alpha 1_x}(y)=0.2$ and $\overline{\beta 1_y}(x)=0.2$ thus $\overline{\alpha 1_x}(y)\wedge\overline{\beta 1_y}(x)=0.2<\alpha\wedge\beta$ ∴ (X, t^*) is T_0

But (X, t^*) is not T_0; since $\overline{1_y}(x)\wedge\overline{1_x}(y)=0.2\neq 0$.

Example: $T_0 \not\Rightarrow T_0'$

Consider a fuzzy supra topological space (X, t^*) where X={x, y} and t^*= {o, u, 1} ∪ {constants}; u(x)=0, u(y)=0.5= Let α =0.6 , then $\overline{\alpha 1_x}(y)=0.5<0.6$ ∴ (X, t^*) is T_0' again let β = 0.8, then $\overline{\beta 1_y}(x)=$ constant fuzzy set with value β thus $\overline{\alpha 1_x}(y)\wedge\overline{\beta 1_y}(x)=0.6\wedge 0.8=0.6\not<0.6\wedge 0.8=0.6$ this implies that (X, t^*) is not T_0 .

Example: $T_0''' \not\Rightarrow T_0''$

Consider a fuzzy supra topological space (X, t^*) where X={x, y} and t^* = {o, u, v, 1} ∪ {constants}; u(x)= 0= v(y) , u(y) = 0.5, v(x) =0.6. Let α =0.5, then $\overline{\alpha 1_x}(y)=0.5$ and $\overline{\alpha 1_y}(x)=0.4$, thus $\overline{\alpha 1_x}(y)\wedge\overline{\alpha 1_y}(x)=0.4<0.5$. This implies that (X, t^*) is T_0''' Now let β = 0.4. Then $\overline{\alpha 1_x}(y)=0.5\not<\alpha$ and $\overline{\beta 1_y}(x)=0.4\not<\beta$. This implies that (X, t^*) is not T_0'' .

Example: $WT_0 \not\Rightarrow T_0'''$

Consider a fuzzy supra topological space (X, t^*) where X={x, y} and t^* = {o, u, v, 1} ∪ {constants}; u(x)= 0= v(y) , u(y) = 0.5, v(x) =0.5. Now u' (x) = 1, u'(y) =0.5, v'(y) =1, v'(x) =0.5; Thus $\overline{1_x}=u'$, $\overline{1_y}=v'$Let α =0.5, then we have $\overline{1_y}(x)\wedge\overline{1_x}(y)$= 0.5 ∧0.5=0.5 <1 but $\overline{\alpha 1_x}(y)\wedge\overline{\alpha 1_y}(x)$= 0.5 ∧0.5=0.5≮0.5. Hence $WT_0 \not\Rightarrow T_0'''$

4.1.9. Lemma: Let (X, t^*) and (X, t_1^*) be two fuzzy supra topological spaces, where t^*, and t_1^* be two fuzzy supra topology on X. Let t^* is finer than t_1^*, then we show that if (X, t_1^*) is fuzzy supra T_0 space then (X, t^*) is also fuzzy supra T_0 space.
Proof: Let x, y be two distinct points on X. Since t_1^* is fuzzy supra T_0 space, then $\exists$ u $\in t_1^*$ s.t. u(x) =0, u(y) =1 or v$\in$ t^* s. t v(x) = 1, v(y) =0. Also since $t_1^* \subset t^*$ so for t^*, u(x) =0, u(y) = 1 or v$\in$ t^* s. t v(x) =1, v(y) =0 is hold. Hence t^* is also a fuzzy supra T_0 space.

4.2 .Good extension properties:

In this section we show that all $FST_0(k)$ (i ≤ k ≤ ix) properties are good extensions of their supra topological counter parts;.

4.2.1. Theorem:

(a) If (X, T^*) is an ST_0-space, then (X, ω (T^*)) satisfies $FST_0(k)$ (i ≤ k ≤ ix).
(b) If (X, ω (T^*)) satisfies $FST_0(K)$ (i ≤ k ≤ ix) then (X, T^*) is an ST_0-space.

Proof: (a) Let(X, t^*) is a supra T_0-topological space. We shall prove that $(X, \omega(t^*))$ is a fuzzy supra T_0(i) space. Let x, y$\in$X, with x$\neq$y, since(X, t^*) is a supra T_0, $\exists$ U, V$\in$ t^* such that x$\in$ U y$\notin$ U, or x$\notin$ V, y$\in$U but from definition of lower semi continuous, we have $1_U \in \omega(t^*)$ and $1_U(x)$ =1, $1_U(y)$ =0.or $1_V(y) = 0$, $1_V(x) = 1$. Hence we see that $(X, \omega(t^*))$ is a fuzzy supra T_0(i) space.

Conversely suppose that $(X, \omega(t^*))$ is a fuzzy supra T_0(i) space we will prove that (X, t*) is a supra T_0-topological space, since $(X, \omega(t^*))$ is a fuzzy supra T_0(i) space , so $\exists$ u$\in$ t^* s.t. u(x) =0, u(y)=1 or$\exists$ v$\in$ t^* s.t v(x)=1, v(y)=0. Suppose u$\in$ t^* s.t. u(x) =0, u(y) =1, x$\notin u^{-1}(\alpha, 1]$ and y$\in u^{-1}(\alpha, 1]$, and by the definition of lsc. $u^{-1}(\alpha, 1] \in t^*$.
Hence (X, t^*) is a supra T_0-topological space.

Proof: (b) Let (X, t^*) is a supra T_0-topological space. We shall prove that $(X, \omega(t^*))$ is a fuzzy supra T_0(ix) space. Let x, y$\in$X, with x$\neq$y, since(X, t^*) is a supra T_0, $\exists$ U$\in$ t^* such that x$\in$ U y$\notin$ U, but from definition of lower semi-continuous, we have $1_U \in \omega(t^*)$ and $1_U(x)$ =1, $1_U(y)$ =0.So $\exists$ fuzzy point x_t, y_r $\in$X, such that $x_t \in u^{-1}(\alpha, 1]$, $y_r \notin$

$u^{-1}(\alpha, 1]$. Since $cl(x_t)$ is the smallest closed set containing x_t, so $cl(x_t) \subseteq u^{-1}(\alpha, 1]$, and hence $cl(y_r) \not\subseteq u^{-1}(\alpha, 1]$, so we conclude that $(X, \omega(t^*))$ is a fuzzy supra T_0(ix) space. Converse is similar as above.

All other proofs are similar so omitted.

Thus it is seen that $FST_0(k)$ $(i \leq k \leq ix)$ is a good extension of its supra topological counter part.

4.2.2. Theorem: Let (X, t^*) be a fuzzy supra topological space and $I_\alpha(t^*) = \{u^{-1}(\alpha, 1]: u \in t^*, \alpha \in I_1\}$ then (X, t^*) is $FST_0(k) \Rightarrow (X, I_\alpha(t^*))$ is supra T_0, $k \in \{i, ii, iii, iv, v, vi, vii, viii\}$

Proof: (a) Let (X, t^*) is FST_0(i)we shall prove that $(X, I_\alpha(t^*))$ is supra T_0. Let $x, y \in X$; $x \neq y$, since (X, t^*) is FST_0 (i) space then $\exists$ $u \in t^*$ s.t. $u(x) = 1$, $u(y) = 0$ or $\exists$ $v \in t^*$ s. t $v(y) = 1$, $v(x) = 0$ .Let $u \in t^*$ s.t. . $u(x) = 1$, $u(y) = 0$, since $u^{-1}(\alpha, 1] \in I_\alpha(t^*)$ and $x \in u^{-1}(\alpha, 1]$ and $y \notin u^{-1}(\alpha, 1]$. Hence we have $(X, I(t^*))$ is supra T_0.(i)

Proof (b) Let (X, t^*) is FST_0(iv)we shall prove that $(X, I_\alpha(t^*))$ is supra T_0. Let $x, y \in X$; $x \neq y$, since (X, t^*) is FST_0 (i) space then $\exists$ $u \in t^*$ s.t. $u(x) = 1$, $u(y) < \alpha$ or $\exists$ $v \in t^*$ s. t $v(y) = 1$, $v(x) < \alpha$. Let $u \in t^*$ s.t. $u(x) = 1$, $u(y) < \alpha$, since $u^{-1}(\alpha, 1] \in I_\alpha(t^*)$ and $x \in u^{-1}(\alpha, 1]$ and $y \notin u^{-1}(\alpha, 1]$. Hence we have $(X, I(t^*))$ is supra T_0. Similarly one can prove the other cases.

4.3. Heredity, productive and homeomorphic properties of FST_0 Spaces.

4.3.1. Definition: The initial fuzzy topology on X for the family of topological spaces $(X_j, t_j)_j \in J$ and the family of function $f_j : X \rightarrow (X_j, t_j)_j \in J$,is the smallest fuzzy topology on x, making each function f_j fuzzy continuous.

4.3.2. Definition: If G is any fuzzy set in a set X and $0 \leq \alpha < 1$ $(0 < \alpha \leq 1)$ then $\alpha(G) = \{x \in X : G(x) > \alpha\}$ is called an α-level set in X. **[48]**

4.3.1. Theorem: The properties $FST_0(k)$, $k \in \{i, ii, iii, iv, v, vi, vii\}$ are initial, i,e if $(f_j : X \rightarrow (X_j, t^*_j)_j \in J$, is a source in fsts. where all $(X_j, t^*_j)_j \in_J$ are $FST_0(k)$ and α-level,where $\alpha \in I_1$ then the initial topology is also $FST_0(k)$.

Proof: Since $\alpha({}_it^*)$ is finer than $\alpha(t^*_j)$ for all $\alpha \in I_1$ and the spaces $(X_j, t^*_j)_{j\in J}$ are $FST_0(k)$, k∈{i, ii. ......vii}; so clearly the initial fuzzy supra topological space (X, t^*) is also $FST_0(k)$, k∈{i, ii. ......vii}.

4.3.2. Theorem: The properties $FST_0(k)$, k∈{i, ii, iii, iv, v, vi, vii} are initial, i,e if $(f_j : X\rightarrow (X_j, t^*_j))_{j\in J}$, is a source in fsts. where all $(X_j, t^*_j)_{j\in J}$ are $FST_0(k)$ then the initial topology is also $FST_0(k)$.

(a) Let $\{(X_i, t^*_i)_{i\in J}\}$ be a family of FST_0(iii), and $\{f : X \rightarrow (X_i, t^*_i)_{i\in J}\}$ be a family of functions and t^* be the initial fuzzy supra topology on X induced by the family $\{f_i : i\in J\}$. Let x, y∈ X, x≠y and $\exists\ \lambda\in t^*$ such that $\lambda(y) < \lambda(x)$. We can find basic t^*- supra open sets λ_i, i∈J such that $\lambda = \text{Sup}\{\lambda_i, i\in J\}$. Also λ_i must be expressible as $\lambda_i = \text{Inf}\{f^{-1}_{ik}(\lambda_{ik}) : 1\leq k\leq n\}$ where $\lambda_{ik} \in t^*_{ik}$ and ik∈J. Now we can find some k, $(1\leq k\leq n)$, sayk_1 such that $f^{-1}_{ik_1}(\lambda_{ik_1})(y) < f^{-1}_{ik_1}(\lambda_{ik_1})(x) \Rightarrow \lambda_{ik_1} f_{ik_1}(y) < \lambda_{ik_1} f_{ik_1}(x)$. Since $(X_{ik_1} t^*_{ik_1})$ is FST_0(iii),or there exists $V_{ik_1} \in t^*_{ik_1}$ such that $V_{ik_1} f_{ik_1}(x) < V_{ik_1} f_{ik_1}(y)$ $\Rightarrow f^{-1}_{ik_1}(V_{ik_1})(x) < f^{-1}_{ik_1}(V_{ik_1})(y)$. Put $V = f^{-1}_{ik_1}(V_{ik_1}) \in t^*$.Thus $v(x) < v(y)$. Hence (X, t^*) is FST_0(iii)

Similarly we can proof for the other.

4.3.3. Theorem: All the properties $FST_0(k)$ of subspace topology where $(i\leq k \leq ix)$ are hereditary.

Proof: Consider the fsts (X, t^*), Let A⊂X. $t^*_A = \{u\wedge A: u \in t^*\}$, We have to show that, the subspace (A, t^*_A) has $FST_0(k)$ $(i\leq k \leq ix)$ if (X, t^*) has $FST_0(k)$ $(i\leq k \leq ix)$.

(i) Let x, y ∈ A, x ≠y, so that x, y ∈ X as A⊂X, since (X, t^*) is FST_0(i), so, ∃ u∈ t^* s.t. u(x) =0, u(y) =1, Again from definition of subspace we have $u\wedge A \in t^*_A$ and $(u\wedge A)(x) = 0$, $(u\wedge A)(y)=1$ as x, y∈ A. This implies that, (A, t^*_A) has FST_0(i).

(ii) Let x, y ∈ A, x ≠y, so that x, y ∈ X as A⊂X, since (X, t^*) is FST_0(iv), so, ∃ u ∈ t^* with $\alpha \in I_1$, s. t. u(x) =1, $u(y)<\alpha$, Again from definition of subspace we have u∧

$A \in t_A^*$ and $(u \wedge A)(x) = 1$, $(u \wedge A)(y) < \alpha$ as x, y$\in$ A. This implies that, (A, t_A^*) has FST_0(iv).

(iii) Let x, y $\in$ A, x $\neq$y, so that x, y $\in$ X as A$\subset$X since (X, t^*) is FST_0(ix) $\exists$ u$\in$ t^* s. t. cl u(x) $\neq$ cl u(y). We have, t^*-cl(1_x) $\cap 1_A$= t_A^*-cl(1_x). Since (X, t^*) is FST_0(ix) so, $\exists$ u$\in$ t^*, so t^*-cl($1_{u(x)}$)$\neq$ t^*-cl($1_{u(y)}$) . Hence t^*-cl($1_{u(x)}$)$\cap 1_A \neq t^*$-cl($1_{u(y)}$)$\cap 1_A$. This implies that, (A, t_A^*) has FST_0(ix).

All other proofs are similar and straight forward.

4.3.4. Theorem: The properties FST_0(k), k $\in$ {i, ii. ……vii, viii}.are productive, i.e. if $(X_i, t_i^*)_{i\in J}$ is a family of fuzzy supra topological spaces, each of them having the property FST_0(k), if and only if the product space $\left(X = \prod_{i\in J} X_i, t^*\right)$ also has FST_0(k).

Proof: Let $(X_i, t_i^*)_{i\in J}$ is fuzzy supra T_0(iv)-topological space. We shall prove that (X, t^*) is a fuzzy supra T_0(i) space. Let x, y$\in$X, with x$\neq$y, then $x_i \neq y_i$ for some i$\in$J, since (X_i, t_i^*) is FST_0(iv), then $\alpha \in I_1$, $\exists$ $u_i\in t_i^*$ s.t $u_i(x_i) = 1$, $u_i(y_i) < \alpha$ .But we have $\pi_i(x) = x_i$, and $\pi_i(y) = y_i$. Then $u_i(\pi_i(x)) = 1$ and $u_i(\pi_i(y)) < \alpha$, i.e $(u_i \circ \pi_i)(x) = 1$and $(u_i \circ \pi_i)(y) < \alpha$, It follows that $\exists$ $(u_i \circ \pi_i) \in t^*$ such that $(u_i \circ \pi_i)(x) = 1$ and $(u_i \circ \pi_i)(y) < \alpha$. Hence it follows that (X, t^*) is FST_0(iv).

Conversely suppose that (X, t^*) is FST_0(iv), we shall prove that $(X_i, t_i^*)_{i\in J}$ is FST_0(iv) space. Let a_i be a fixed element in X_i, suppose that A_i= {x$\in$X $=\prod_{i\in\Delta} X_i$ / $x_j = a_j$ for some i$\neq$j}. So that A_i is a subset of X, and hence (A_i, $t_{A_i}^*$) is also a subspace of (X, t*) , since(X, t*) is FST_0(iv), then (A_i , $t_{A_i}^*$) is also FST_0(iv). Now we have A_i is a homeomorphic image of X_i. Hence it is clear that $(X_i, t_i^*)_{i\in J}$ is also FST_0(iv).

Similarly we can prove for k $\in$ {i, ii, iii, v, vi, vii, viii}.

4.3.5. Theorem: Let (X, t_1^*) and (Y, t_2^*) be two fuzzy supra topological spaces, and f:(X, t_1^*)$\rightarrow$(Y, t_2^*) be a one-one , onto and supra open map then , (X, t_1^*) is FST_0(k) $\Rightarrow$ (Y, t_2^*) is FST_0(k), k $\in$ {i, ii, iii, v, vi, vii, viii, ix}.

Proof: Suppose (X, t_1^*) be FST_0(iv), we shall prove that (Y, t_2^*) is FST_0(iv). Let y_1,

$y_2 \in$Y, be an arbitrary point s .t $y_1 \neq y_2$. Since f is onto ⇒ ∃x_1, x_2 ∈X. with f(x_1) = y_1 and f(x_2) = y_2 where $x_1 \neq x_2$ as f-one-one, Again since (X, t_1^*) is FST_0(iv) so for $\alpha \in I_1$ ∃ u∈t^* s.t. u(x_1) =1, u(x_2) <α.

Now f(u)(y_1) ={Sup u(x_1): f(x_1) = y_1 }=1

f(u)(y_2) ={Sup u(x_2): f(x_2) = y_2 } <α

Since f is supra open then f(u)∈ t_2^* as u∈ t_1^* . We observe that ∃ f(u)∈ t_2^* such that f(u)(y_1) =1, and f(u)(y_2) < α. Hence it is clear that(Y, t_2^*) is FST_0(iv). Similarly we can show for others.■

4.3.6. Theorem: Every homeomorphic image of FST_0(k) is also an FST_0(k), (i≤k≤viii).

Proof:. Let f:(X, t_1^*)→(Y, t_2^*) be a homeomorphism between fsts, where (X, t_1^*) has FST_0(i), , Let y_1, $y_2 \in$Y, be an arbitrary point s.t $y_1 \neq y_2$. Since f is one-one onto ⇒ ∃x_1, x_2 ∈X. s .t $f^{-1}(y_1)$ = x_1 and $f^{-1}(y_2) = x_2$.Then $x_1 \neq x_2$.Again since (X, t_1^*) is FST_0(i) so ∃ u∈ t^* s.t. u(x_1) =0, u(x_2) =1. Now f(u)(y_1)= u ($f^{-1}(y_1)$)=. u (x_1) =0 and f(u)(y_2) = u ($f^{-1}(y_2)$)= u(x_2) =1. Since u∈ $t^*$and f is continuous, so f(u) ∈ t_2^* Hence it is clear that that (Y, t_2^*) is an FST_0(i) All other proof FST_0(ii) - FST_0(viii) are similar. Not ascertain about FST_0(ix) .

CHAPTER- V

Fuzzy Supra T_1- topological spaces

5. Introduction:

In this chapter we introduced and studied some definitions of fuzzy supra T_1topological spaces. This chapter also contains two sections ; first section is on different types of definitions, implications and non-implications among these definitions with some lemmas and counter- examples, second section is on good extension property, heredity and productive properties, and on homeomorphic property of fuzzy supra T_1 topological spaces. Throughout this chapter we use FST_1 to indicate fuzzy supra T_1 topological spaces.

5.1. Definitions of FST_1 Spaces

5.1.1. Definitions: Let (X, t^*) be fuzzy supra topological space, T_1-properties of (X, t^*) as follows:

(a) **FST_1 (i)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s.t. u(x) =0, u(y) = 1 and $\exists$ v$\in$ t^* s.t v(x) =1, v(y) =0.

(b) **FST_1 (ii)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s. t. u(x) = 0, u(y)>0 and $\exists$ v$\in$ t^* s. t v(x)>0, v(y) =0.

(c) **FST_1 (iii)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s. t. u(x) <u(y) and $\exists$ v$\in$ t^* s. t v(y) <v(x).

(d) **FST_1 (iv)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, with $\alpha \in I_1$, $\exists$ u$\in$ t^* s. t. u(x)=1, u(y)<α,and $\exists$ v$\in$ t^* s. t v(y) =1, v(x) <α.

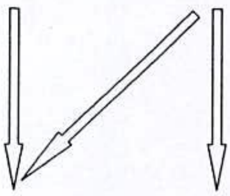
(e) **FST_1 (v)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, with $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t. u(x)=0, u(y)=α, and $\exists$ v$\in$ t^* s. t. v(y)= 0, v(x)=α.

(f) **FST_1 (vi)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, with $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t. u(x)= 0, u(y)>α, and $\exists$ v$\in$ t^* s. t. v(y)=0, v(x)>α.

(g) **FST_1 (vii)** space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s. t. $0\leq u(x)\leq \alpha<u(y)\leq 1$, and $\exists$ v$\in$ t^* s. t $0\leq v(y)\leq \alpha < v(x)\leq 1$

5.1.1Theorem: The following implications are true:

(a) $\Rightarrow$(b) $\Rightarrow$(c)



(e) (d)$\Rightarrow$ (g)$\Leftarrow$ (f)

From (a) $\Rightarrow$(b) $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s.t. u(x) = 0, u(y) =1 and $\exists$ v$\in$ t^* s.t v(x)=1, v(y)=0 ,then clearly u(x)= 0, u(y)>0, or v(x)>0, v(y)=0, which is (b).

(b) $\Rightarrow$(c) is obvious , since from (b) $\forall$ x, y$\in$ X , x$\neq$y, $\exists$ u$\in$ t^* s. t. u(x)= 0, u(y)>0 and $\exists$ v$\in$ t^* s.t v(x)>0, v(y)= 0.So$\exists$ u$\in$ t^* s.t u(x) <u(y) or v(y) <v(x) , which is (c).

(c)$\Rightarrow$(e) and **(c)$\Rightarrow$(f)** are straight forward, **(a)$\Rightarrow$(e)** is obvious for α=1

(d)$\Rightarrow$(g)

Let (X, t^*) be a fuzzy supra topological spaces having properties FST_1(iv) space. We shall prove that (X, t^*) is FST_1(vii) space. Let x, y$\in$ X, x$\neq$y, since (X, t^*) is FST_1(iv) space, for $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t u(x)=1, u(y)<α, and v$\in$ t^* s.t v(y)=1, v(x)<α, it follows that $0\leq u(y) < \alpha<u(x)\leq 1$ and $0\leq v(x) < \alpha<v(y)\leq 1$ Hence it is clear that (X, t^*) is FST_1(vii).

(f)$\Rightarrow$(g)

Let (X, t^*) be a fuzzy supra topological spaces having properties FST_1(vi) space. We shall prove that (X, t^*) is FST_1(vii) space. Let x, y$\in$ X, x$\neq$y, since (X, t^*) is FST_1(vi) space, for $\alpha \in I_1$, $\exists$ u$\in$ t^* s.t . u(x) =0, u(y)>α and $\exists$ v$\in$ t^* s. t v(y)=0, v(x)=α. it follows that $0\leq u(x) < \alpha<u(y)\leq 1$,and $0\leq v(y) < \alpha<v(x)\leq 1$. Hence it is clear that (X,t^*) is FST_1(vii).

We show the non-implications among FST_1(k), k$\in$ {i-vii} with some examples:

Example.1: FST_1(ii) $\not\Rightarrow$ FST_1(i).

Let X={x, y, z} and t^*= {1, 0, u= {(x, .75), (y, .5), (z, 0)}, w = {(x, .75), (y, .5), (z, 1)},

$v = \{(x, 0), (y, .5), (z, 1)\}\}$ on X. Here $u(x)>0$, $u(z)=0$, and also $v(x)=0$, $v(z)>0$; So it is clear that(X, t^*) is $FST_1(ii)$ but there does not exist $u, v \in t^*$ such that $v(x)=0$, $v(z)=1$ and $u(x)=1$, $u(z)=0$. So $FST_1(ii) \nRightarrow FST_1(i)$.

Example.2: $FST_1(iii) \nRightarrow FST_1(i)$.

Let $X=\{x, y, z\}$ and $t^*= \{1, 0, u= \{(x, .75), (y, .5), (z, 0)\}, w= \{(x, .75), (y, .5), (z, 1)\}, v= \{(x, 0), (y, .5), (z, 1)\}\}$ on X. Here $u(x)>0$, $u(z)=0$, so $u(z)<u(x)$ and also $v(x)< v(z)$; So it is clear that(X, t^*) is $FST_1(iii)$ but there does not exist $u, v \in t^*$ when $v(x)=0$, $v(z)=1$ then $u(x)=1$, $u(z)=0$. So $FST_1(iii) \nRightarrow FST_1(i)$.

Example.4: $FST_1(iv) \nRightarrow FST_1(v)$.

Let $X=\{x, y\}$ and $u, v \in t^*$ are defined by $u(x) =1$, $u(y) =0.4$ and $v(x) = 0.4$, $v(y) =1$; consider the fuzzy supra topology t^* on X generated by $\{0, u, v, 1\} \cup \{constants\}$; for $\alpha=0.5$, we see that (X, t^*) is $FST_1(iv)$ but (X, t^*) is not $FST_1(v)$. So $FST_1(iv) \nRightarrow FST_1(v)$.

Example.5: $FST_1(vii) \nRightarrow FST_1(vi)$.

Consider a fuzzy supra topological space (X, t^*) where $X=\{x, y\}$ and let $t^*= \{o, u, v, 1\} \cup \{constants\}$; where $u(x)= 0.6$, $u(y) = 0.8$; $v(x) = 0.8$, $v(y)=0.6$. Let $\alpha = 0.7$, it is clear $\exists\ u\in t^*$ s. t. $0\leq u(x)\leq \alpha<u(y) \leq 1$, and $\exists\ v\in t^*$ s. t $0 \leq v(y)\leq \alpha < v(x) \leq 1$; so (X, t^*) is FST_1 (vii) space. But (X, t^*) is not $FST_1(vi)$. Since $u(x) = v(y)\neq 0$; also (X, t^*) is not $FST_1(vi)$ because $u(x) =v(y)\neq 1$.

5.1.2 Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) $FST_1(ii)$ space, i.e $\forall\ x, y\in X, x\neq y, \exists\ u\in t^*$ s.t. $u(x)= 0$, $u(y)>0$ and $\exists\ v\in t^*$ s.t $v(x)>0$, $v(y)=0$.

(b) WT_1: For every pair $x, y\in X$, $x \neq y$, $\overline{1_x}(y) < 1$. **[36]**

(a)⟹(b)

Proof: From (a) we have $\forall\ x, y\in X, x\neq y, \exists\ u\in t^*$ s.t. $u(x) =0$, $u(y)>0$ and $\exists\ v\in t^*$ s.t $v(x)>0$, $v(y) =0$. Then $1-u(x) =1$, $1-u(y) <1$; and $1-v(x) <1$, $1-v(y) =1$; since $1-u(x)$ and $1-v(y)$ is closed, so we see that $\overline{1_x}(y) < 1, and\ \overline{1_y}(x) < 1$. Hence $\overline{1_x}(y) < 1$.

(b)⟹(a)

Let $\overline{1_x}(y) < 1 \Rightarrow$ either $\overline{1_x}(y) < 1$, and $\overline{1_y}(x) < 1 \Rightarrow$ 1- $\overline{1_x}(y) > 0$ and 1- $\overline{1_y}(x) > 0$. Let 1- $\overline{1_x}$ =u, then u$\in$ t^* and u(y)>0 also u(x) =0. Similarly from 1- $\overline{1_y}(x) > 0$, we can show that v(x)>0, and v(y) =0. ■

5.1.3. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) FST_1 (iii) space i.e $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s.t. u(x) <u(y) and $\exists$ v$\in$ t^* s.t v(y) <v(x).

(b) T_1' : If for every pair x, y$\in$X, x $\neq$ y and $\forall$ $\alpha \in I_0$; $\overline{\alpha 1_x}(y) < \alpha$ [36]

Proof: (a)$\Rightarrow$(b)

Suppose for every x, y$\in$ X, x $\neq$ y, there exists a t^*-supra open set v such that v (y) < v (x). Let $\beta = v(y)$, then clearly $\overline{\beta 1_y}(x) < \beta$. Hence by the same way we prove that if u(x) <u(y) then $\overline{\alpha 1_x}(y) < \alpha$.

(b) $\Rightarrow$(a)

Let for every pair x, y$\in$X, x $\neq$ y and $\forall$ $\alpha \in I_0$; $\overline{\alpha 1_x}(y) < \alpha$ hence by a lemma there exists $\beta \in I_0$ such that $\overline{\beta 1_y}(x) < \beta$. So suppose e $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s.t. u(x) <u(y), let β= u(x) then $\overline{\beta 1_y}(x) < \beta$ by (b) there exist t^* supra closed set v such that v(x)<v(y), hence t^* supra open set v such that v(y)<v(x). ■

5.1.4. Lemma: For any fuzzy supra topological space (X, t^*), the following are equivalent:

(a) {x} , $\forall$ x $\in$X, is fuzzy supra closed is in X.

(b) FST_1 (i) space $\Rightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t* s.t. u(x) =0, u(y)= 1 and $\exists$ v $\in$ t* s. t. v(x)=1, v(y)=0.

(c) T_1: If for every pair x$\in$X, $\overline{1_x} = 1_x$. [46]

Proof: The proof is easy, so omitted.

5.1.5. Lemma: For any fuzzy supra topological space (X, t^*), the following T_1 properties are equivalent:

(a) T_1: For every $x \in X$, $\overline{1_x} = 1_X$

(b) T_1' : If for every pair x, $y \in X$, $x \neq y$ and $\forall\, \alpha \in I_0$; $\overline{\alpha 1_x}(y) < \alpha$

(c) WT_1: For every pair x, $y \in X$, $x \neq y$, $\overline{1_x}(y) < 1$.

Proof: $T_1 \Rightarrow T_1'$

Let (X, t^*) is T_1, Now $\overline{\alpha 1_x}(y) < \overline{1_x}(y) = 1_x(y) = 0 < \alpha$. Thus we get $T_1 \Rightarrow T_1'$

$T_1' \Rightarrow WT_1$.

Again let (X, t^*) is T_1'. Then for every pair x, $y \in X$, $x \neq y$ and $\forall\, \alpha \in I_0$; $\overline{\alpha 1_x}(y) < \alpha$

Taking $\alpha = 1$, we have $\overline{1_x}(y) < 1$. Thus we see that $T_1' \Rightarrow WT_1$.

$T_1 \Rightarrow WT_1$

Since (X, t^*) is T_1, i.e for every pair $x \in X$, $\overline{1_x} = 1_X$, now $\overline{1_x}(y) = 1_x(y) = 0 < 1$; Thus we see that $T_1 \Rightarrow WT_1$.

5.1.6. Theorem: A fuzzy supra topological space (X, t^*) is $FST_1(1)$ if and only if, $\forall$ x, $y \in X$, $x \neq y$, $\exists\ v \in t^*$ s. t. nbhd v(x) $\ni x \notin$ nbhd v(y) and $\exists\ u \in t^*$ s. t. nbhd u(y) $\ni y \notin$ nbhd u(x).

Proof: Suppose (X, t^*) satisfy $FST_1(1)$, then $\exists\ v \in t^*$ s.t v(x)=1, v(y)=0. Then clearly nbhd. v(x) $\ni x \notin$ nbhd. v(y) and $\exists\ u \in t^*$ s. t u(x) =0, u(y) = 1, so nbhd u(y) $\ni y \notin$ nbhd u(x).

5.2: Good extension, Heredity, Productive, and Homeomorphic properties of FST_1 spaces.

In this section we show that all $FST_1(k)(i \leq k \leq vii)$ properties are good extensions of their supra topological counter parts.

5.2.1. Theorem:

(a) If (X, T^*) is an ST_1-space, then (X, ω (T^*)) satisfies $FST_1(k)$, $k \in \{i, ii, iii, iv, vi, vii\}$ **[65]**;

(b) If $(X, \omega(T^*))$ satisfies $FST_1(k)$ $k\in\{i, ii, iii, iv, vi, vii\}$ then (X, T^*) is an ST_1-space.

Proof: (a) Let (X, t^*) is a supra T_1-topological space. We shall prove that $(X, \omega(t^*))$ is a fuzzy supra T_1(i) space. Let $x, y \in X$, with $x \neq y$, since (X, t^*) is a supra T_1, $\exists$ U, V $\in t^*$ such that $x \in U$, $y \notin U$, and $x \notin V$, $y \in V$; but from definition of lower semi continuous, we have $1_U \in \omega(t^*)$ and $1_U(x) = 1$, $1_U(y) = 0$.and $1_V(y) = 0$, $1_V(x) = 1$. Hence we see that $(X, \omega(t^*))$ is a fuzzy supra T_1(i) space.

(b) Conversely suppose that $(X, \omega(t^*))$ is a fuzzy supra T_1(i) space we will prove that (X, t^*) is a supra T_1-topological space, since $(X, \omega(t^*))$ is a fuzzy supra T_1(i) space , so $\exists$ $u \in t^*$ s.t. $u(x) = 0$, $u(y) = 1$ and $\exists$ $v \in t^*$ s.t $v(x) = 1$, $v(y) = 0$. Suppose $u \in t^*$ s.t. $u(x) = 0$, $u(y) = 1$, so $x \notin u^{-1}(\alpha, 1]$ and $y \in u^{-1}(\alpha, 1]$, and $y \notin v^{-1}(\alpha, 1]$ and $x \in v^{-1}(\alpha, 1]$, by the definition of lsc. $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1]$, $\in t^*$. Hence (X, t^*) is a supra T_1-topological space.

Similarly we can show that if (X, T^*) is an ST_1-space, then $(X, \omega(T^*))$ satisfies $FST_1(k)$, $k\in\{$ ii, iii, iv, v, vi, viii$\}$ also its converse. Here $(X, \omega(T^*))$ is a good extension of its topological counter parts.

5.2.2. Theorem: Let (X, t^*) be a fuzzy supra topological space. Then the following implications hold.

(a) $(X, \omega(t^*))$ is FST_1 (i) $\Rightarrow$ $(X, \omega(t^*))$ is FST_1 (ii).

(b) $(X, \omega(t^*))$ is FST_1 (ii) $\Rightarrow$ $(X, \omega(t^*))$ is FST_1 (iii).

(c) $(X, \omega(t^*))$ is FST_1 (i) $\Rightarrow$ $(X, \omega(t^*))$ is FST_1 (iv).

(d) $(X, \omega(t^*))$ is FST_1 (ii) $\Rightarrow$ $(X, \omega(t^*))$ is FST_1 (vi).

(e) $(X, \omega(t^*))$ is FST_1 (iv) or FST_1 (vi) $\Rightarrow$ $(X, \omega(t^*))$ is FST_1 (vii).

(f) $(X, \omega(t^*))$ is FST_1 (iii) $\Rightarrow$ $(X, \omega(t^*))$ is FST_1 (v).

Proof (a): Suppose that $(X, \omega(t^*))$ is a fuzzy supra T_1(i) space we will prove that $(X, \omega(t^*))$ is a fuzzy supra T_1(ii)-topological space, since $(X, \omega(t^*))$ is a fuzzy supra T_1(i) space . Let $x, y \in X$, with $x \neq y$, since (X, t^*) is a fuzzy supra T_1(i) $\exists$ $u \in t^*$ s.t. $u(x) = 0$, $u(y) = 1$ and $\exists$ $v \in t^*$ s.t $v(x) = 1$, $v(y) = 0$.We have from definition of lower semi-continuous, we have 1_u, $1_v \in \omega(t^*)$ and since $(X, \omega(t^*))$ is a fuzzy supra $T_1(i)$, $1_u(x) = 0$, $1_u(y) = 1$.and $1_V(y) = 0, 1_v(x) = 1$. Also it is clear that $1_v(x) = 1 > 0$, $1_V(y) = 0$, and $1_u(x) = 0$, $1_u(y) = 1 > 0$; hence we see that$(X, \omega(t^*))$ is a fuzzy supra $T_1(ii)$ space. Which is (a)

Proof (b): Suppose that $(X, \omega(t^*))$ is a fuzzy supra $T_1(ii)$ space we will prove that $(X, \omega(t^*))$ is a fuzzy supra $T_1(iii)$-topological space, since $(X, \omega(t^*))$ is a fuzzy supra $T_1(ii)$ space . Let x, $y \in X$, with $x \neq y$, since (X, t^*) is a fuzzy supra $T_1(ii)$ $\exists$ $u \in t^*$ s. t. $u(x) = 0$, $u(y) > 0$ and $\exists$ $v \in t^*$ s. t $v(x) > 0$, $v(y) = 0$. We have from definition of lower semi- continuous,, we have 1_u, $1_v \in \omega(t^*)$ and since $(X, \omega(t^*))$ is a fuzzy supra $T_1(ii)$, $1_u(x) = 0$, $1_u(y) > 0$, and $1_v(y) = 0, 1_v(x) > 0$. Also it is clear that $1_v(x) = 1 > 0$, $1_V(y) = 0$, and $1_u(x) = 0$, $1_u(y) = 1 > 0$; so$1_u(x) < 1_u(y)$ and $1_v(y) < 1_v(x)$, hence we see that$(X, \omega(t^*))$ is a fuzzy supra $T_1(iii)$ space. Which is (b)

Proof (c) Suppose that $(X, \omega(t^*))$ is a fuzzy supra $T_1(i)$ space we will prove that $(X, \omega(t^*))$ is a fuzzy supra $T_1(iv)$-topological space, since $(X, \omega(t^*))$ is a fuzzy supra $T_1(i)$ space . Let x, $y \in X$, with $x \neq y$, since (X, t^*) is a fuzzy supra $T_1(i)$ $\exists$ $u \in t^*$ s.t. $u(x) = 0$, $u(y) = 1$ and $\exists$ $v \in t^*$ s.t $v(x) = 1$, $v(y) = 0$.We have from definition of lower semi-continuous, we have 1_u, $1_v \in \omega(t^*)$ and since $(X, \omega(t^*))$ is a fuzzy supra $T_1(i)$, $1_u(x) = 0$, $1_u(y) = 1$.and $1_V(y) = 0, 1_v(x) = 1$. Also $1_u(x) = 1$, $1_u(y) = 0 < \alpha \in I_1$, $1_v(y) = 1$, $1_V(x) = 0 < \alpha \in I_1$, hence $(X, \omega(t^*))$ is a fuzzy supra $T_1(iv)$ space. Which is (c)

Proof(d): Suppose that $(X, \omega(t^*))$ is a fuzzy supra $T_1(ii)$ space we will prove that $(X, \omega(t^*))$ is a fuzzy supra $T_1(iii)$-topological space, since $(X, \omega(t^*))$ is a fuzzy supra $T_1(ii)$ space . Let x, $y \in X$, with $x \neq y$, since (X, t^*) is a fuzzy supra $T_1(ii)$ $\exists$ $u \in t^*$ s. t. $u(x) = 0$, $u(y) > 0$ and $\exists$ $v \in t^*$ s. t $v(x) > 0$, $v(y) = 0$. We have from definition of lower semi- continuous,, we have 1_u, $1_v \in \omega(t^*)$ and since $(X, \omega(t^*))$ is a fuzzy supra $T_1(ii)$, $1_u(x) = 0$, $1_u(y) > 0$, and $1_v(y) = 0, 1_v(x) > 0$. Also it is clear that $1_u(x) = 0$, $1_u(y) > \alpha \in I_1$ and $1_v(y) = 0$, $1_v(x) > 0 \Rightarrow 1_v(y) = 0$, $1_v(x) > \alpha \in I_1$. Hence $(X, \omega(t^*))$ is a fuzzy supra $T_1(vi)$ space.

Proof(e) Suppose that $(X, \omega(t^*))$ is a fuzzy supra $T_1(iv)$ space we will prove that $(X, \omega(t^*))$ is a fuzzy supra $T_1(vii)$-topological space, since $(X, \omega(t^*))$ is a fuzzy supra $T_1(iv)$ space . Let x, $y \in X$, with $x \neq y$, since (X, t^*) is a fuzzy supra $T_1(iv)$ with $\alpha \in I_1$,

$\exists$ u$\in$ t^* s. t. u(x)=1, u(y)<α,and $\exists$ v$\in$ t^* s. t v(y) =1, v(x)<α.We have from definition of lower semi- continuous,, we have 1_u, $1_v \in \omega(t^*)$ and since $(X, \omega(t^*))$ is a fuzzy supra T_1(iv); $1_u(x) = 1$, $1_u(y) < \alpha$, and $1_v(y) = 1$, $1_v(x) < \alpha$. Also it is clear that $0 \le 1_u(y) \le \alpha < 1_u(x) \le 1$, or $\exists$ v $\in t^*$ s. t $0 \le 1_v(x) \le \alpha < 1_v(y) \le 1$. Hence (X, $\omega(t^*)$) is a fuzzy supra T_1(vi) space. Similarly we can show that if $(X, \omega(t^*))$ is supra T_1(vi) space then it is fuzzy supra T_1(vii) space. Which is (e)

Proof (f): Suppose that $(X, \omega(t^*))$ is a fuzzy supra T_1(iii) space we will prove that (X, $\omega(t^*)$) is a fuzzy supra T_1(v)-topological space, since $(X, \omega(t^*))$ is a fuzzy supra T_1(iii) space . Let x, y$\in$X, with $x \neq y$, since (X, t^*) is a fuzzy supra T_1(iii) $\exists$ u$\in$ t^* s. t. u(x) <u(y) and $\exists$ v$\in$ t^* s. t v(y) <v(x). We have from definition of lower semi-continuous,, we have 1_u, $1_v \in \omega(t^*)$ and since $(X, \omega(t^*))$ is a fuzzy supra T_1(iii), so $1_u(x) < 1_u(y)$ and $1_v(y) < 1_v(x)$, hence we can treat with $\alpha \in I_1$, $1_u(x) = 0$, and $1_u(y) = \alpha$ and $\exists$ v$\in$ t^* s. t. $1_v(y) = 0$, $1_v(x) = \alpha$. Thus (X, $\omega(t^*)$) is a fuzzy supra T_1(v) space.

5.2.3. Theorem : Let (X, t^*) be a fuzzy supra topological space, and $I_\alpha(t^*) = \{ u^{-1}(\alpha, 1] : u \in t^* \}$, then **[64]**

(a) (X, t^*) is FST_1(i), $\Rightarrow$ (X , $I_\alpha(t^*)$) is T_1.

(b) (X, t^*) is FST_1(iv), $\Rightarrow$ (X , $I_\alpha(t^*)$) is T_1.

(c) (X, t^*) is FST_1(vi), $\Rightarrow$ (X, $I_\alpha(t^*)$) is T_1.

(d) (X, t^*) is FST_1(vii), $\Leftrightarrow$ (X , $I_\alpha(t^*)$) is T_1.

Proof: (a) Let (X, t^*) is FST_1(i)we shall prove that (X, $I_\alpha(t^*)$) is supra T_1. Let x, y $\in$X; x$\neq$y, since (X, t^*) is FST_1 (i) space $\Leftrightarrow$ $\forall$ x, y$\in$ X, x$\neq$y, $\exists$ u$\in$ t^* s. t. u(x) =0, u(y) = 1 and $\exists$ v$\in t^*$ s. t. v(x) =1, v(y) =0. for $\alpha \in I_1$, $\exists$ u , v $\in t^*$such that u(y) = 1 , u(x) < α and v(y) <α , v(x) = 1.Since (X , t^*) is FST_1(i), Since $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$, and it is clear that $y \in u^{-1}(\alpha, 1]$, $x \notin u^{-1}(\alpha, 1]$ and $y \notin v^{-1}(\alpha, 1]$, $x \in v^{-1}(\alpha, 1]$. Hence (X, $I_\alpha(t^*)$) is supra T_1- space.

Proof: (b) . Let (X, t^*) be FST_1(iv). We shall prove that (X, $I_\alpha(t^*)$ is supra T_1. Let x, y $\in$ X with x $\neq$ y .Since (X , t^*) is FST_1(iv), for $\alpha \in I_1$, $\exists$ u , v $\in t^*$such that u(x) = 1 , u(y) < α and v(x) <α , v(y) = 1. Since $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$, and it is clear that $x \in u^{-1}(\alpha, 1]$, $y \notin u^{-1}(\alpha, 1]$ and $x \notin v^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$. Hence $(X, I_\alpha(t^*))$ is supra T_1- space .

Proof: (c) Again, suppose that (X, t^*) is FST_1(vi) space .We shall prove that $(X, I_\alpha(t^*))$ is supra T_1- space . Let $x, y \in X$ with $x \neq y$, since (X, t^*) is FST_1(vi) space, for $\alpha \in I_1$, $\exists\, u, v \in t^*$ such that $u(x) = 0$, $u(y) > \alpha$ and $v(x) > \alpha$, $v(y) = 0$. Since $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$ and it is clear that $x \notin u^{-1}(\alpha, 1]$, $y \in u^{-1}(\alpha, 1]$ and $x \in v^{-1}(\alpha, 1]$, $y \notin v^{-1}(\alpha, 1]$. Hence $(X, I_\alpha(t^*))$ is supra T_1 - Space.

Proof: (d). Further, suppose that (X, t^*) is FST_1(vii). We shall prove that $(X, I_\alpha(t^*))$ is supra T_1.Let $x, y \in X$ with $x \neq y$. Since (X, t^*) is FST_1(vii)., for $\alpha \in I_1$, $\exists\, u, v \in t^*$ such that $0 \leq u(x) \leq \alpha < u(y) \leq 1$ and $0 \leq v(y) \leq \alpha < v(x) \leq 1$. Since $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$ and it is clear that $x \notin u^{-1}(\alpha, 1]$, $y \in u^{-1}(\alpha, 1]$ and $x \in v^{-1}(\alpha, 1]$, $y \notin v^{-1}(\alpha, 1]$. Hence $(X, I_\alpha(t^*))$ is supra T_1 -Space.

Conversely, suppose that $(X, I_\alpha(t^*))$ is supra T_1- space. We shall prove that (X, t^*) is FST_1(vii) . Let $x, y \in X$ with $x \neq y$, since $(X, I_\alpha(t^*))$ is supra T_1- space so $\exists\, M, N \in I_\alpha(t^*)$ such that $x \in M$, $y \notin M$ and $x \notin N$, $y \in N$, where $M = u^{-1}(\alpha, 1]$, $N = v^{-1}(\alpha, 1]$, where $u, v \in t^*$. So it is clear that $u(x) > \alpha$, $u(y) < \alpha$ and $v(x) < \alpha$, $v(y) > \alpha$. This implies $0 \leq u(x) \leq \alpha < u(y) \leq 1$ and $0 \leq v(x) \leq \alpha < v(y) \leq 1$. Hence (X, t^*) is FST_1(vii).

Now, we give an example.

5.2.1. Example :- Let $X = \{x, y\}$and $u, v, w \in I^X$, where u, v, w are defined by $u(x) = 1$, $u(y) = 0$, $v(x) = 0.42$, $v(y) = 0.95$, $w(x) = 0.15$, $w(y) = 0.32$. Consider the fuzzy supra topology t^* on X generated by $\{0, u, v, w, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.61$, it is cleat that (X, t^*) is not FST_1(iv) and (X, t^*) is not FST_1(vi). Now $I_\alpha(t^*) = \{X, \phi, \{x\}, \{y\}\}$. Then we see that $I_\alpha(t^*)$ is a topology on X and $(X, I_\alpha(t^*))$ is supra T_1 space.

This completes the proof.

5.2.4 Theorem: All the properties $FST_1(k)$ of subspace topology where ($i \leq k \leq vii$) are hereditary.

Proof: Consider the fsts (X, t^*), Let $A \subset X$. where $t_A^* = \{u \wedge A : u \in t^*\}$, we have to show that, if (X, t^*) has FST_1(k) (i≤ k ≤ vii) then the subspace (A, t_A^*) has FST_1(k) (i≤ k ≤vii).

(i) Let x, y ∈ A, x ≠y, so that x, y ∈ X as A⊂X, since (X, t^*) is FST_1(i), so, ∃ u∈ t^* s.t. u(x) =0, u(y) =1, and ∃ v∈ t^* s.t v(x)=1, v(y)=0. Again from definition of subspace we have u∧A ∈t_A^* , (u∧ A)(x) = 0, (u∧ A)(y)=1 and v∧A ∈t_A^* , with (v∧ A)(x) = 1, (v∧ A)(y)=0, as x, y∈ A. This implies that, (A, t_A^*) has FST_1(i).

(ii) Let x, y ∈ A, x ≠y, so that x, y ∈ X as A⊂X, since (X, t^*) is FST_0(iv), so, ∃ u∈ t^* with α ∈I_1, s. t. u(x) =1, u(y)<α , and ∃ v∈ t^* s. t. v(y) =1, v(x)<α. Again from definition of subspace we have u∧ A ∈t_A^* and (u∧ A)(x) =1, (u∧ A)(y)<α also v∧A ∈ t_A^* , with (v∧ A)(x) <α, (v∧ A)(y) =1, as x, y∈ A. This implies that, (A, t_A^*) has FST_1(iv).

All other proofs are similar and easier so omitted.

5.2.5. Theorem : - Let (X_i , t_i^*) , i ∈ J be fuzzy topological spaces and X = $\Pi_{i \in J} X_i$. Let t^* be the product fuzzy supra topology on X, then ∀ i ∈ J, (X_i , t_i^*) is FST_1(K), i≤k≤vii if and only if (X, t^*) is FST_1(K).

Proof : - Suppose that ∀ i ∈ J, (X_i , t_i^*) is FST_1(iv). We shall prove that (X, t^*) is FST_1(iv). Let x, y ∈ X with x ≠ y, then $x_i \neq y_i$, for some i ∈ J. Since (X_i, t_i^*) is FST_1(iv), for α∈ I_1, ∃ u_i , v_i ∈ t_i^* , i ∈ Λ, such that $u_i(x_i)$ = 1 , $u_i(y_i)$ < α and $v_i(x_i)$ <α , $v_i(y_i)$ = 1 . But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$. Then $u_i(\pi_i(x)) = 1$, $u_i(\pi_i(y)) < \alpha$ and $v_i(\pi_i(x)) < \alpha$, $v_i(\pi_i(y)) = 1$. It follows that ∃ $u_i o \pi_i$, $v_i o \pi_i$ ∈ t^* such that (u_i o π_i) (x) = 1 , (u_i o π_i) (y) <α and (v_i o π_i) (x) <α , (v_i o π_i) (y) = 1 . Hence it is clear that (X, t^*) is FST_1(iv) .

Conversely, suppose that (X, t^*) is FST_1(iv). We shall prove that (X_i , t_i^*), i ∈ Λ is FST_1(iv). Let for some i ∈ J, a_i be a fixed element in X_i, suppose that A_i = { x ∈ X = $\Pi_{i \in \Lambda} X_i$: $x_j = a_j$ for some i ≠ j } . So that A_i is a subset of X , and hence (A_i , $t_{A_i}^*$)

is also a subspace of (X, t^*). Since (X, t^*) is $FST_1(iv)$, then $(A_i, t^*_{A_i})$ is also $FST_1(iv)$. Now we have A_i is a homeomorphic image of X_i Hence (X_i, t^*_i), $i \in J$, is $FST_1(iv)$. Similarly we can be proving for other conditions.

5.2.6. Theorem:- Let (X, t^*) and (Y, s^*) be two fuzzy supra topological spaces and $f : X \longrightarrow Y$ be a one-one, onto and supra open map. Then, (X, t^*) is $FST_1(k)$, $i \leq k \leq vii \Rightarrow (Y, s^*)$ is also $FST_1(k)$, $i \leq k \leq vii$.

Proof: Suppose (X, t^*) be $FST_1(i)$,. We shall prove that (Y, s^*) is $FST_1(i)$,. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is onto then $\exists\ x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$ and $x_1 \neq x_2$ as f is one-one. Again since (X, t^*) is $FST_1(i)$, for $\alpha \in I_1$, $\exists\ u, v \in t^*$ such that $u(x_1) = 1$, $u(x_2) = 0$ and $v(x_1) = 0$, $v(x_2) = 1$.

Now $f(u)(y_1) = \{ \text{Sup } u(x_1) ; f(x_1) = y_1 \}$

$= 1.$

$f(u)(y_2) = \{ \text{Sup } u(x_2) ; f(x_2) = y_2 \}$

$= 0$

and $f(v)(y_1) = \{ \text{Sup } v(x_1) ; f(x_1) = y_1 \}$

$= 0$

$f(v)(y_2) = \{ \text{Sup } v(x_2) ; f(x_2) = y_2 \}$

$= 1$

Since f is supra open then $f(u), f(v) \in s^*$. Now it is clear that $\exists\ f(u), f(v) \in s^*$ such that $f(u)(y_1) = 1$, $f(u)(y_2) = 0$ and $f(v)(y_1) = 0$, $f(v)(y_2) = 1$. Hence (Y, s^*) is $FST_1(i)$

Similarly for $K \in \{ii, iii, iv, v, vi, vii\}$ we can prove the theorem.

5.2.7. Theorem:- Let (X, t^*) and (Y, s^*) be two fuzzy supra topological spaces and $f : X \longrightarrow Y$ be a continuous and one-one map then , Then, (Y, s^*) is $FST_1(k)$, $i \leq k \leq vii \Rightarrow (X, t^*)$ is also $FST_1(k)$, $i \leq k \leq vii$.

Proof :- Suppose (Y, s^*) be $FST_1(iv)$, We shall prove that (X, t^*) is $FST_1(iv)$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ in Y as f is one-one . Since (Y, s^*) is $FST_1(iv)$,, for $\alpha \in I_1$, $\exists\ u, v \in t^*$ such that $u(f(x_1)) = 1$, $u(f(x_2)) < \alpha$ and $v(f(x_1)) < \alpha$, $v(f(x_2)) = 1$. This implies that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) < \alpha$ and f^-

${}^{1}(v)(x_1) < \alpha$, $f^{-1}(v)(x_2) = 1$, since f is continuous and u , $v \in s^*$ then $f^{-1}(u)$, $f^{-1}(v) \in t^*$. Now it is clear that $\exists$ $f^{-1}(u)$, $f^{-1}(v) \in t^*$ such that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) < \alpha$ and $f^{-1}(v)(x_1) < \alpha$, $f^{-1}(v)(x_2) = 1$. Hence (X, t^*) is FST_1(iv),Similarly we can prove the theorem for $K \in \{i, ii, iii, v, vi, vii\}$.

CHAPTER-VI

Fuzzy supra T_2 topological spaces

6. Introduction:

Separation axioms in fuzzy topological spaces were studied by Lowen, R., and Wuyts, P., [36], Pao. M. P., and Ying, [47], Shostak; A.P., [56], Gantner, T.E., Steinlage, R.C., and Warren, R.H., [24], Benchalli, S.S. and Malghan, S.R. [19], Rodabaugh, S.E., [49], Hutton, B., and Reilly, I., [30], Sarkar, M., [52] and Srivastava, R. Lal S.N., and Srivastava, A.K., [58]. Cho, S.K.,[23], Ali, D.M., and Hossain, M.S., [15]. The important separation property namely Hausdorffness Concept has been defined and studied by many Mathematicians from different view of points. At present not less than ten approaches to the definition of Hausdorff fuzzy topological spaces are known. Some of them differ negligibly; but others do basically. In this chapter we introduce and study on some Hausdorffness Concepts of fuzzy supra topological spaces and discuss some properties in this connection. We symbolize a fuzzy supra T_2 topological space by FST_2.

6.1. Definitions of FST_2 spaces

6.1.1. Definition: A fuzzy supra topological space (X, t^*) is called

FST_2(i) : iff for every $x, y \in X$, $x \neq y$, there exist $u, v \in t^*$ such that $u(x)= 1= v(y)$ and $u \wedge v= 0$. [Ganter-Steinlage-Warren T_2[**24**]]

FST_2(ii) : iff for every $x, y \in X$, $x \neq y$, and for every $\alpha, \beta \in I_0$, there exist $u, v \in t^*$ such that $u(x)>\alpha$, $v(y)>\beta$ and $u \wedge v= 0$. [SS-T_2 [**58**]].

FST_2(iii) : iff for every $x, y \in X$, $x \neq y$, there exist $u, v \in t^*$ such that $u(x)>0$, $v(y)>0$ and $u \wedge v=0$.

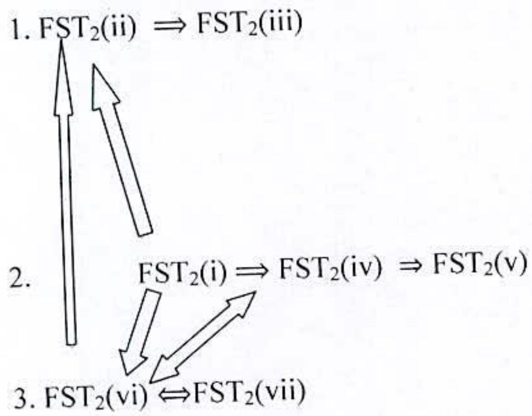
FST_2(iv) : iff for every $x, y \in X$, $x \neq y$, there exist $u, v \in t^*$ such that $u(x)= 1= v(y)$ and $u \leq 1-v$.[MB[**19**]]

FST_2(v) : iff for every $\alpha, \beta \in I_0$, there exist $u \in t^*$ such that $\alpha \lesssim u(x)$ and $\beta \nleq \bar{u}(y)$ [Sarker[**52**]].

FST_2(vi) : iff for every x, y $\in$X, x$\neq$y, there exist u, v $\in t^*$ such that u(x)= 1= v(y), u(y) =0 =v(x) and u $\subset \bar{v}$ [Ali- Srivastava-T_2[**5**]]

FST_2(vii) : iff for every x, y $\in$X, x$\neq$y, and for every α, β $\in I_0$, there exist u, v $\in t^*$ such that $\bar{v}(y) \geq$ u(x)$\geq\alpha$, $\bar{u}(y)\geq$v(x)$\geq\beta$ and u $\subseteq \bar{v}$. [Ghanim- Kerre-Mashhour [**26**]]

6.1.1. Theorem: The following implications are true.



Proof: FST_2(i) $\Rightarrow$$FST_2$(ii)

Let (X, t*) be a fuzzy supra topological spaces and let α, β $\in I_0$, then for every x, y $\in$X, x$\neq$y, there exist u, v $\in t^*$ such that u(x) = 1 = v(y) and u$\wedge$v=0, we have $\alpha \leq$1= u(x), $\beta \leq$ 1=v(y); hence clearly FST_2(i) $\Rightarrow$$FST_2$(ii).

(ii) **FST_2(ii) $\Rightarrow$$FST_2$(iii)**

Let (X, t^*) be a fuzzy supra topological spaces and let α, β $\in I_0$, then there exist u, v $\in t^*$ such that u(x)>α, v(y)>β and u$\wedge$v =0, since α, β $\in I_0$, so α>0 and β>0 , hence clearly FST_2(ii) $\Rightarrow$ FST_2(iii)

FST_2(i) $\Rightarrow$ FST_2(vi) and FST_2(iv) $\Leftrightarrow$ FST_2(vi) is straight forward.

FST_2(i) $\Rightarrow$ FST_2(iv) is direct since u$\wedge$v=0 $\Rightarrow$ u$\leq$1-v. Now we show that

(iii) FST_2(iv) $\Rightarrow$ FST_2(v).

Let x, y $\in$X, x$\neq$y, and α, $\beta \in I_0$ then by FST_2(iv) there exist u, v $\in t^*$ such that u(x)= 1=

v(y) and u$\leq$ 1-v, i.e $\bar{u} \leq$ 1-v.Thus $\bar{u}(y) = 0 = \bar{v}(x)$ hence we see that $\alpha \lesssim$ u(x) and $\beta \lesssim \overline{u(y)}$ implying (X, t^*) is FST_2(v).

(iv) FST_2(vi) $\Leftrightarrow$$FST_2$(vii)

First let FST_2(vii) $\Rightarrow$ FST_2(vi), then let (X, t^*) be a fuzzy supra topological spaces and $\alpha, \beta \in I_0$, then there exist u, v $\in t^*$ such that $\bar{v}(y) \geq$ u(x)$\geq \alpha$, $\bar{u}(y) \geq$ v(x)$\geq \beta$ and u $\subseteq \bar{v}$ .Setting α=1, β=0 then $\bar{v}(y) \geq$ u(x) $\geq$1, $\bar{u}(y) \geq$ v(x) $\geq$0.and u $\subseteq \bar{v}$. Again From these conditions we obtain u(x) =1=v(y) and u(y) =0=v(x).

FST_2(vi) $\Rightarrow$$FST_2$(vii)

Let (X, t^*) be a fuzzy supra topological space, and $\alpha, \beta \in I_0$, then there exist u, v $\in t^*$ such that u(x)=1=v(y), u(y) =0 =v(x) then $\alpha \leq 1 = \mu_u(x) = \mu_v(y)$ and $\beta \geq 0 = \mu_u(y) = \mu_v(x)$ and since u $\subseteq \bar{v}$ hence $\bar{v}(y) \geq$ u(x)$\geq\alpha$, and $\bar{u}(y) \geq$ v(x)$\geq\beta$, so (X, t^*) is FST_2(vii).

FST_2(vi) $\Rightarrow$ FST_2(ii) is obvious.

Now we give some examples:

1. **FST_2(ii) $\not\Rightarrow$$FST_2$(i).**

Example: Let X ={x, y} and u, v, w $\in I^X$, where u, v, w are defined by u(x) =0.7, u(y) =0; v(x) =0 and v(y) =0.8; w(x) =0.7, w(y) =0.8, Consider the fuzzy supra topology t^* on X generated by {1, 0, u, v, w}. Now for α= 0.5, β=0.6, it is clear that (X, t*) is FST_2(ii) but (X, t*) is not FST_2(i).

2. **FST_2(iii) $\not\Rightarrow$$FST_2$(i), FST_2(ii)**

Example: Let X ={x, y} and u, v, w $\in I^X$, where u, v, w are defined by u(x) =0.7, u(y) =0; v(x) =0 and v(y) =0.8; w(x) = 0.7, w(y) = 0.8, Consider the fuzzy supra topology t^* on X generated by {1, 0, u, v, w}.Here u, v $\in t^*$ such that u(x)>0, v(y)>0 and u$\wedge$v=0.So it is clear that (X, t*) is FST_2(iii) but not FST_2(i).Again for α= 0.8, β=0.9, it is clear that (X, t*) is not FST_2(ii) but (X, t*) is FST_2(iii).

2. $FST_2(iv) \nRightarrow FST_2(iii)$

Let X be an infinite set and for any x, y $\in$X , we define u_{xy}, a fuzzy set in X , as follows: $u_{xy}(x)=1$, $u_{xy}(y)=0$, $u_{xy}(z)=0.5\forall z \in X, z \neq x, y$. Now consider the fuzzy supra topology t^* on X generated by { u_{xy}: x, y $\in$X, $x \neq y$ } , it is clear that (X, t^*) is $FST_2(iv)$. However (X, t^*) is not $FST_2(iii)$, As the intersection of any two non-trivial supra open sets can not be zero. Also $FST_2(i)\Rightarrow FST_2(ii)\Rightarrow FST_2(iii)$, so $FST_2(iv).\nRightarrow FST_2(i)$.and $FST_2(iv).\nRightarrow FST_2(ii)$.

4. $FST_2(vi) \nRightarrow FST_2(i)$

Example: Let X ={x, y,z} and u, v, w $\in I^X$, where u, v, w are defined by u(x) =1, u(y) =0, u(z)= .4 ; v(x) =0 and v(y) =1, v(z)= .4 ; w(x)= 1, w(y)=1,w(z) = .4. Let us consider the fuzzy supra topology t^* on X generated by {1, 0, u, v, w}.Then clearly (X, t^*) is $FST_2(vi)$ but not $FST_2(i)$ because $u \wedge v \neq 0$.

6.1.2. Theorem: Let (X, t^*) be a supra topological space, then show that the flowing conditions are equivalents. **[23]**

(1) (X, t^*) is $FST_2(ii)$.

(2) For any x$\in$X, x_1 is fuzzy supra closed in X. (If u is a fuzzy supra open neighborhood of x then $x_1 = 1-u$.)

(3) For x, y $\in$ X with $x \neq y$, there exists a fuzzy supra open nbd U of x_1 such that U(y) = 0.

Proof: Firstly let (X, t^*) is $FST_2(ii)$ i. e for every x, y $\in$X, x$\neq$y, and for every $\alpha, \beta \in I_0$, there exist u, v $\in t^*$ such that $u(x)>\alpha$, $v(y)>\beta$ and $u\wedge v=0$.We shall prove that (1)$\Rightarrow$(2) . Let x$\in$X, then for every y$\neq$x, choose a fuzzy a supra open set V_y such that $V_y(x) = 0$ and $V_y(y) =1$; Let $V= \cup_{y\neq x} V_y$

$$\text{Then } V(z) = \begin{cases} 0 & \text{when } z\neq x, \\ 1 & \text{when } z=x. \end{cases}$$

Since V is fuzzy supra open in X, we have $x_1 = 1-V$ is closed in X. Hence (1)$\Rightarrow$(2).

(2)$\Longrightarrow$(3) Let x; y be distinct points of X and let $U = 1_x - y_1$.Then U(x) = 1 and U(y) = 0. Since y_1 is fuzzy supra closed in X, U is fuzzy supra open.

(3)⇒ (1)

Let x_α and y_β be distinct fuzzy points. For every $z \neq x$, choose a fuzzy supra open nbd U_z of x_1 such that $U_z(z) = 0$. Since X is finite the fuzzy set $U = \cap_{z \neq x} u_z$ is fuzzy supra open. Also,

$$U(z) = \begin{cases} 0 \text{ when } & z \neq x, \\ 1 \text{ when } & z = x. \end{cases}$$

Similarly for any supra open set V_z and U and V are fuzzy supra open nbd. of x_α and y_β .and $U \wedge V=0$. Hence the theorem is proof.

6.1.3.Theorem : The following are equivalent.

(a) Let (X, t*) be a FSTS. then for all $x, y \in X$, and $\alpha, \beta \in I^X$, with $x \neq y$, there are $u, v \in t^*$ such that $\alpha \in u(x)$, $\beta \in v(y)$ and $u \wedge v=0$.

(b) FST_2 (i)

Proof: The proof is easy and so, omitted.

6.1.4. Theorem: The following implication is hold

(a) Let (X, t*) be a FSTS. then for all $x, y \in X$, and $\alpha, \beta \in I^X$, with $x \neq y$, there are $u, v \in t^*$ such that $\alpha \in u(x)$, $\beta \in v(y)$ and $u \wedge v=0$

(b) FST_2 (v). Then, (a) ⟹ (b)

Proof: The proof is easy and so, omitted.

6.2. Good extension of fuzzy supra Hausdorff spaces.

6.2.1. Theorem: Let (X, t^*) be a supra topological space , and (X, t^*) is supra Hausdorff space iff (X, $\omega(t^*)$) is FST_2(j) , where j=i, ii, iii, iv, v, vi, vii.

Proof: Let (X, t^*) be a Supra Hausdorff or T_2 topological space, we shall prove that (X, $\omega(t^*)$) is FST_2(i) . Let $x, y \in X$, with $x \neq y$, Since (X, t^*) is T_2 topological space, there exist $u, v \in t^*$ such that $x \in u$ and $y \in v$ and $u \wedge v=0$. From the definition of lower semi continuous function $1_u, 1_v \in \omega(t^*)$ and $1_u(x) =1$, $1_v(y) =1$ and $1_u \wedge 1_v=0$, If $1_u \wedge 1_v \neq 0$, then $\exists$ $z \in X$ such that $1_u \wedge 1_v(z) \neq 0 \Rightarrow 1_u(z) \neq 0$, $1_v(z) \neq 0 \Rightarrow z \in u$, $z \in v \Rightarrow z \in u \wedge v \Rightarrow u \wedge v \neq 0$, a contradiction . So that $1_u \wedge 1_v=0$, and consequently (X, $\omega(t^*)$) is FST_2(i).

It is clear by the implications between the definitions of FST_2 space, if (X, t^*) is supra Hausdorff topological space, then that $(X, \omega(t^*))$ is $FST_2(j)$,where j=i, ii, iii, iv, v, vi, vii.

Next suppose that $(X, \omega(t^*))$ is $FST_2(iii)$ and x, y $\in$X, $x \neq y$, there exist u, v $\in \omega(t^*)$ such that u(x)>0, v(y)>0 and $u \wedge v=0$. Suppose $\alpha, \beta \in I_{0,1}$ such that $u(x)>\alpha$, and $v(y)>\beta$ then $u^{-1}(\alpha,1]$, $v^{-1}(\beta,1] \in t^*$ and $x \in u^{-1}(\alpha,1]$, $y \in v^{-1}(\beta,1]$. Moreover $u^{-1}(\alpha,1] \wedge v^{-1}(\beta,1]=\phi$ and if not let $z \in u^{-1}(\alpha,1] \wedge v^{-1}(\beta,1]$ then $z \in u^{-1}(\alpha,1]$, and $z \in v^{-1}(\beta,1]$ then $(u \wedge v)(z)>0$. Contradicting that $u \wedge v=0$. Hence (X, t^*) is supra Hausdorff space.

Again suppose that $(X, \omega(t^*))$ is $FST_2(v)$ and x, y $\in$X, $x \neq y$, there exist u $\in \omega(t^*)$ such that u(x) =1 and $\bar{u}(y)<1$, Suppose $\alpha \in I_{0,1}$ then $\bar{u}(y)<\alpha<1$ and $x \in u^{-1}(\alpha,1] \in t^*$. Again we have $u^{-1}(\alpha, 1] \subseteq \overline{u^{-1}(\alpha,1)} \subseteq (\bar{u})^{-1}(\alpha, 1]$. Now since $\bar{u}(y)<\alpha<1$, so by definition of lower semi continuous function, $y \notin (\bar{u})^{-1}(\alpha, 1]$, and hence $y \notin \overline{u^{-1}(\alpha,1)}$ .Therefore (X, t^*) is supra Hausdorff space.

Finally suppose that $(X, \omega(t^*))$ is $FST_2(vi)$ and x, y $\in$X, $x \neq y$, there exist u, v $\in \omega(t^*)$ such that u(x)=1=v(y), u(y)=0=v(x) and $u \subset \bar{v}$ i.e $u \wedge v=0$. Suppose $\alpha, \beta \in I_{0,1}$ such that $u(x)>\alpha$, and $v(y)>\beta$ then $u^{-1}(\alpha,1]$, $v^{-1}(\beta,1] \in t^*$ and $x \in u^{-1}(\alpha,1]$, $y \notin u^{-1}(\alpha,1]$ again $x \notin v^{-1}(\beta,1]$, $y \in v^{-1}(\beta,1]$. Moreover $u^{-1}(\alpha, 1] \wedge v^{-1}(\beta,1]=\phi$ and if not let $z \in u^{-1}(\alpha,1] \wedge v^{-1}(\beta,1]$ then $z \in u^{-1}(\alpha, 1]$ and $z \in v^{-1}(\beta,1]$ then $(u \wedge v)(z)>0$. Contradicting that $u \wedge v=0$. Hence (X, t^*) is supra Hausdorff space. The proofs are similar for other cases.

6.3. Initiality, hereditary, and productive properties of FST_2 spaces.

6.3.1. Theorem: The properties FST_2 (k) , k $\in$ {i, ii, iii, v, vi, vii} are initial.

Proof : (a) Let $\{(X_i, t_i^*)_{i \in J}\}$ be a family of FST_2 (i) , and $\{ f : X \rightarrow (X_i, t_i^*)_{i \in J} \}$ be a family of functions which separates by points and t^* be the initial fuzzy supra topology on X induced by the family $\{ f_i : i \in J\}$. Let x, y$\in$ X, $x \neq y$ and since $(X_i, t_i^*)_{i \in J}$ FST_2 (i) then there exist u, v $\in t^*$ such that u(x) = 1 = v(y). We can find basic t^*- supra open sets u_i i$\in$J such that u= Sup{ u_i i$\in$J }. Also u_i must be expressible as u_i = Inf { $f_{ik}^{-1}(u_{ik})$: $1 \leq k \leq n$} where $u_{ik} \in t_{ik}^*$ and ik$\in$J. Now we can find some k, $1 \leq k \leq n$} say k_1 such that

$f_{ik_l}^{-1}(u_{ik_l})(x) = 1, \Rightarrow u_{ik_l}\, f_{ik_l}(x) = 1$. Since (X_{ik_l}, t_{ik}^{*}) is , FST_2 (i) there exists $V_{i_{k_1}} \in t_{ik_1}^{*}$ such that $V_{i_{k_1}} f_{ik_l}(y)=1$, $\Rightarrow f_{ik_l}^{-1}(v_{i_{k_1}})(y)=1$, Put V= $f_{ik_1}^{-1}(v_{i_{k_1}}) \in t^{*}$. Thus v(y) =1.

Hence (X, t^{*}) is FST_2 (i) .So the property of FST_2 (i) is also initial.

(b) Let $\{(X_i, t_i^{*})_{i\in J}\}$ be a family of FST_2 (v) , and $\{ f: X \rightarrow (X_i, t_i^{*})_{i\in J}\}$ be a family of functions which separates by points and t^{*} be the initial fuzzy supra topology on X induced by the family $\{ f_i : i \in J\}$. Let Let x, y∈ X, x≠y and $\alpha, \beta \in I_0$ since $\{ f_i : i \in J\}$ separates by points then there exist j ∈J such that $f_j(x) \neq f_j(y)$. As (X_j, t_j^{*}) is FST_2 (v) , there exist $u_j \in t_j^{*}$ such that $\alpha \lesssim u_j\, f_j(x) = f_j^{-1}(u_j)(x)$ and $\beta \lesssim \overline{u_j}\, f_j(y) = f_j^{-1}(\overline{u_j})(y)$. $\overline{f_j^{-1}}(u_j) \leq f_j^{-1}(\overline{u_j})$ .Then u∈ t^{*} that $\alpha \lesssim u(x)$ and $\beta \not\lesssim \overline{u(y)}$ implying and (X, t^{*}) is FST_2 (v) .So the property of FST_2 (v) is also initial.

Similarly we can proof the other.

6.3.2. Theorem: All the properties $FST_2(k)$ of subspace topology where (i≤ k ≤ vii) are hereditary.

Proof: Consider the fsts (X, t^{*}), Let A⊂X. where t_A^{*} = {u∧A: u ∈ t^{*}}, We have to show that, if (X, t^{*}) has $FST_2(k)$ (i≤ k ≤ vii) then the subspace (A, t_A^{*}) has $FST_2(k)$ (i≤ k ≤ vii).

(1). Let x, y ∈ A, x ≠y, so that x, y ∈ X as A⊂X, since (X, t^{*}) is FST_2(i), so for every x, y ∈X, x≠y, there exist u, v ∈t^{*} such that u(x)= 1= v(y) and u∧v=0. Again from definition of subspace we have u∧A, v∧ A∈t_A^{*} , (u∧ A)(x) = 1, (v∧ A)(y)=1 and (u∧ A)∧ (v∧A) = $(u \wedge v) \wedge A \in t_A^{*}$, with $(u \wedge v) \wedge A = 0$, as x, y∈ A. This implies that, (A, t_A^{*}) has FST_2(i).

(2). Let x, y ∈ A, x ≠y, so that x, y ∈ X as A⊂X, since (X, t^{*}) is FST_2(ii), so, for every x, y ∈X, x≠y, and for every $\alpha, \beta \in I_0$, there exist u, v ∈t^{*} such that $u(x) > \alpha$, $v(y) > \beta$ and u∧v=0. Again from definition of subspace we have u∧ A, v∧A ∈t_A^{*} and (u∧ A)(x) >α, (v∧ A)(y)>α , and (u∧ A)∧ (v∧A) = $(u \wedge v) \wedge A \in t_A^{*}$, with $(u \wedge v) \wedge A$

= 0, as x, y$\in$ A, and since u and v are disjoint , so u$\wedge$ A and v$\wedge$A are also disjoint. This implies that, (A, t_A^*) hasFST$_2$(ii).

All other proofs are similar and easier and hence omitted.■

6.3.3. Theorem :- Given { (X_i , t_i^*) , i $\in$ J } be fuzzy supra topological spaces and X = $\Pi_{i \in J}$ X_i and t^* be the product fuzzy topology on X . Then ($\forall$ i $\in$ J, (X_i , t_i^*) is an FST$_2$(k) $\Leftrightarrow$ (X , t^*) is an FST$_2$(k) . where i$\leq$ k$\leq$vii)

Proof: (1) Suppose $\forall$ i$\in$J, (X_i , t_i^*) be an FST$_2$(iii) . We shall prove that (X, t^*) is FST$_2$(iii). Let x, y be two distinct points in X = $\Pi_{i \in J}$ X_i , then there exist an $x_i \neq y_i$ in X_i . Since (X_i , t_i^*) is an FST$_2$(iii) , $\exists$ u_i , $v_i \in t_i^*$ such that $u_i(x_i) > 0$, $v_i(y_i) > 0$ and $u_i \wedge v_i = 0$. But we have $\pi_i(x) = x_i$, $\pi_i(y) = y_i$, then $u_i(\pi_i(x)) > 0$, $v_i(\pi_i(y)) > 0$ and ($u_i \wedge v_i$) o $\pi_i \geq 0$. Hence (u_i o π_i) (x) > 0 , (v_i o π_i) (y) > 0 and (u_i o π_i) $\wedge$ (v_i o π_i) = 0.Put u = u_i o π_i , v = v_i o π_i , then u , v $\in$ t* with u(x) > 0 , v(y) > 0 and u $\wedge$v = 0. Hence it is clear that (X, t^*) is T$_2$(iii) .

Conversely, suppose that (X, t^*), is FST$_2$(iii). We shall prove that (X_i, t_i^*) is FST$_2$(iii), for i $\in$ J . For some i $\in$ J, let a_i be a fixed element in X_i . Suppose that A_i = { x $\in$ X = $\Pi_{i \in J}$ X_i : $x_j = a_j$ for some i $\neq$ j }. Then A_i is a subsets of X and therefore (A_i, $t^*_{A_i}$) is a subspace of (X, t^*), Since (X, t^*), is FST$_2$(iii) space. Then, we have also (A_i,$t^*_{A_i}$) is also FST$_2$(iii) space . Furthermore, A_i is homeomorphic image of X_i. Hence it is clear that (X_i, t_i *) is FST$_2$(iii) space.

(2) Suppose $\forall$ i $\in$ J,(X_i , t_i^*) be an FST$_2$(vi) . We shall prove that (X, t^*) is FST$_2$(vi). Let x, y be two distinct points in X = $\Pi_{i \in J}$ X_i, then there exist an $x_i \neq y_i$ in X_i. Since (X_i , t_i^*) is an FST$_2$(vi), $\exists$ u_i , $v_i \in t_i^*$ such that $u_i(x_i)$ = 1=$v_i(y_i)$, $u_i(y_i)$ =0= $v_i(x_i)$ and $u_i \subset \overline{v_i}$ But we have $\pi_i(x) = x_i$, $\pi_i(y) = y_i$,then $u_i(\pi_i(x))$ =1=$v_i(\pi_i(y))$; $u_i(\pi_i(y_i))$ =0 = $v_i(\pi_i(x_i))$ and $u_i \circ \pi_i \subset \overline{v_i}$ o π_i. Hence (u_i o π_i) (x) =1 = (v_i o π_i) (y) , (u_i o π_i) (y) =0 = (v_i o π_i) (x) and $u_i \circ \pi_i \subset \overline{v_i}$ o π_i. .Put u = u_i o π_i , v = v_i o π_i , then u , v $\in$ t* with u(x)=1= v(y), u(y) =0 = v(x) and u $\subset \overline{v}$. Hence it is clear that (X, t^*) is T$_2$(vi).

Conversely, suppose that (X, t^*), is $FST_2(vi)$. We shall prove that (X_i, t_i^*) is $FST_2(vi)$, for $i \in \Lambda$ .For some $i \in J$, let a_i be a fixed element in X_i. Suppose that $A_i = \{ x \in X = \Pi_{i \in J} X_i : x_j = a_j \text{ for some } i \neq j \}$. Then A_i is a subsets of X and therefore $(A_i, t^*_{A_i})$ is a subspace of (X, t^*), . Since (X, t^*), is $FST_2(vi)$ space, we have also $(A_i, t^*_{A_i})$ is also $FST_2(vi)$ space . Further more, A_i is homeomorphic image of X_i. Hence it is clear that (X_i, t_i^*) is $FST_2(vi)$ space.

Other proofs are similar.

6.4. Mapping between two FST_2 Spaces.

6.4.1. Theorem:- Let (X, t^*) and (Y, s^*) be two fuzzy supra topological spaces and $f : X \to Y$ be a one-one, onto and supra open map, then $(X, t^*)$is $FST_2(k) \Rightarrow (Y, s^*)$ is $FST_2(k)$, $k \in \{i, ii, iii, iv, vi\}$.

Proof:- Suppose $(X, t^*)$is $FST_2(i)$. We shall prove that (Y, s^*) is $FST_2(i)$. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is onto then, $\exists\, x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$ and $x_1 \neq x_2$ as f is one-one. Again since $(X, t^*)$is $FST_2(i)$, there exist $u, v \in t^*$ such that $u(x)= 1 = v(y)$ and $u \wedge v = 0$.

Now $f(u)(y_1) = \{ \text{Sup } u(x_1) \;:\; f(x_1) = y_1 \}$

$= 1$

$f(v)(y_2) = \{ \text{Sup } v(x_2) \;:\; f(x_2) = y_2 \} = 1$

and $f(u \cap v)(y_1) = \{ \text{Sup } (u \cap v)(x_1) \;:\; f(x_1) = y_1 \} = 0$

$f(u \cap v)(y_2) = \{ \text{Sup } (u \cap v)(x_2) \;:\; f(x_2) = y_2 \} = 0$

Hence $f(u \cap v) = 0 \Rightarrow f(u) \cap f(v) = 0$.

Since f is supra open then $f(u)$, $f(v) \in s^*$. Now it is clear that $\exists\, f(u)$, $f(v) \in s^*$ such that $f(u)(y_1) = 1$, $f(v)(y_2) = 1$ and $f(u) \cap f(v) = 0$. Hence (Y, s^*) is $FST_2(i)$.

Other proofs are similar.

6.4.2. Theorem: Let f and g be continuous function of a fuzzy supra topological space (X, t^*) into a fuzzy supra Hausdorff space (Y, s^*), where (Y, s^*) be $FST_2(p)$ $p \in \{i, ii, iii, iv, v, vi, vii\}$. let A be a fuzzy dense subset of X , and let $f(x)=g(x)\ \forall x \in A$, then f=g.

Proof: We know that $FST_2(i) \Rightarrow FST_2(ii) \Rightarrow FST_2(iii)$; $FST_2(i) \Rightarrow FST_2(iv) \Rightarrow$

$FST_2(v)$ also $FST_2(i) \Longrightarrow FST_2(vi) \Rightarrow FST_2(vii)$. We shall prove the theorem only for the case of $FST_2(i)$, and $FST_2(vi)$.

Given f and g be continuous function of a fuzzy supra topological space (X, t^*) into a fuzzy supra Hausdorff space (Y, s^*) and A be a fuzzy dense subset of X. We have to proof that f=g. Suppose (Y, s^*) is $FST_2(i)$, If $f \neq g$ then for $x \in X\text{-}A$, $f(x) \neq g(x)$, and there exist u, v $\in t^*$ such that $uf(x)= 1$, $vg(y)=1$ also $u \wedge v=0$. Now since f, g continuous, so $f^1(u)$, $g^{-1}(v) \in t^*$, and hence $f^1(u) \wedge g^{-1}(v) \in t^*$. Also Sup $(f^1(u) \wedge g^{-1}(v)) =1$, since A is dense $(f^1(u) \wedge g^{-1}(v))(a) \neq 0$ for some $a \in A$, this contradicts '$u \wedge v=0$' since $f(a)=g(a)$ Next let (Y, s^*) is $FST_2(vi)$, If $f \neq g$ then $\exists$ $x \in X\text{-}A$, $f(x) \neq g(x)$, and there exist u, v $\in t^*$ such that $uf(x)= 1$, $vg(y)=1$; $uf(y) =0$, $vg(x)=0$; and $u \subset \bar{v}$ or $u \wedge v=0$. Now since f, g continuous, so $f^1(u)$, $g^{-1}(v) \in t^*$, and hence $f^1(u) \wedge g^{-1}(v) \in t^*$. Also Sup $(f^1(u) \wedge g^{-1}(v)) =1$, since A is dense in (X, t^*), $(f^1(u) \wedge g^{-1}(v))(a) \neq 0$ for some $a \in A$, This contradicts '$u \wedge v=0$' since $f(a) =g(a)$.

6.5. Some α- Types of fuzzy supra Hausdorff spaces.

6.5.1 Definition [15]: Let (X, t*) be a fuzzy supra topological space and $\alpha \in I_1$ then (X, t*) is

$\alpha-$**FST₂(i) :** iff for every x, $y \in X$, $x \neq y$, there exist u, $v \in t^*$ such that $u(x)= 1= v(y)$ and $u \wedge v \leq \alpha$ (α^* -$FST_2(i)$, if $u(x)= 1= v(y)$ and $u \wedge v< \alpha$)

$\alpha-$**FST₂(ii) :** iff for every x, $y \in X$, $x \neq y$, there exist u, $v \in t^*$ such that $u(x)>\alpha$,

$v(y)>\alpha$ and $u \wedge v =0$. (α^* -$FST_2(ii)$, if $u(x) \geq \alpha$, $v(y) \geq \alpha$ and $u \wedge v=0$)

$\alpha-$**FST₂(iii):** iff for every x, $y \in X$, $x \neq y$, there exist u, $v \in t^*$ such that $u(x)>\alpha$, $v(y)>\alpha$ and $u \wedge v \leq \alpha$, (α^*-$FST_2(iii)$, if $u(x) \geq \alpha$, $v(y) \geq \alpha$ and $u \wedge v< \alpha$)

6.5.2. Definition: Let (X, t*) be a fuzzy supra topological space and $\alpha \in I_1$ or I0 then the family t^*_α= {α(u): u∈t*} of all subsets of X of the form $\alpha(u)$ ={x∈X: $u(x)>\alpha$}, or

α^*(u) ={x∈X: u(x)≥ α}, forms a fuzzy supra topology on X, this fuzzy supra topology is called α-level or α^*-level fuzzy supra topology on X respectively. **[18]**

6.5.3. Lemma: Let (X, t*) be a α- FST_2(p) space, where p∈{i, ii, iii}, then (X, t_α^*) is T_2 space.

Proof: (1) Let, there exist u, v ∈ t* such that u(x) = 1= v(y) and u∧v < α, hence there exist u, v ∈ t*such that u(x)≥ α, v(y)≥ α and u∧v < α . Therefore (X, t*) is α^*-FST_2(i).From definition α^*(u) and α^*(v) are supra open in t_α^* and x∈ α^*(u), y∈ α^*(v) also α^*(u)∧ α^*(v) <α, since u∧v < α .Hence (X, t_α^*) is α^*-level supra T_2 space.

(2) Let x, y∈ X, x≠y, there exist u, v ∈t* such that u(x) >α , v(y)>α and u∧v =0. Therefore (X, t*) is α- FST_2(ii). From definition α(u) and α (v) are supra open in t_α^* and x∈ α (u), y∈ α(v) also α(u)∧ α(v) =0 , since u∧v =0 .Hence (X, t_α^*) is α-level supra T_2 space.

Other proofs are same.

CHAPTER- VII

Fuzzy supra Regular Topological spaces

7. Introduction:

The concepts of Fuzzy regularly spaces were studied by several authors, e.g. Ghanim et al. [26], Hutton, B., and Reilly, I., [30], Sarkar, M., [52], Benchalli, S.S., and Malghan, S.R., [19], Wang, G.J., [60] and Ali, D. M., [6, 13]. In this chapter, we introduce and study some properties of Fuzzy supra regular spaces. First we note that for x∈X, $\lambda \in I^X$ then $\alpha \in I_0 = (0, 1]$, $\alpha \lesssim \lambda(x)$ means that $\alpha < \lambda(x)$ if $\alpha \neq 1$ and $\lambda(x) = 1$ if α=1and symbolize FSR for Fuzzy Supra regular topological spaces.

7.1. Definitions of FSR

7.1.1. Definition: An (X, t^*) is called

FSR (i) ⇔ $\alpha \in I_0$, $\lambda \in t^{*c}$, x∈X and $\alpha \lesssim 1-\lambda(x)$ imply that ∃ u, v $\in t^*$ with $\boldsymbol{\alpha} \lesssim u(x)$, $\lambda \leq v$ and $u \leq 1-v$. [M.H. **Ghanim, [25]**].

FSR(ii) ⇔ $\alpha \in I_0$, $\lambda \in t^{*c}$, x∈X and $\alpha \lesssim 1-\lambda(x)$ imply that ∃ u, v $\in t^*$ with $\alpha \lesssim u(x)$, $\lambda \leq v$ and $u \leq 1-v$. [**Sarkar , M.,[52]**]

FSR (iii) ⇔ each u $\in t^*$ is a supremum of u_j j∈J, where for each j∈J, $u_j \in t^*$ and $\overline{u_j} \lessdot u$. [**Hutton, B., and Reilly, I., [30]**].

FSR(iv) ⇔ $\lambda \in t^{*c}$, x∈X and $\lambda(x)=0$, imply that ∃ u, v $\in t^*$ with u(x) =1, $\lambda \leq v$ and $u \leq 1-v$. [**Benchalli, S.S., and Malghan, S.R., [19]**]

FSR(v) ⇔ $\lambda \in t^{*c}$, x∈X and $1-\lambda(x) > 0$, imply that ∃ u, v $\in t^*$ with u(x)>0, $\lambda \leq v$ and $u \leq 1-v$. [**D.M. Ali [10]**].

FSR(vi) ⇔ $\alpha \in I_0$, $\lambda \in t^{*c}-\{0\}$, $\lambda(x) > 0$, x∈X ,imply that ∃ u, v $\in t^*$ with $\boldsymbol{\alpha} > 1-u(x)$, $\lambda(y) > 1-v(y)$ ∀ y ∈ $\lambda^{-1}(0, 1]$, and $u \wedge v = 0$. [**Wang, G.J., [60]**].

FSR (vii) ⇔ $\alpha \in I_0$, u $\in t^*$, x∈X and $\alpha \lesssim u(x)$ imply that ∃ v $\in t^*$ with $\alpha \lessdot v(x)$, and $\bar{v} \leq u$. [**Adnadjevic D. [3]**].

FSR (viii) $\Leftrightarrow$ $\alpha \in I_0$, u $\in t^*$, x$\in$X and 1- $\alpha \lesssim$ u(x) imply that $\exists$ u, v $\in t^*$ with $\alpha \lesssim$u(x), $\lambda \subseteq$v and u$\wedge$v=0.

FSR (ix) $\Leftrightarrow$ $\alpha \in I_0$, $\lambda \in t^{*}c$, x$\in$X and $\alpha <1-\lambda(x)$ imply that $\exists$ u, v $\in t^*$ with $\alpha <$u(x), $\lambda \leq$v and u $\leq$1-v.

7.1.1. Theorem: The following implications are hold for FSR.

FSR(viii)

$\Uparrow$

FSR(i)$\Rightarrow$ FSR(vii)$\Leftrightarrow$ FSR(ii) $\Rightarrow$ FSR(ix)$\Leftrightarrow$ FSR(iii) $\Leftrightarrow$ FSR(v)

$\Downarrow$

FSR(iv)

Proof:

<u>FSR(i)$\Rightarrow$ FSR(vii)</u>

Let $\alpha \in I_0$, u $\in t^*$, x$\in$X and $\alpha \lesssim$ u(x) $\Rightarrow \alpha <$u(x) if $\alpha \neq 1$, and u(x) = 1, if $\alpha =1$, as $\alpha \in I_0$, clearly $\alpha \lesssim 1- \lambda(x)$, $\lambda \in t^{*c}$. Since (X, t^*) is FSR (i) space then $\alpha \in I_0$, $\lambda \in t^{*c}$, x$\in$X and $\alpha \lesssim 1-\lambda(x)$ imply that $\exists$ u, v $\in t^*$ with $\boldsymbol{\alpha} \lesssim$u(x), $\lambda \leq$v and u $\leq$1-v.And so clearly $\exists$ v$\in t^*$ with $\alpha \preccurlyeq$v(x), and $\bar{v} \leq$u.

FSR (vii)$\Leftrightarrow$ FSR (ii)

Let $\alpha \in I_0$, $\lambda \in t^{*c}$, x$\in$X and $\alpha \lesssim 1-\lambda(x)$. Put 1-$\lambda$= u then $u \in t^*$ and $\alpha \lesssim u$ (x). As(X, t^*) is FSR(vii) space then $\alpha \in I_0$, u $\in t^*$, x$\in$X and $\alpha \lesssim$ u(x) imply that $\exists$ v $\in t^*$ with $\alpha \preccurlyeq$v(x), and $\bar{v} \leq$u. So clearly (ii) imply that $\exists$ v$\in t^*$with $\alpha \preccurlyeq$v(x) and $\bar{v} \leq u$. Setting 1-$\bar{v}$ = v, then v$\in t^*$and λ = 1- $u \subseteq$1-$\bar{v}$ = v. i.e $\lambda \leq$v. Also we have u $\subseteq \bar{v}$=1-u, i.e. v$\leq$1-u. Hence clearly (X, t^*) is FSR (ii). Similarly, we can show converse.

FSR (ii)$\Rightarrow$ FSR(viii)

Let $\alpha \in I_0$, u$\in t^*$, x$\in$X and 1 - $\alpha \lesssim$ u(x), Putting, u =1-λ, then $\lambda \in t^{*c}$ and 1- $\alpha \lesssim$ u(x), i.e. $\alpha \gtrsim 1 - u(x), i.e\ \alpha \gtrsim \lambda(x)$, and hence $\alpha \lesssim 1-\lambda(x)$. Since (X, t^*) is FSR(ii) space $\Rightarrow \alpha \in I_0$, $\lambda \in t^{*c}$, x$\in$X and $\alpha \lesssim 1-\lambda(x)$ imply that $\exists$ u, v $\in t^*$ with $\alpha \lesssim$u(x), $\lambda \leq$v and u $\leq$1-v.So clearly $\lambda \subseteq$v and u$\wedge$v=0.

FSR (ii) $\Rightarrow$ FSR(ix)

Let, $\alpha \in I_0$, $\lambda \in t^{*c}$, x$\in$X and $\alpha <1-\lambda(x)$, also $\alpha \lesssim \lambda(x)$ means that $\alpha < \lambda(x)$ if $\alpha \neq 1$ and $\lambda(x)$ = 1 if α=1. i. e. $\lambda(x) = 0$ if α = 1 .Since (X, t^*) is FSR (ii) space $\Rightarrow$ $\alpha \in I_0$,

$\lambda \in t^{*c}$, x∈X and $\alpha \lesssim 1-\lambda(x)$ imply that ∃ u, v ∈t^* with $\alpha \lesssim u(x)$, $\lambda \leq v$ and $u \leq 1-v$. So clearly, for $\alpha \neq 1$, $\alpha < u(x)$ and $\lambda \leq v$, $u \leq 1-v$.

FSR (ix) ⇔ FSR (iii)

Let $\alpha \in I_0$, $\lambda \in t^{*c}$ and x∈X be such that $\alpha < 1-\lambda(x)$, Now by (iii) ∃ $u_j \in t^*$, j∈J with $\overline{u_j}$ < 1-λ (x), for each j∈J and 1-λ = $\sup_{j \in J}$ .Clearly for some j∈J, $u_j(x) > \alpha$. Setting v =1- $\overline{u_j}$ so that v∈$t^*$and $u_j \leq 1-v$, Moreover $\lambda \leq 1- \overline{u_j}$ =v, imply that FSR (ix) ⇒FSR (iii). Conversely let (X, t^*) is FSR (iii) and u∈t^*- {0}, then, u= Sup $\{\alpha \wedge 1_x : \alpha \in I_0, u(x)>\alpha\}$, Now for $\alpha \in I_0$, $u_\alpha(x)>\alpha$ then $\exists\, u_\alpha(x)$, v∈t^*,$\alpha < u_\alpha(x)$, 1-u≤v and $u_\alpha(x) \leq 1$-v,

Hence $\bar{u_\varepsilon} \leq 1-v \leq u$, then u = Sup{ $u_\alpha(x)$: u(x)>α , x∈X, $\alpha \in I_0$}.

Hence FSR (iii) ⇒ FSR (ix), so we conclude that, FSR (ix) ⇔ FSR (iii) ■

Similarly, we prove the other conditions.

N.B: Since the formulation of FSR (vi) is different from others, so we shall not include this in our present study.

7.1.1. Few examples to show the non- implications among FSR

Example (a) FSR (i)⇏FSR(vi)

Let X={x, y}, and u, v ∈I^X, with u = {(x, 0), (y, .5)}, v = {(x, 1), (y, .5)}. Let, t^*= { 0, 1, u = {(x, 0), (y, .5)}, v = {(x, 1), (y, .5)}} be a fuzzy supra topology on X. Then it is easy to show that (X, t^*) is FSR (i). Now we choose pseudo crisp $\lambda \in t^{*c}$ - {0}, with $\lambda(x) = 0$, Then for any α>0, there does not exist u, v ∈ t^* with α> 1-u(x), $\lambda(y)$> 1-v(y) for y∈ $\lambda^{-1}(0,1]$ and u∧v= 0, Thus (X, t^*) is not FSR (vi).

Example (b) FSR(iii)⇏FSR(iv) [7]

Let X be a set with at least two elements and, t^*= {u∈I^X: $u^{-1}\{0\} \neq \phi \Rightarrow u = 0$} be a fuzzy supra topology on X. For a fix x∈X and $\alpha \in I_{0,1}$, let $\lambda = 1_{X-\{x\}}$. Then $\lambda \in t^{*c}$ and $\lambda(x) = 0$, but there does not exist v, w ∈ t^* with v(x) =1, $\lambda \leq$w, v≤1-w as 1-w(x)=1 ⇒ w(x)= 0 ⇒ w = 0. Thus (X, t^*) is not FSR (iv).However it can easily verified that (X, t^*) is FSR(iii).

Example (c) FSR(iv)⇏FSR(v)

Let X={x, y}, with a fuzzy supra topology FSR (iv) space, let, t^* = {0, 1, u = {(x, 1), (y, .25)}, v = {(x, 0), (y, .65)}, w = {(x, 1), (y, .65)}} be a fuzzy supra topology on X. Then the class of all fuzzy supra closed sets of $t^*$is t^{*c}= {1, 0, {(x, 0), (y, .75)}, {(x, 1), (y, .35)}, λ = {(x, 0), (y, .35)}} clearly (X, t^*) is FSR (iv). Here $\lambda \in t^{*c}$, x∈X and $\lambda(x) = 0 \Rightarrow \exists$ v, u ∈ t^* u(x) =1, $\lambda \leq$v, u≤1-v. But (X, t^*) is not FSR (v) for if λ= 1-u then 1-λ(y) >0, but ∄ v, w∈t^*, ∋ $\lambda \in t^{*c}$, x∈X and 1- λ(x) > 0 ⇒ v(x)>0, $\lambda \leq$w and v ≤1-w.

Example (d) FSR(v) ⇏FSR(iv)

Let X={x, y}, with a fuzzy supra topology, t^* = {0, 1, u = {(x, .8), (y, .5)}, v = {(x, .2), (y, .5)}} on X, then if λ= 1-u then λ(x) =.2, λ(y) =.5, then $\lambda \in t^{*c}$ $\lambda \leq$v and u≤1-v. So (X, t^*) is FSR (v) but ∄ no such λ for which and λ(x) = 0. Hence (X, t^*) is not FSR (iv).

Example (e) FSR(v)⇏FSR(iii)

Let (X, t^*) be a fuzzy supra topological space, and X={x, y} where t^*= < {w}∪ {Constant} > . w is defined as w(x) =.8 and w(y)=1; Now if λ= 1-w , 1 − λ(x) >0, $\lambda \in t^{*c}$, x∈X then ∃ u, v ∈t^* with u(x)>0, $\lambda \leq$v and u ≤1-v So (X, t^*) is FSR (v) . Now if α = .85 , then α < 1- λ(y), but $t^*$is not FSR (iii).because if u is constant u(x) = 1, u(y) = 1; since u ≮ 1-v.

Example (f) FSR(ix)⇏FSR(ii)

Let X={x, y}, with a fuzzy supra topology, t^* = {0, 1, u = {(x, 1), (y, .25)}, v = {(x, 0), (y, .65)}, w = {(x, 1), (y, .65)}} on X. let λ=1-u, α= .25, α <1-λ(x), and α <u(x), $\lambda \leq$v and u ≤1-v. but if α=1, α ≴u(x),but λ ≰w and u ≰1-w.

7.1.2. Non-implications: In general topology it is always true that "Every supra regular space is supra R_1 space, but it is not true in fuzzy supra regular topological spaces."

Example (a) : Let X={a, b}, with a fuzzy supra topology , t^* = {0, 1, u = {(a, .6), (b, .4)}, v = {(a, .4), (b, .5)}, w = {(a, .6), (b, .5)}} on X . If λ=1-u, then λ(a) =.4, λ(b) =.6 ; then $\lambda \in t^{*c}$, and 1- λ(a) >0 with u(a)>0, $\lambda \leq$v and u$\leq$1-v, so (X, t^*) is FSR (v) but (X,

t^*) is not FSR_1 (xvi) because v(a)≠v(b), and u, w ∈t^*, such that u⊆ 1-w but u(a)≠1≠ w(b)

Example (b) : Let X={a, b}, with a fuzzy supra topology , t^* = {0, 1, u = {(a, 1), (b, .5)}, v = {(a, 0), (b, .5)}, w = {(a, 1), (b, .5)}} on X . Then the class of all fuzzy supra closed sets of $t^*$is t^{*c}= {1, 0, λ = {(a, 0), (b, .5)}, {(a, 1), (y, .35)}, {(a, 0), (b, .35)}}, clearly (X, t^*) is FSR (iv). But (X, t^*) is not FSR_1 (iv) because v(a) ≠v(b), and u, w ∈t^*, such that u ⋢1-w but u(a)=1≠ w(b).

Example (c) : Let X={a, b}, with a fuzzy supra topology , t^* = {0, 1, u = {(a, .6), (b, .4)}, v = {(a, .4), (b, .5)}, w = {(a, .6), (b, .5)}} on X . If λ=1-u, then λ(a) =.4, λ(b) =.6 ; then $\lambda \in t^{*c}$, and 1- λ(a) >0 with u(a)>0, $\lambda \leq$v and u$\leq$1-v, so (X, t^*) is FSR (ix) but(X, t^*) is not FSR_1 (xvi) because v(a) ≠ v(b), and u, w ∈t^*, u(a)>0, w(b)>0, u∧w ≠ 0.
Similarly, we can show the other cases.

7.2. Initial, productive and hereditary properties of FSR.

7.2.1. Theorem: The Properties (iii), (v) and (ix) are initial.

Proof: (a) Let $\{(X_j , t_j^*) : j \in J\}$ be a family of FSR (iii), and $\{ f_j: X \to (X_j, t_j^*) : j \in J\}$ be a family of functions and t^* be the initial fuzzy topology on X induced by the family $\{f_j : j \in J \}$. Let x ∈ X, $\alpha \in I_{0,1}$ and u∈t^* such that u(x)> $\alpha \in I_{0,1}$, let u∈t^*, there exist basic t^* -supra open set, u_p such that u= sup {u_p :p∈j}. Also each must be expressible as $u_p = \inf \{ f_{p_k}^{-1} u_{p_k} : p_k \in J , 1\leq p \leq n\}$, we can find some k (1≤k ≤n) say k' such that $\inf_{1 \leq k \leq n} f_{p'_k}^{-1} u_{p'_k}(x) > \alpha$ and $\inf_{1 \leq k \leq n} f_{p'_k}^{-1} u_{p'_k}(x) \leq u$. This implies that $f_{p_{k'}}^{-1}$ $u_{p'_k}(x) > \alpha$, so that $u_{p'_k} f_{p_{k'}}^{-1}(x) > \alpha$. Since $(X_{p'_k}, t^*_{p'_k})$ is FSR (iii), there exist, $v_{p_{k'}}$ $\in t^*_{p_{k'}}$ such that $\alpha < v_{p_{k'}} f_{p_{k'}}^{-1}(x)$ and $v_{p_{k'}} \leq u_{p_{k'}}$ for all k, 1≤k ≤n, where $v_{p_{k'}}$ is a local base of closed α - nhds of $f_{p_{k'}}^{-1}(x)$. Therefore $f_{p_{K'}}^{-1} v_{p_{K'}}$, is closed. Now, we have $\inf_{1 \leq k \leq n} f_{p_{K'}}^{-1} v_{p_{K'}}(x) > \alpha$, and $\inf_{1 \leq k \leq n} f_{p_{K'}}^{-1} v_{p_{K'}}(x) \leq u$. Since each $v_{p_{k'}}$

being a member of $L_{p_{k^/}}$ is a closed fuzzy set. Thus $\{ f^{-1}_{p_{K^/}} v_{p_{K^/}} : v_{p_{k^/}} \in L_{p_{k^/}}, p_{k^/} \in J$ $\}$ is a local sub base of closed α - nhds of x. Hence (X, t^*) is an FSR(iii) space

.**(b)** Let $\{(X_j, t^*_j) : j \in J\}$ be a family of FSR (ix), and $\{ f_j : X \to (X_j, t^*_j) : j \in J\}$ be a family of functions and t^* be the initial fuzzy topology on X induced by the family $\{f_j : j \in J\}$. Let $\alpha \in I_0$, $\lambda \in t^{*c}$, x∈X and $\alpha < 1\text{-}\lambda(x)$. Let u∈$t^*$, there exist basic t^* -supra open set, u_p such that u= sup $\{u_p : p \in j\}$. Also each must be expressible as $u_p = \inf\{ f^{-1}_{p_k} u_{p_k} : 1 \leq p \leq n\}$ as , we can find some k (1≤k ≤n) say k/ such that $\inf_{1 \leq k \leq n} f^{-1}_{p^/_k} u_{p^/_k}(x) > \alpha$ and $\inf_{1 \leq k \leq n} f^{-1}_{p^/_k} u_{p^/_k}(x) \leq u$. This implies that $f^{-1}_{p^/_k} u_{p^/_k}(x) > \alpha$, so that $u_{p^/_k} f^{-1}_{p_{k^/}}(x) > \alpha$, Since $(X_{p^/_k}, t^*_{p^/_k})$ is FSR (ix), there exist $u_{p_{k^/}}$, $v_{p_{k^/}}$ $t^*_{p_{k^/}}$ such that $\alpha < u_{p_{k^/}}(x)$, $\lambda \leq v_{p_{k^/}}$ and $u_{p_{k^/}} \leq 1 - v_{p_{k^/}}$ .where $v_{p_{k^/}}$ is a local base of closed α - nhds of $f^{-1}_{p_{k^/}}(x)$. Therefore $f^{-1}_{p_{K^/}} v_{p_{K^/}}$, is closed. Therefore, $\inf_{1 \leq k \leq n} f^{-1}_{p_{K^/}} v_{p_{K^/}}(x) > \alpha$, and $\inf_{1 \leq k \leq n} f^{-1}_{p_{K^/}} u_{p_{K^/}}(x) \leq 1 - v_{p_{k^/}}$,Hence (X, t^*) is an FSR(ix) space .

Other proof is similar.

We do not yet know, whether the properties The properties FSR (i), FSR (ii), FSR (iv), FSR (vi), FSR (vii), FSR (viii) are initial and productive or not. But fuzzy supra regular topological spaces FSR (iii), FSR (v) and FSR (ix), are productive.

7.2.2. Theorem: Every subspace of a fuzzy supra regular space is fuzzy supra regular and hence fuzzy supra regular is hereditary.

Proof: Let (X, t^*) be a fuzzy supra topological space. Let A⊂X, where $t^*_A = \{u \wedge A : u \in t^*\}$, we have to show that, if (X, t^*) is FSR(i) then the subspace (A, t^*_A) has FSR(i) , let $\alpha \in I_0$, $\lambda \in t^{*c}_A$, x∈A and $\alpha \lesssim 1\text{-}\lambda(x)$, we shall prove that the subspace (A, t^*_A) has FSR(i). Given λ is a t^*_A closed set with $\alpha \lesssim 1\text{-}\lambda(x)$. Let $\overline{\lambda t^*}$ = Closure of λ with respect to the fuzzy supra topology t^* and

$\overline{\lambda t_A^*}$ = Closure of λ with respect to the fuzzy supra topology t_A^*. However, we know that $\overline{\lambda t_A^*} = \overline{\lambda t^*} \wedge A$. Since, λ is t_A^* closed $\Rightarrow \lambda = \overline{\lambda t_A^*} \Rightarrow \overline{\lambda t^*} \wedge A$, so $\alpha \lesssim 1-\lambda(x) \Rightarrow \alpha \lesssim 1-(\overline{\lambda t^*} \wedge A)(x)$.

Since (X, t^*) is FSR(i), so for $\alpha \in I_0$, $\overline{\lambda t^*} \in t^{*c}$, $x \in X$ and $\alpha \lesssim 1-\overline{\lambda t^*}(x)$ imply that $\exists$ u, v $\in t^*$ with $\alpha \lesssim u(x)$, $\overline{\lambda t^*} \leq v$ and

$$u \leq 1-v \qquad (1)$$

Again from definition of subspace $u \wedge A$, $v \wedge A \in t_A^*$, so for $\alpha \in I_0$, $\lambda \in t_A^{*c}$, $x \in A$ and $\alpha \lesssim 1-(\overline{\lambda t^*} \wedge A)(x) = 1-\overline{\lambda t_A^*}(x) \Rightarrow u \wedge A$, $v \wedge A \in t_A^*$, with $\alpha \lesssim (u \wedge A)(x)$, $\lambda = \overline{\lambda t^*} \wedge A \leq (v \wedge A)$ and $(u \wedge A) \leq 1-(v \wedge A)$, as $x \in X$. This implies that (A, t_A^*) is FSR(i). Similarly, we can show the hereditary, for the other definitions.

7.3. α- Fuzzy Supra Regular Spaces.

7.3.1. Definition: Let (X, t^*) be a fuzzy supra topological space and $\alpha \in I_1$,

(a) (X, t^*) is an α- FSR(i) space $\Leftrightarrow \forall\, w \in t^{*c}$, $\forall\, x \in X$, with $w(x) < 1$, $\exists$ u, v $\in t^*$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$.

(b) (X, t^*) is an α- FSR(ii) space $\Leftrightarrow \forall\, w \in t^{*c}$, $\forall\, x \in X$, with $w(x) < 1$, $\exists$ u, v $\in t^*$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$.

(c) (X, t^*) is an α- FSR(iii) space $\Leftrightarrow \forall\, w \in t^{*c}$, $\forall\, x \in X$, with $w(x) = 0$, $\exists$ u, v $\in t^*$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$.

(d) (X, t^*) is an α- FSR(iv) space $\Leftrightarrow \forall\, w \in t^{*c}$, $\forall\, x \in X$, with $w(x) = 0$, $\exists$ u, v $\in t^*$ such that $u(x) > \alpha$,, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$.

7.3.1. Theorem: The following implications are true:

(X, t^*) is α- FSR(i) ⇒ (X, t^*) is α- FSR(ii) ⇒ (X, t^*) is α- FSR(iv)

(X, t^*) is α- FSR(i) ⇒ (X, t^*) is α- FSR(iii) ⇒ (X, t^*) is α- FSR(iv)

Proof: First, suppose that (X, t^*) is α - FSR(i). We shall prove that (X, t^*) is α - FSR(ii). Let $w \in t^{*c}$, $x \in X$, with $w(x) < 1$. Since (X, t^*) is α- FSR(i), for $\alpha \in I_1$, $\exists$ u, v

$\in t^*$ such that u (x) = 1 , v(y) =1, $y \in w^{-1}$ {1} and u , v$\in t^*$ such that u(x)= 1, v(y) =1, y$\in$w^{-1}{1} and u$\wedge$ v $\leq\alpha$. Now we see that u(x)>α, v(y) =1, y$\in$w^{-1}{1} and u$\wedge$ v $\leq\alpha$. Hence it is clear that (X, t^*) is α- FSR(ii) .

Next suppose that (X, t^*) is α - FSR(i). We shall prove that (X, t^*) is α - FSR(iii). Let w $\in t^{*c}$, x $\in$ X, with w (x) = 0, then we have w (x) < 1. Since (X, t^*) is α- FSR(i), for $\alpha \in I_1$,$\exists$ u, v $\in t^*$ such that u (x) = 1 , v(y) =1, $y \in w^{-1}$ {1} and u$\wedge$ v $\leq\alpha$. Now it is clear that (X, t^*) is α- FSR(iii) .

Again suppose that (X, t^*) is α - FSR(ii). We shall prove that (X, t^*) is α - FSR(iv). Let w $\in t^{*c}$, x $\in$ X, with w (x) = 0. Then we have w (x) < 1. Since (X, t^*) is α- FSR(ii), for $\alpha \in I_1$,$\exists$ u, v $\in t^*$ such that u (x) >α , v(y) =1, $y \in w^{-1}$ {1} and u$\wedge$ v $\leq\alpha$. Hence it is clear that (X, t^*) is α- FSR(iv) .

Finally, suppose that (X, t^*) is α - FSR(iii). We shall prove that (X, t^*) is α - FSR(iv).

Let w $\in t^{*c}$, x $\in$ X, with w (x) =0. . Since (X, t^*) is α- FSR(iii), for $\alpha \in I_1$,$\exists$ u, v $\in t^*$ such that u (x) = 1 , , v(y) =1, $y \in w^{-1}$ {1} and u$\wedge$ v $\leq\alpha$. Hence, we see that (X, t^*) is α- FSR(iv) .

7.3.1. Non- implications among α-FSR.

Example (a) : α- FSR(ii) $\nRightarrow$ α- FSR(i)

Let X= {a, b}, with a fuzzy supra topology t^*, on X is generated by t^* = {0, 1, u = {(a, 0.9), (b, .5)}, v = {(a, .5), (b, 1)}, β = {(a, .9), (b, 1)}}. For w =1-u and α = 0.7, we see that (X, t^*) is α- FSR(ii) but (X, t^*) is not α- FSR(i).

Example (b) : α- FSR(iii) and α- FSR(iv) $\nRightarrow$ α- FSR(ii)

Let X= {x, y} and u, v $\in I^X$, where u, v are defined by u(x) = 1, u(y) = 0.7; v(x) = 0.8, v(y) = 1; w(x) = 1, w(y) = 0, let the fuzzy supra topology t^* on X generated by {0, u, v, w, 1} $\cup$ {constants}. For p=1-w and α = 0.9. We see that (X, t^*) is α- FSR(iii) and (X, t^*) is α- FSR(iv) but (X, t^*) is not α- FSR(ii).As there do not exist any w(x) <1 with u, v $\in t^*$ such that u(x)>α, v(y) =1, y$\in$w^{-1}{1} and u$\wedge$ v $\leq\alpha$.

Example (c) : α- **FSR(iv)** $\nRightarrow$ α- **FSR(iii)**

Let X= {x, y} and u, v, w $\in I^X$, where u, v and w are defined by u(x) = .9, u(y) = 0; v(x) = 0.5, v(y) = 1; w(x) = 1, w(y) = 0, let the fuzzy supra topology t^* on X generated by {0, u, v, w, 1} ∪ {constants}. For p=1-w and α = 0.6. We see that (X, t^*) is α-FSR(iv) but (X, t^*) is not α- FSR(iii).As there do not exist any w(x) = 0 with u, v $\in t^*$ such that u(x) = 1, v(y) = 1, y$\in w^{-1}${1} and u$\wedge$ v $\leq\alpha$. This completes the proof.

7.3.2. Theorem: Let $0 \leq\alpha \leq\beta <1$, then

(a) (X, t^*) is α – FSR (i) $\Rightarrow$(X, t^*) is β – FSR (i).

(b) (X, t^*) is α - FSR (iii) $\Rightarrow$(X, t^*) is β – FSR (iii).

Proof: First, suppose that (X, t^*) is α - FSR(i). We shall prove that (X, t^*) is β - FSR (i). Let w $\in t^{*c}$ and x $\in$ X, with w (x) < 1. Since (X, t^*) is α- FSR (i), for $\alpha \in I_1$, $\exists$ u, v $\in$ t^* such that u (x) = 1, v(y) =1, y $\in w^{-1}$ {1} and u $\wedge$v $\leq \alpha$. Since $\alpha \leq \beta$, then u $\wedge$v $\leq\beta$, so it is observed that (X, t^*) is β - FSR (i).

Next suppose that (X, t^*) is α – FSR (iii). We shall prove that (X, t^*) is β – FSR (iii). Let w $\in t^{*c}$, x $\in$ X, with w (x) = 0.Since (X, t^*) is α - FSR (iii), for $\alpha \in I_1$, $\exists$ u, v $\in t^*$ such that u (x) =1, u(y) = 1, y $\in w^{-1}$ {1} and u $\wedge$v $\leq \alpha$. Since $0 \leq\alpha \leq\beta <1$ then u $\wedge$v $\leq\beta$, Now it can be written as w$\in t^{*c}$, $\forall$ x$\in$ X, with w(x) =0, $\exists$ u, v$\in t^*$ such that u(x) = 1, v(y) =1, y$\in w^{-1}${1} and u$\wedge$ v $\leq \beta$. so it is observed that (X, t^*) is β - FSR (i).

7.3.2. Example: Let X= {x, y} and u, v $\in I^X$, where u, v are defined by u(x) = 1, u(y) = 0; v(x) = 0.7, v(y) = 1; let we consider the fuzzy supra topology t^* on X generated by {0, u, v, 1} ∪ {constants}. For w=1-u and α = 0.75. β =0.6. We see that (X, t^*) is β - FSR(i) and (X, t^*) is α- FSR(iii) but (X, t^*) is not α- FSR(i).and (X, t^*) is not α- FSR(iii). As there do not exist any w(x) <1 with u, v $\in t^*$ such that u(x)>α, v(y) =1, y$\in w^{-1}${1} and u$\wedge$ v $\leq\alpha$.

7.3.2. Definition: Let (X, T^*) be a supra topological space, this space is to be a supra regular space if:

Given an element x ∈X and closed set W⊂X s. t. x ∉W, ∃ disjoint open sets U, V⊂X s.t. x∈U, W⊂V.

7.3.3. Theorem: Let (X, t^*) be a fuzzy supra topological space and $I_\alpha(t^*) = \{u^{-1}(\alpha, 1]: u \in t^*\}$, then (X, t^*) is 0– FSR (i) ⇒$(X, I_0(t^*)))$ is Supra Regular. **[14]**

Proof: We consider (X, t^*) be a 0 – FSR (i). We shall prove that $(X, I_0(t^*)))$ is Supra Regular. Let V be a closed set in $I_0(t^*))$ and x ∈ X be such that x ∉V, then $V^c \in I_0(t^*))$ and x∈ V^c So by definition of $I_0(t^*))$, there exists an u ∈ t^* such that $V^c = u^{-1}(0, 1]$. i, e. u (x) >0 . Since u ∈ t^* then u^c is closed fuzzy set in $t^*$and $u^c(x) < 1$. Since (X, t^*) is 0 – FSR (i), ∃ v, w∈ t^* such that v(x) = 1, w≥ $1_{(u^c)^{-1}\{1\}}$ and v∧ w = 0.

(a) Since v, w∈ t^* then $v^{-1}(0, 1]$, $w^{-1}(0, 1] \in I_0(t^*))$ and x ∈ $v^{-1}(0, 1]$

(b) Since $w \geq 1_{(u^c)^{-1}\{1\}}$ then $w^{-1}(0, 1] \geq (1_{(u^c)^{-1}\{1\}})^{-1}(0,1]$

(c) v∧ w = 0, mean $(v\wedge w)^{-1}(0, 1] = v^{-1}(0, 1] \wedge w^{-1}(0, 1] = 0$

Now, we have

$(1_{(u^c)^{-1}\{1\}})^{-1}(0, 1] = \{x: 1_{(u^c)^{-1}\{1\}}(x) \in (0,1]\}$

$= \{x: 1_{(u^c)^{-1}\{1\}}(x) = 1\}$

$= \{x: x \in (u^c)^{-1}\{1\}\}$

$= \{x: u^c(x) = 1\}$

$= \{x: u(x) = 0\}$

$= \{x: x \notin V^c\}$

$= \{x: x \in V\}$

= V, now taking W= $v^{-1}(0, 1]$ and $W^* = w^{-1}(0, 1]$, then x ∈ W, $W^* \supseteq V$ and $W \cap W^* = \varphi$ .Hence it clear that $(X, I_0(t^*)))$ is Supra Regular.

7.3.4. Theorem: Let (X, t*) be a fuzzy supra topological space, A⊆X and t^*_A = {u∧ A: u ∈ t*} then $1_{((u\wedge A)c)-1\{1\}}(x) = (1(u^c)^{-1}\{1\} \wedge A)(x)$ ∀ x∈ X.

Proof: Let w be a closed fuzzy set in t^*_A i.e w$\in t^{*c}_A$, then $u\wedge A = w^c$, where u$\in t^*$

Now we have

$$1_{((u\wedge A)^c)^{-1}\{1\}}(x) = \begin{cases} 0 \text{ if } x \notin ((u\wedge A)^c)^{-1}\{1\} \\ 1 \text{ if } x \in ((u\wedge A)^c)^{-1}\{1\} \end{cases}$$

$$= \begin{cases} 0 \text{ if } x \notin \{ y : (u\wedge A)^c)(y) = 1\} \\ 1 \text{ if } x \in \{ y : (u\wedge A)^c)(y) = 1\} \end{cases}$$

$$= \begin{cases} 0 \text{ if } (u\wedge A)^c)(x) < 1\} \\ 1 \text{ if } (u\wedge A)^c)(x) = 1\} \end{cases}$$

$$= \begin{cases} 0 \quad \text{if} \quad w(x) < 1 \\ 1 \quad \text{if} \quad w(x) = 1 \end{cases}$$

Again $1_{(u^c)^{-1}\{1\}}(x) = \begin{cases} 0 \text{ if } x \notin (u^c)^{-1}\{1\} \\ 1 \quad \text{if } x \in (u^c)^{-1}\{1\} \end{cases}$

$$= \begin{cases} 0 \text{ if } x \notin \{y : u^c(y) = 1\} \\ 1 \text{ if } x \in \{ y : u^c(y) = 1\} \end{cases}$$

$$\begin{cases} 0 \quad \text{if } u^c(x) < 1 \end{cases}$$

$$= \quad 1 \quad \text{if } u^c(x) = 1$$

$$\text{Now } (1_{((u)^c)^{-1}\{1\}} \wedge A)(x) = \begin{cases} 0 & \text{if } (u^c \wedge A)(x) < 1 \\ 1 & \text{if } (u^c \wedge A)(x) = 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } (u \wedge A)^c(x) < 1 \\ 1 & \text{if } (u \wedge A)^c(x) = 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } w(x) < 1 \\ 1 & \text{if } w(x) = 1 \end{cases}$$

Hence it is clear that $1_{((u\wedge A)^c)^{-1}\{1\}}(x) = (1_{(u^c)^{-1}\{1\}} \wedge A)(x)\ \ \forall\, x \in X$.

7.3.5. Theorem: Let (X, t^*) be a fuzzy supra topological space, $A \subseteq X$ and $t^*_A = \{u \wedge A : u \in t^*\}$ then

(a) (X, t^*) is α – FSR (i) $\Rightarrow$ (A, t^*_A) is α – FSR (i).

(b) (X, t^*) is α – FSR (ii) $\Rightarrow$ (A, t^*_A) is α – FSR (ii).

(c) (X, t^*) is α – FSR (iii) $\Rightarrow$ (A, t^*_A) is α – FSR (iii).

(d) (X, t^*) is α – FSR (iv) $\Rightarrow$ (A, t^*_A) is α – FSR (iv).

Proof: (b) (X, t^*) is an α- FSR(ii) space $\Leftrightarrow$ $\forall\, w \in t^{*c}$, $\forall\, x \in X$, with $w(x) < 1$, $\exists\, u, v \in t^*$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$.

Let (X, t^*) is α - FSR(ii). We shall prove that (A, t^*_A) is α – FSR (ii). Let w be a closed fuzzy set in t^*_A and $x^* \in A$ such that $w(x^*) < 1$. This implies that $w^c \in t^*_A$ and $w^c(x^*) > 0$. So there exists an $u \in t^*$ such that $u \wedge A = w^c$ and clearly u^c is closed in t^* and $u^c(x^*) = (u \wedge$

$A)^c(x^*) = w(x^*) < 1$, i.e $u^c(x^*) < 1$. Since (X, t^*) is an α- FSR(ii) space, so for $\alpha \in I_1$ $\exists$ v, $v^* \in t^*$ such that $v(x)>\alpha$, $v^*(y) \geq 1_{(u^c)^{-1}\{1\}}$ and $v \wedge v^* \leq \alpha$. Since v, $v^* \in t^*$, then $v \wedge A$, $v^* \wedge A$ $\in t^*_A$ and $(v\wedge A)(x^*) > \alpha$, $(v^* \wedge A) \geq 1_{(,u^c)^{-1}\{1\}} \wedge A$ and $(v\wedge A) \wedge (v^* \wedge A) = (v \wedge v^*) \wedge A \leq \alpha$. But $1_{(,u^c)^{-1}\{1\}} \wedge A = 1_{((u\wedge A)^c)^{-1}\{1\}} = 1_{w^{-1}\{1\}}$, then $(v^* \wedge A) \geq 1_{w^{-1}\{1\}}$ .Hence it is clear that (A, t^*_A) is α – FSR (ii).

The proofs of (a), (c), (d) are similar.

7.3.6. Theorem: Let (X, T^*) be a Supra topological space, considering the following statements,

(1) (X, T^*) is a Supra Regular space.

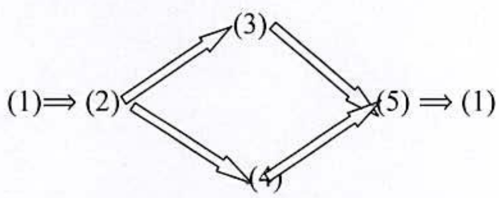
(2) $(X, \omega(T^*))$ is α – FSR (i)

(3) $(X, \omega(T^*))$ is α – FSR (ii)

(4) $(X, \omega(T^*))$ is α – FSR (iii)

(5) $(X, \omega(T^*))$ is α – FSR (iv)

Then



Proof: First, suppose that (X, T^*) be supra regular space. We shall prove that $(X, \omega(T^*))$ is α - FSR(i). Let w be a fuzzy supra closed set in $\omega(T^*)$ and $x \in X$ such that $w(x)<1$,then $w^c \in \omega(T^*)$ and $w^c(x)>0$. Now we have $(w^c)^{-1}(0, 1] \in T^*$, $x \in (w^c)^{-1}(0, 1]$. Also it is clear that $[(w^c)^{-1}(0, 1]]^c = w^{-1}\{1\}$ be a closed set in T and $x \notin w^{-1}\{1\}$. Since (X, T^*) is supra regular, then $\exists$ V, $V^* \in T^*$ such that $x \in V$, $V^* \supseteq w^{-1}\{1\}$ and $V \cap V^* = \phi$. But by the definition of lower semi continuous function 1_V, $1_V^* \in \omega(T^*)$ and $1_V(x) = 1$, $1_V^* \supseteq 1_{w^{-1}\{1\}} = 1_{V\cap V*} = 0$, let $u = 1_V$ and $v = 1_V^*$, then it is clear that $u(x)=1$, $v \supseteq 1_{w-1\{1\}}$ and $u \wedge v \leq \alpha$. Hence $(X, \omega(T^*))$ is α – FSR (i) .i.e. (i)⇒ (2).

Now show that (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (5), (2) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (5), and finally (5) $\Longrightarrow$ (1).

(2)⇒(3)

Let (X, $\omega(T^*)$) is α - FSR(i), We shall prove that (X, $\omega(T^*)$) is α - FSR(ii). Let w be a fuzzy supra closed set in $\omega(T^*)$ and $x \in X$ such that $w(x) < 1$, Since (X, $\omega(T^*)$) is α - FSR(i), for $\alpha \in I_1$ $\exists$ u, v$\in \omega(T^*)$ such that that u(x)=1, v(y) =1, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$. By the definition of lower semi continuous function $u(x) > \alpha$, v(y) =1, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$. So, it is clear that (X, $\omega(T^*)$) α - FSR(ii).

(3) $\Longrightarrow$ (5),

Let (X, $\omega(T^*)$) is α - FSR(ii), We shall prove that (X, $\omega(T^*)$) is α - FSR(iv). Let w be a fuzzy supra closed set in $\omega(T^*)$ and $x \in X$ such that $w(x) < 1$,then we have w(x)=0 Since (X, $\omega(T^*)$) is α - FSR(ii), for $\alpha \in I_1$ $\exists$ u, v$\in \omega(T^*)$ such that that $u(x) > \alpha$, v(y) =1, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$. So, it is clear that (X, $\omega(T^*)$) α - FSR(iv).

(2)⇒ (4)

Let (X, $\omega(T^*)$) is α - FSR(i), We shall prove that (X, $\omega(T^*)$) is α - FSR(iii). Let w be a fuzzy supra closed set in $\omega(T^*)$ and $x \in X$ such that $w(x) < 1$,then we have w(x)=0, Since (X, $\omega(T^*)$) is α - FSR(i), for $\alpha \in I_1$ $\exists$ u, v$\in \omega(T^*)$ such that that u(x)=1, v(y) =1, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$. Hence it is clear that (X, $\omega(T^*)$) is α - FSR(iii).

(4)⇒(5)

Let (X, $\omega(T^*)$) is α - FSR(iii), We shall prove that (X, $\omega(T^*)$) is α - FSR(iv). Let w be a fuzzy supra closed set in $\omega(T^*)$ and $x \in X$ such that w(x)=0 Since (X, $\omega(T^*)$) is α - FSR(iii), for $\alpha \in I_1$ $\exists$ u, v$\in \omega(T^*)$ such that that $u(x) > \alpha$, v(y) =1, $y \in w^{-1}\{1\}$ and $u \wedge v \leq \alpha$. Hence one can observe that (X, $\omega(T^*)$) α - FSR(iv).

(5) ⇒(1)

Let (X, $\omega(T^*)$) is α – FSR (iv). We shall prove that (X, T^*) is Supra Regular space. Let $x \in X$, V be a closed set in T^*, such that $x \notin V$. This implies that $V^c \in T^*$ and $x \in V^c$. But from definition of $\omega(T^*)$, $1_V{}^c \in \omega(T^*)$, and $(1_V{}^c)^c = 1_V$ closed fuzzy set in $\omega(T^*)$ and $1_V(x) = 0$. Since (X, $\omega(T^*)$) is α - FSR(iv) ,for $\alpha \in I_1$, $\exists$ u, v$\in \omega(T^*)$ such that $u(x) > \alpha$,

$v(y) \geq (1_V)^{-1}\{1\}$, and $u \wedge v \leq \alpha$. Since $u, v \in \omega(T^*)$, then $u^{-1}(\alpha, 1], v^{-1}(\alpha, 1] \in T^*$ and $x \in u^{-1}(\alpha, 1]$. Since $v \geq 1_v$, then $v^{-1}(\alpha, 1] \supseteq (1_V)^{-1}(\alpha, 1] = V$, and $u \wedge v \leq \alpha$. Implies $(u \cap v)^{-1}(\alpha, 1] = u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$. Now from above it is clear that (X, T^*) is supra regular space.

Thus it is seen that α - FSR(p) is a good extension of its Supra topological counterpart. (p = i, ii, iii, iv).

7.3.7. Theorem: Let (X, t^*) and (Y, s^*) be two fuzzy supra topological spaces and $f: X \longrightarrow Y$ be continuous, one-one, onto and supra open map then,

(a) (X, t^*) is α – FSR (i) $\Rightarrow (Y, s^*)$ is α – FSR (i).

(b) (X, t^*) is α – FSR (ii) $\Rightarrow (Y, s^*)$ is α – FSR (ii).

(c) (X, t^*) is α – FSR (iii) $\Rightarrow (Y, s^*)$ is α – FSR (iii).

(d) (X, t^*) is α – FSR (iv) $\Rightarrow (Y, s^*)$ is α – FSR (iv).

Proof: Suppose (X, t^*) be α – FSR (i). We shall prove that (Y, s^*) is α – FSR (i). Let $w \in s^{*c}$ and $p \in Y$ such that $w(p) < 1$, $f^{-1}(w) \in t^{*c}$ as f is continuous and $x \in X$ such that $f(x) = p$ as f is one-one and onto. Hence $f^{-1}(w)(x) = w(f(x)) = w(p) < 1$. Since (X, t^*) is α – FSR (i), for $\alpha \in I_1$, then $\exists$ $u, v \in t^*$ such that $u(x) = 1$, $v(y) = 1$, $y \in \{f^{-1}(w)\}^{-1}\{1\}$ and

$u \wedge v \leq \alpha$. This implies that $f(u)(p) = \{\text{Sup } u(x) : f(x) = p\} = 1$.
And $f(v)\ f(y) = \{\text{Sup } v(y)\} = 1$ as $f(f^{-1}(w)) \subseteq w \Longrightarrow f(y) \in w^{-1}\{1\}$
Again $f(u \wedge v) \leq \alpha$ as $u \wedge v \leq \alpha \Rightarrow f(u) \wedge f(v) \leq \alpha$.
Now it is clear that $\exists$ $f(u), f(v) \in s^*$ such that $f(u)(x) = 1$, $f(v)(f(y)) = 1$, $f(y) \in w^{-1}\{1\}$ and $f(u) \wedge f(v) \leq \alpha$. Hence (Y, s^*) is α-FSR (i).
Similarly (b), (c) and (d) can be proved.

7.3.8. Theorem: Let (X, t^*) and (Y, s^*) be two fuzzy supra topological spaces and $f: X \longrightarrow Y$ be continuous, one-one, onto and supra closed map then,

(a) (Y, s^*) is α – FSR (i) $\Rightarrow (X, t^*)$ is α – FSR (i)

(b) (Y, s^*) is α – FSR (ii) $\Rightarrow (X, t^*)$ is α – FSR (ii)

(c) (Y, s^*) is α – FSR (iii) $\Rightarrow (X, t^*)$ is α – FSR (iii)

(d) (Y, s^*) is α – FSR (iv) ⇒ (X, t^*) is α – FSR (iv)

Proof: Suppose (Y, s^*) is α – FSR (i). We shall prove that (X, t^*) be α – FSR (i). Let $w \in t^{*c}$ and $x \in X$ such that $w(x) < 1$, $f(w) \in t^{*c}$ as f is closed and we find $p \in Y$ such that $f(x) = p$ as f is one-one. Hence we have $f(w)(p) = \{\text{Sup } w(x): f(x) = p\} < 1$. Since (Y, s^*) is α – FSR (i)., for $\alpha \in I_1$, then ∃ u, v ∈ s^* such that $u(f(x)) = 1$, $v(y) = 1$, $y \in (f(w))^{-1}\{1\}$ and $u \wedge v \leq \alpha$. This implies that $f^{-1}(u), f^{-1}(v) \in t^*$ as f is continuous. Now u, v ∈ s^*, then $f^{-1}(u)(x) = u(f(x)) = u(p) = 1$ and $f^{-1}(v)(q) = v(f(q)) = v(y) = 1$ as $f(q)=y$, $y \in (f(w))^{-1}\{1\}$ i.e $f(p) \in (f(w))^{-1}\{1\} \Rightarrow q \in w^{-1}\{1\}$ and $f^{-1}(u) \wedge f^{-1}(v) \leq \alpha$ as $u \wedge v \leq \alpha$.. Now we observe that ∃ that $f^{-1}(u), f^{-1}(v) \in t^*$ such that $f^{-1}(u)(x) = 1$, $f^{-1}(v)(q) = 1$, $q \in w^{-1}\{1\}$ and $f^{-1}(u) \wedge f^{-1}(v) \leq \alpha$, Hence (X, t^*) is α-FSR (i).

Similarly (b), (c) and (d) can be proved.

CHAPTER- VIII

Compactness in Fuzzy Supra topological spaces

8. Introduction:

The concept of compactness in [0, 1]-fuzzy set theory was first introduced by Chang, C.L., [21],and latter many topologists studied the concept of fuzzy compactness such as Lowen, R. [33, 34] who introduced an improved Version of fuzzy compactness, fuzzy strong compactness and fuzzy ultra compactness, Wong, C.K., [63, 64] introduced the concept of local compactness ; Gantner, T.E., Steinlage, R.C., and Warren, R.H., [24] introduced the concept of α- compactness in fuzzy topological spaces Choubey, A., and Srivastava, A.K., [22] obtained some characterizations of α-compactness. Wang G.J [60] extended the Lowen fuzzy compactness into L-fuzzy topology. In 1988, Mao-kang L. [37] introduced S^*- paracompactness and S-paracompactness concept in fuzzy topological spaces. In this chapter we introduce compactness in fuzzy supra topological spaces and also establish a number of characterizations in this regard.

8.1. Definitions

8.1.1. Definition: Let (X, t^*) be a fsts. A family F of fuzzy supra open sets is a cover of a fuzzy set μ if and only if $\mu \subset \vee\{\mu_i \mid \mu_i \in F, i \in J\}$. It is also called a cover of X, if $\overset{n}{\underset{i=1}{\vee}} \mu_i = 1$.If there exist subset J_1 of J such that $\mu \subset \vee\{\mu_i \mid \mu_i \in F, i \in J_1\}$, then $\{\mu_i \mid \mu_i \in F, i \in J\}$ is called a subcover.

8.1. 2. Definition: A fuzzy supra topological space (X, t^*) is fuzzy supra compact, if every supra open cover of X by members of t* contains a finite sub cover, that is if $\mu_i \in t^*$ for all $i \in J$, (J an index set) then there are finitely many indices $i_1, i_2, i_3, i_4, i_5, i_6, \ldots, i_n \in J$ such that $\overset{n}{\underset{J=1}{\vee}} \mu_{ij} = 1$.

8.1.3. Definition: Let (X, T*) be a supra topology on X, and $t^* = \omega(T^*)$be a fuzzy supra topology on X. Let B be the family of fuzzy supra open sets such that $\forall$ $\mu \in$ B, $B \subset \omega(T^*) = L(\mu^{\in})$ be the set of all lower semi continuous functions from X×R to I_α with usual topology, where $\sup\limits_{\mu \in B} \mu \geq \alpha$, Thus $L(\mu^{\in}) = \{(x, r): \mu^{\in}(x) > r\}$. $r \in [0, \alpha] = I_\alpha$, $\alpha > \varepsilon > 0$, $\alpha \in I_0$ and $\mu^{\varepsilon} = \mu + \in$.

8.1.4. Definition: Let (X, t^*) be a fsts. Let $\alpha \in I$ and $\mu = \{\mu_i : i \in J\}$ be an fuzzy supra open subset of t^* then $t^*_\alpha = \{x \in X, \mu_i(x) \geq \alpha\}$.

8.1.1. Theorem: Let (X, t^*) be a fuzzy supra topological space. Then the following conditions are equivalent.

(1) $\{\mu_i\}$, $i \in J$ is a cover of X.

(2) $\vee_{i \in J} \mu_i = 1$ where $i \in J$ $\forall$ $x \in X$.

(3) $\wedge_{i \in J} \mu_i = 0$ where $i \in J$ $\forall$ $x \in X$. **[65]**

Proof: (1) $\Rightarrow$ (2).

It is clear from the definition (8.1.2) of a Cover, since $\{\mu_i\}$, $i \in J$ is a cover of X means that $\vee_{i \in J} \mu_i = 1$, where $i \in J$, $\forall$ $x \in X$.

(2) $\Rightarrow$ (3).

Since $\wedge_{i \in J} \mu_i = \inf\{\mu_i\}$ where $i \in J$, $\forall$ $x \in X$.

$= 1 - \sup\{\mu_i\}$, where $i \in J$, $\forall$ $x \in X$

$= 1 - 1 = 0$.

(3) $\Rightarrow$ (1).

From (3) as above it can be shown that $\vee_{i \in J} \mu_i = 1$. Which implies that $\{\mu_i\}$ is a cover of X.

8.1.2 Theorem: Let (X, T^*), (Y, S^*) be two fuzzy supra topological spaces, with (X, T^*) fuzzy supra compact. Let *f*: X$\rightarrow$Y be a fuzzy supra continuous surjection. Then (Y, S^*) is fuzzy supra compact. **[65]**

Proof: Let ui $\in S^*$ for each $i \in J$ with $\vee_{i \in J} u_i = 1$. Since f is fuzzy supra continuous, so $f^{-1}(u_i) \in T^*$. As (X, T^*) is supra compact, we have for each $x \in X$, $\vee_{i \in J} f^{-1}(u_i)(x) = 1$.

So we see that $\{f^{-1}(u_i)\}$, $i \in J$ is a cover of X. Hence $\exists$ finitely many indices $i_1, i_2, i_3, i_4, i_5, i_6$ $i_n \in I$ such that $\vee_{i \in J} f^{-1}(u_{ij}) = 1$. Let u be a fuzzy set in Y. Since f is a surjection we observe that for any $y \in Y$

$f(f^{-1}(u))(y) = \text{Sup}\{f^{-1}(u)(z) : z \in f^{-1}(y)\}$

$= \text{Sup}\{u(f(z)) : f(z) = y\} = u(y)$ so that

$f(f^{-1}(u)) = u$. This is true for any fuzzy set in Y. Hence

$$1 = f(1) = f(\vee_{i \in J} f^{-1}(u_{ij})) = \vee_{i \in J} f(f^{-1}(u_{ij})) = \vee_{i \in J} u_{ij}.$$

Therefore (Y, S^*) is fuzzy supra compact.

8.1.3.: Theorem Let (X, T^*), (Y, S^*) be two fuzzy supra topological spaces, and let *f*: $X \rightarrow Y$ be a fuzzy supra continuous surjection. Let A is a fuzzy supra compact set in (X, T^*), Then *f*(A) is also fuzzy supra compact in (Y, S^*).

Proof: Let $B = \{G_i : i \in J\}$, where $\{G_i\}$ be a fuzzy supra open cover of f(A). Then by definition of fuzzy supra continuity $\{f^{-1}(G_i) : i \in J\}$ is the fuzzy supra open cover of A. Since A is fuzzy supra compact, then exists a finite sub cover of A, that is

G_{ik}, k=1, 2, 3, ……… n, such that $A \subseteq \vee_{i=1}^{n} f^{-1}(G_{ik})$.

Hence $f(A) \subseteq f(\vee_{i=1}^{n} f^{-1}(G_{ik})) = \vee_{i=1}^{n} f(f^{-1}(G_{ik})) \subseteq \vee_{k=1}^{n} G_{ik}$.

Therefore $f(A)$ is fuzzy supra compact.

8.1.4. Theorem: Let (X, T^*), (Y, S^*) be two fuzzy supra topological spaces. Then the product $(X \times Y, \delta^*)$ is fuzzy supra compact if and only if (X, T^*) and (Y, S^*) are fuzzy supra compact.

Proof: First suppose that $(X\times Y, \delta^*)$ where $\delta^* = \{G_i\times H_i\colon G_i\in T^*$ and $H_i\in S^*\}$ is fuzzy supra compact, then we can define a fuzzy continuous surjection mapping κ_I and π_I from $(X\times Y, \delta^*)$ to (X, T^*) and (Y, S^*) respectively. Now by the theorem 8.1.3., (X, T^*) and (Y, S^*) are fuzzy supra compact.

Conversely let (X, T^*) and (Y, S^*) are fuzzy supra compact. Since $\delta^* = \{G_i\times H_i : G_i\in T^*$ and $H_i\in S^*$ for $i\in J\}$ where G_i and H_i are fuzzy supra open set. We claim that $\{G_i\colon i \in J\}$ is a cover of X, and $\{Hi\colon i\in J\}$ is a cover of Y. That is if $\vee_{i\in J} G_i(x) = 1$ for all $x\in X$, and if $\vee_{i\in J} H_i(y) = 1$ for all $y\in Y$, then $\vee_{i\in J} \{(G_i\times H_i)(x, y) = \text{Sup}\{\min\{G_i(x), H_i(y)\}\}$. Hence we have finite subset J' of J for which $\vee_{i\in J} G_i(x) = 1$ or $\vee_{i\in J} H_i(y) = 1$. Hence we have $\delta^* = \{G_i\times H_i\colon G_i\in T^*$ and $H_i\in S^*$ for $i\in J'\}$ is a finite sub cover of $(X\times Y, \delta^*)$. Hence $(X\times Y, \delta^*)$ is fuzzy supra compact.

8.1.1. Cor: If $(x_i, \delta_i)_{i\in J}$ is a family of fuzzy supra compact topological spaces then $(\Pi_{i\in J} X_i, \Pi_{i\in J}\delta_i)$ is also fuzzy supra compact **[35]**.

8.1.5. Theorem: The fuzzy supra topological space $(X, \omega(t^*))$ is fuzzy supra compact if and only if (X, t^*) is fuzzy supra compact.

Proof: Firstly suppose that (X, t^*) is fuzzy supra compact, let $B \subset \omega(t^*)$ be such that $\sup_{\mu\in B} \mu \geq \alpha$. Then $\forall\ \mu \in B$ and taking $\mu^{\varepsilon} = \mu + \epsilon$, the lower semi continuous functions $L(\mu^{\epsilon}) = \{(x, r)\colon \mu^{\epsilon}(x) > r\}$ is an supra open set of $X\times R$, $r\in[0, \alpha] = I\alpha$, $\alpha > \varepsilon > 0$. Now $\sup_{\mu\in B} L(\mu^{\epsilon}) \supset X\times I\alpha$, we know that $X\alpha \times I\alpha$ is fuzzy supra compact. Hence $\exists$ finite subfamily $B_i\subset B$, which covers X $\Rightarrow (X, \omega(t^*))$ is fuzzy supra compact.

Conversely suppose fuzzy supra topological space $(X, \omega(t^*))$ is a fuzzy supra compact. Then from definition of fuzzy supra compactness $\exists\ B_i\subset B$ and $\mu_i \in B_i$ such that $\text{Sup}\,\mu_i = 1$. Hence (X, t^*) is supra compact.

8.1.6. Theorem: Let (X, t^*) be a fuzzy supra topological compact space then there exist a fuzzy supra compact topology $\omega(t^*)$ in which every closed fuzzy set is also fuzzy supra compact.

Proof: Let α and $\alpha^c \in \omega$ (t*), and $B \subset \omega$ (t*) such that $\sup_{\mu\in\beta} \mu \geq \alpha$. Now $\alpha^c \in \omega$ (t*) $\Rightarrow 1-\alpha \in \omega$ (t*), Hence the collection $T(\alpha) = \{(x, r): \alpha(x) < r\}$ is fuzzy supra open in $X\times I$. Therefore $T(\alpha)^c$ is fuzzy supra compact. Choosing $\epsilon>0$ and taking $\mu^{\epsilon} = \mu+\epsilon$, we have $\sup_{\mu\in B} L(\mu^{\epsilon}) \geq T(\alpha)^c$, so there exist finite subfamily $B_0 \subset B$ such that $\sup_{\mu\in B} L(\mu^{\epsilon}) \geq T(\alpha)^c$. So in $\omega(t^*)$ in which every closed fuzzy set is also fuzzy supra compact. Hence the proof of the theorem is complete.

8.1.7. Theorem: Let (X, t^*) be a fuzzy supra topological space, then (X, t^*) is fuzzy supra Hausdorff space iff fuzzy supra compact disjoint fuzzy subsets of X can be separated by disjoint fuzzy supra open sets.

Proof: Let (X, t*) be a fuzzy supra Hausdorff space. Let λ, μ be two compact fuzzy subsets of X with $\lambda \wedge \mu=0$. To prove λ, μ are separated by disjoint fuzzy supra open sets. It is sufficient to prove that there exist u, v $\in t^*$ such that $\lambda \subset u$ and $\mu \subset v$. By the Hausdorff Property FST_2(iii) : iff for every x, y $\in X$, $x \neq y$, there exist u, v t*such that $u(x)>0$, $v(y)>0$ and $u\wedge v= 0$. Let $\{u_i : i \in J$ and $x\in\lambda\}$ s.t. $u_i(x)>0$, is a cover of λ. Since is λ fuzzy supra compact then there are finitely many indices $i_1, i_2, i_3, i_4, i_5, i_6, \ldots, i_n \in J$ such that $\lambda \subset \bigvee_{J=1}^{n} u_i(x) = u(x)$ say and similarly we can prove that $\mu \subset v$. Now we shall prove that u and v are disjoint. Let $\bigvee_{J=1}^{n} u_i(x) = u_k(y)$ and $v_k(y) = \wedge \{u_i(x): i_1, i_2, i_3, i_4, i_5, i_6, \ldots, i_n \in J\}$ then u_k and v_k are disjoint supra open sets. Thus u and v are fuzzy supra open sets s.t $\lambda \subset u$ and $\mu \subset v$.

Similarly we can prove for the other property.

8.1.8. Theorem: Every fuzzy supra compact subset of a fuzzy supra T_2 –space is closed.

Proof: Let (X, t^*) is a fuzzy supra T_2 (iii)-space and β is a fuzzy subset of X. We shall prove that A is closed. If $A^C= 0$, then A is closed. Since 0 is an fuzzy supra open set, and if $A^C\neq 0$ and let x $\in A^C$ Now since (X, t*) is a fuzzy supra T_2 (iii)-space (then x, y $\in$X, x ≠ y, ∃ λ, μ ∈ t* such that λ(x)>0, μ(y)>0 and λ∧μ=0). So for each y∈A, ∃ fuzzy supra neighborhoods λ_{y_i} and μ_{y_i} containing x and y respectively such that $\lambda_{y_i} \wedge \mu_{y_i}$ =0, since X is supra compact so finite number of points y_1, y_2, y_3,..........y_n such that A $\subset \bigcup_{i=1}^{n} \{\mu_{y_i}\}$. Let $\lambda = \bigcap_{i=1}^{n} \{\lambda_{y_i}\}$ and $\mu = \bigcap_{i=1}^{n} \{\mu_{y_i}\}$.Since each λ_{y_i} is an fuzzy supra open neighbourhood of x and the finite intersection of neighbourhoods is a neighbourhood, it follows that λ is a fuzzy supra neighbourhood of x. If x ∈ λ ⇒ x∈ λ_{x_i} for some y_i ∈A ⇒ x ∉ μ_{y_i}. Since λ∧μ=0 ⇒ x∉ $\{\mu_{y_i}\}$ = μ .Now A ⊂λ and λ∧μ=0. ⇒ μ ⊂ A^c. This shows that A^c is neighbourhood of its points. Therefore A^c fuzzy supra open .Hence A is closed.

Similarly we can prove for the other property.

8.2. Fuzzy supra α- compactness

Now we study several features of fuzzy supra α-compactness.

8. 2. 1. Definition:. Let (X, t*) be a fuzzy supra topological space and α ∈[0, 1]. A family F (F⊂ I^X) of fuzzy supra open subsets of X is called a supra α - shading of X if for each point x ∈ X and μ∈ F such that μ (x)>α.

8.2.2. Definition: Let (X, t^*) be a fsts. Let $\alpha \in I$ then (X, t^*) is said to be α-supra compact if every fuzzy supra open α-shading of the space has a finite α-sub shading **[24].**

8.2.1. Theorem Let $0 \le \alpha \le 1$, then a fsts. (X, t^*) is fuzzy α - supra compact, iff (X, t^*_α) is α - supra compact **[65].**

Proof: Let (X, t^*) be fuzzy α- supra compact, Let $\mu = \{\mu_i : i \in \Lambda\}$ be an supra open shading of (X, t^*_α).To show (X, t^*_α) is α - supra compact, we have to prove that every α-open shading has a finite sub shading. Since μ is a α-supra open shading of (X, t^*_α) then by definition of t^*_α there exist $x \in X$ and $\mu_{i_0} \in \mu$ be such that μ_{i_0}(x)> α.Again by definition of fuzzy α-supra compactness of (X, t^*) each μ_i has a sub shading say $\{\mu_i\}_{i=1}^{i=n}$, Hence (X, t_α^*) is α- supra compact.

Conversely let (X, t_α^*) is α- supra compact then by t_α^* shading it is clear that (X, t^*) is fuzzy α- supra compact.

8.2.3. Definition: Let A be a fuzzy subset of a fuzzy topological space X. A is said to be fuzzy α-open if A⊂Int Cl IntA. The set of all fuzzy α -open subsets of X will be denoted by $F_\alpha(x)$ **[28]**.

8.2.4. Definition: Let (X, t*) be an fsts. A family F of fuzzy supra open subsets of X is a α- cover of X if and only if $F \subset \{\vee \mu_i : \mu_i \in F_\alpha(x)\}$ and F covers X i.e if $\mu_i \in F$, such that $\overset{n}{\underset{i=1}{\vee}} \mu_i = 1$.

8.2.1. Example: Let X={x, y, z} and let u, v, be the fuzzy sets on X defined by u(x)=0.3, u (y)=0.5, u (z)=0.7 v(x) =0.5, v(y)=0.6, v(z)=0.9 and let fuzzy supra topology on X is defined by

t*={0, 1, u, v} then for α=0.4; {y, z} are t_α^* supra open set and which is not t*, α-supra open.

8.2.5. Definition: Let (X, t*) be a fuzzy supra topological space and α ∈ [0, 1]. Let f: (X, t*)→ (Y, s*) is said to be fuzzy α-supra continuous, if $f^{1}(u)$ is α-supra open in t* for each supra open set u in s*.In other words f is fuzzy α-supra continuous. If ∀ u ∈ s*, $f^{1}(u) \in F_\alpha(X)$.

8.2.2. Example: Let X={x, y} and let u, v, be the fuzzy sets on X are defined by u(x)=0.3, u (y)=0.4; v(x) =0.5, v(y)=0.6, and let fuzzy supra topology on X are defined by t*={0, 1, u} and δ^*={0,1, v} and let f: (X, t*)→ (Y, δ^*) be the identity mapping then for $\alpha < 0.2$; f is fuzzy α-supra continuous.

8. 2. 6 .Definition: Let (X, t^*) be a fuzzy supra topological space. A subfamily F of t^* is called a base for t^* iff each member of t^* can be expressed as a supremum of members of F.

8.2.7. Definition: Let (X, t*) be a fuzzy supra topological space and t^*_β is a fuzzy supra topology on X, which has F_α (X) as a base then f: (X, t*) ⟶ (Y, s*) is called fuzzy β supra continuous if f: (X, t^*_β) → (Y, s*) is fuzzy supra continuous.

8.2.2. Theorem: Let (X, t^*) be a fuzzy supra topological space and $t^*{}_\beta$ is a fuzzy supra topology on X, which has F_α (X) as a base, if f: (X, t*) ⟶(Y, s*) is fuzzy α- supra continuous, then f is fuzzy β- supra continuous.

Proof: Since f: (X, t*)→ (Y, s*) is fuzzy α -supra continuous, so fuzzy ∀ supra open set u∈ s*, f^1(u) is α-supra open in t*, i.e f^1(u)∈ F_α (X). Again since F_α (X) is a base for the fuzzy supra topology t^*_β hence $f^1(u) \in t^*_\beta$ so f is fuzzy β supra continuous.

8.2.3. Theorem: Let (X, t^*) and (Y, s^*) be two fsts. Let f: (X, t^*) →(Y, s^*) be fuzzy α - supra continuous and G be α -supra compact in X, then f(G) is α -supra compact in Y.
Proof: Let B= {G_i : i∈J }, where {G_i} be a α -fuzzy supra open shading of f(G). Then by definition of α -fuzzy supra continuity A= { f^{-1}(G_i): i∈J} is the fuzzy α -supra open cover of G. Since G is fuzzy α-supra compact in X, then there exists a finite sub shading of G, that is G_{ik}, k=1, 2, 3, ……… n, such that $G \subseteq \bigvee_{i=1}^{n} f^{-1}(G_{ik})$. Hence f(G)

$$\subseteq f(\bigvee_{i=1}^{n} f^{-1}(G_{ik}))= \bigvee_{i=1}^{n} f(f^{-1}(G_{ik}))\subseteq \bigvee_{k=1}^{n} G_{ik} .$$

Therefore f(G) is fuzzy α - supra compact in Y.

8.2.4. Theorem: Let (X, t^*) and (Y, S^*) be two fsts. Let f: (X, t^*) ⟶ (Y, S^*) be fuzzy α-supra continuous, ⇒f preserves fuzzy α- supra compactness. That is the image of each fuzzy α- supra compact spaces is α- supra compact.

Proof: Since f: (X, t*) ⟶ (Y, s*) is fuzzy α- supra continuous u ∈s*, f-1(u) ∈ F_α (X), where F_α (X) is the set of all α- supra open subsets of X. ⟹∃ μ∈ t^*, such that μ(x)> α, ⟹ ∃ u ∈s*, f^1(u)>α. Now the proof follows from above theorem.

8.2.5. Theorem: Let (X, t*) be a fuzzy supra topological space and ${t^*}_\beta$ be a fuzzy supra topology on X, which has F_α (X) as a base, then(X, t*) is α- supra compact if and only if (X, ${t^*}_\beta$) is α- supra compact

Proof: Let (X, t^*_β) be α - supra compact. Since t^*_β is a fuzzy supra topology on X, which has F_α (X) as a base, then $t^*_\beta \subseteq F_\alpha$ (X), which implies that (X, t*) is α - supra compact.

Conversely let (X, t*) be α - supra compact, which has F_α (X) as a base then

t*$\subseteq F_\alpha$ (X), since $t^*_\beta \subseteq$t*$\subseteq F_\alpha$ (X).Which implies that (X, t^*_β) is α - supra compact.

.2.6. Theorem: Let (X, t*) be a fuzzy supra topological space and t^*_β be a fuzzy supra topology on X, which is α – supra compact. Then each t^*_β closed fuzzy set in X is α – supra compact.

Proof: Let U be any t^*_β closed fuzzy set in X. Let $\{V_{\gamma_i} : \gamma_i \in$I} be a t^*_β supra open cover of U. Since X-U is supra open, so $\{V_{\gamma_i} : \gamma_i \in$ I} $\cup$(X-U) is a ${t^*}_\beta$ supra open cover of X. Since X is ${t^*}_\beta$ compact then by the theorem above there exists a finite subset $I_0 \subset$ I such that X= { V_{γ_i} : $\gamma_i \in I_0$}U (X-U) this implies that U $\subset \{V_{\gamma_i} : \gamma_i \in I_0\}$,

Hence U is α- supra compact, relative to X. This completes the proof of the theorem.

8.2.8. Definition: A fuzzy supra space *(X,* t^*) is called α-supra compact, where α∈ [0, 1), if for every U $\subset t^*$ such that $\vee$ U> α, there is a finite $U_0 \subset$ U satisfying $\vee U_0 > \alpha$. A fuzzy supra space which is α - supra compact for all α ∈ [0, 1) is called fuzzy supra strongly compact. **[54]**

8.2.7. Theorem : Let (X, t^*) be a fuzzy supra topological space which is strongly compact. Then each ${t^*}_\beta$ -closed fuzzy set in *X* is fuzzy supra strongly compact. **[45]**

Proof: Clearly X is fuzzy supra strongly compact with the help of the theorem 8. 2.6. and by the definition 8.2.8 of fuzzy supra strongly compact.

8.2.8. Theorem: Every fuzzy strongly supra compact space is fuzzy supra compact.
Proof: From definition of fuzzy strongly supra compactness it is clear that for all $\alpha \in [0, 1)$, every open cover have a finite sub cover, so every strongly supra compact space is fuzzy supra compact.

8.3. Fuzzy supra paracompactness.

Here we obtain some properties of fuzzy supra paracompactness.
First we give the following definitions.
8.3.1. Definition:. The star of a fuzzy set μ with respect to a cover $\mathcal{F}$ is denoted by st $(\mu, \mathcal{F}) = \vee\{ f_s : \mu q f_s \}$ where $\mathcal{F} = \{ f_s : s \in \mathcal{S}$be a cover of X . The cover $\mathcal{B}= \{b_t : t \in t^*\}$of a set X is a star refinement of a cover $\mathcal{F}$ in the same set X, if st$(b_t, \mathcal{B})$ is a refinement of $\mathcal{F}$ and denoted by st$(b_t, \mathcal{B}) \subset f_s$. Again if {st$(b_t, \mathcal{B})$:b_t is a fuzzy point in X} is a refinement of $\mathcal{F}$, $\forall$fuzzy point b_t, then $\mathcal{B}$ is said to be barycentric refinement

[29].

8.3.2. Definition: A supra open cover of a FSTS (X, t^*) is locally finite (resp, *- locally finite) if every point of the space has a neighborhood U such that U is quasi-coincident (resp intersects) only finitely many sets in the cover. In symbol, Let A be family of fuzzy sets and B is a fuzzy set in a fsts (X, t^*). We say that A is locally finite if and only if, for any fuzzy point p in B, there exists some neighborhood $U_\alpha \in Q(p)$ such that U_α is quasi-coincident {resp. $\alpha \in A$: $U_\alpha \wedge Q(p) \neq 0$} with at most a finite number of fuzzy sets of A.

8.3.4 .Definition: Let (X, t^*) be a fsts. And let $_i(t^*)$be its initial fuzzy supra topology on X, then (X, t^*) is said to be Ultra fuzzy supra compact if and only if $(X, {}_i(t^*))$is supra compact. **[35]**

8.3.5 Definition: A fsts. is said to be strong fuzzy supra compact iff it is supra compact for all $\alpha \in [0, 1)$.

8.3.6. Definition: Let (X, t^*) be a fsts, $\alpha \in$ I and μ be a fuzzy set in (X, t^*). Then μ is said to be α-supra Paracompact (resp α^*- supra paracompact) if each α-shading of μ by fuzzy supra open sets has a locally finite (resp, *- locally finite) α-shading refinement by fuzzy supra open sets.

8.3.7. Definition: Let μ be a fuzzy set in a fuzzy supra topological space (X, t*), we say that μ is fuzzy supra paracompact (*-fuzzy supra paracompact) if each supra open set $\mathcal{B}$ of μ there exist an supra open refinement $\mathcal{L}$ of μ which is locally finite (resp,* locally finite) in μ.

8.3.8. Definition: A family of sets $\mathcal{A}$is called a Q-cover of a set B if for each $x \in$ supp(B), there exist an A$\in\mathcal{A}$ such that A and B are quasi-coincident at x. Let $\alpha \in$ (0, 1], $\mathcal{A}$ is called an α -Q cover of B if $\mathcal{A}$ is Q-cover of $B_{<\alpha>}$.**[60]**

8.3.9: Definition. Let $\alpha \in$ (0, 1], A be an set in fts. A is said to be α - paracompact (α *- paracompact) if for every α -open Q-cover of A, there exists an open refinement of it which is both locally finite (resp,*- locally finite) in A and α - Q-cover of A **[29]**.

8.3.10. Definition: Let (X, t*) be a fsts and $\alpha \in I_0$, A be an fuzzy set in fsts. A is said to be S- supra paracompact (S*- supra paracompact) if for every $\alpha \in$ (0, 1], A is α - supra Paracompact (resp α^* - supra paracompact) **[2]**.

8.3.11. Definition: An fts. (X, $\mathcal{T}$) is called to be regular if for each point e in (X ,$\mathcal{T}$) and each U$\in$Q(e), there exists a V$\in$Q(e)such that $\bar{v} \subset U$ **[29]**.

8. 3.12. Definition: A refinement of a cover of a space X is a new cover of the same space such that every supra open set in the new cover is a subset of some set in the old cover. In symbols, the cover V = {V_β: β in B} is a refinement of the cover U = {U_α: α in A}, (where A is an indexed set) if and only if, for any V_β in V, there exists some U_α in U such that V_β is contained in U_α **[29]**.

8.3.13. Definition: An open cover of a space X is locally finite if every point of the space has a neighborhood which intersects only finitely many sets in the cover. In symbol, Let A be family of fuzzy sets and B is a fuzzy set in a fsts (X, t*) .We say that A is locally finite if and only if, for any fuzzy point p in B, there exists some neighborhood $U_\alpha \in$ Q(p) such that U_α is quasi-coincident {resp α∈A : $U_\alpha \wedge$ Q(p)≠0} with at most a finite number of fuzzy sets of A.

8.3.14. Definition: Let μ be a fuzzy set in a fuzzy supra topological space (X, t*), we say that μ is fuzzy supra paracompact if each supra open set $\mathcal{B}$ of μ there exist an supra open refinement $\mathcal{L}$ of μ which is locally finite in μ.

8.3.1.Theorem: Every metric space is Paracompact **[62]** .

8.3. 15. Definition: An FSTS is paracompact if it is regular **[62]** .

8.3. 16. Definition: Let(X, τ) be a topological space and be the set of all semi continuous function from (X, τ) to the unit interval, I= [0, 1] equipped with the usual topology, then (X, $\omega(\tau)$) is called induced topological space by (X, τ)**[61]**.

8.3.17. Definition:. A fuzzy extension of a topological property is said to be good, when it is possessed by (X, $\omega(\tau)$) if and only if, the original property is possessed by (X, τ).**[32]**

8.3.18. Definition: A fuzzy supra topological space (X, T^*) is called a weekly induced of the crisp supra topological space (X, T^*_0) if [T^*] = T^*_0 and each element of T^* is lower semi-continuous from (X, T^*_0) to [0, 1].

8.3.19: Definition Let (X, τ).be a topological space and $\omega(\tau)$ be the set of all semi continuous functions from (X, τ).to the unit interval equipped with the usual topology, then (X, $\omega(\tau)$) is called the weakly induced fuzzy topological space by (X, τ).**[31]**

8.3.2. Theorem: Let (X, t*) is a weakly induced FSTS. If a Q-cover U of a fuzzy set X is a barycentric refinement of a Q- cover $\mathcal{V}$, and Q- cover $\mathcal{V}$ of a fuzzy set X is a barycentric refinement of a Q-cover $\mathcal{W}$, then U is a fuzzy star refinement of $\mathcal{W}$. **[29]**

Proof: Let $u \in U$, and the fuzzy point x_λ is quasi coincident with u i.e x_λ qu. Since Q-cover $\mathcal{V}$ of a fuzzy set X is a barycentric refinement of a Q-cover $\mathcal{W}$ so we can choose $w \in \mathcal{W}$, so st($x_\lambda, \mathcal{V}$) $\leq$ w. The proof of the theorem is complete if we can show that st(u, U) $\leq$ w. Now by definition 8.3.1, since Q-cover U of a fuzzy set X is a barycentric refinement of a Q-cover $\mathcal{V}$, so st (x_λ, U) = $\vee$ {u: $x_\lambda$qu} and hence u< st (x_λ, U)<v, where $\mathcal{V} = \vee$ v.

Thus we have st(u, U)= $\bigvee_{x_\lambda \in u_s}$ st (x_λ, U)= $\bigvee_{x_\lambda \in u}$ {$\vee$ {u: $x_\lambda$qu$_s$}} $\leq$ $\bigvee_{x_\lambda \in u}$ { v_t }, so

st(u, U) = $\bigvee_{x_\lambda \in u_s}$ st (x_λ,U)$\leq$ $\bigvee_{x_\lambda \in u}$ { v_t }, Again $\bigvee_{x_\lambda \in u}$ { v_t }= st (x_λ, V) and similar way we can prove that st(u, U)< st (x_λ, V) < w_z, thus U is a fuzzy star refinement of $\mathcal{W}$ ■

8.3.3. Theorem: Let $\alpha \in (0, 1]$,

(a) An fsts. is α-supra Paracompact iff for each supra open 1-α shading U of X, there exists an supra open 1-α shading U* of X such that U* is locally finite in X and a refinement of U.

(b) If an fsts X is (1 - α)-supra compact, then X is α-supra paracompact. **[2]**

Proof: Suppose (X, t*) be a fsts and α -supra Paracompact, where $\alpha \in (0, 1]$, then by definition (8.3.6), a fuzzy set μ is said to be α-supra Paracompact if each α-shading of μ by fuzzy supra open sets has a locally finite α-shading refinement by fuzzy supra open sets. Then for every point $x \in X$ there exist $\lambda \in t^*$ such that $\lambda(x) > \alpha \Leftrightarrow -\lambda(x) < (-\alpha)$, $\Leftrightarrow 1-\lambda(x) < 1-\alpha$, $\Leftrightarrow \lambda'(x) < 1-\alpha \Leftrightarrow x_{1-\alpha} \notin \lambda'$.

(a) Now let U is supra open 1-α shading of X, $\Rightarrow$ U(x)>1-α, $\Rightarrow$ U(x) +α >1, $\Rightarrow$ U is intersect with a fuzzy set U* in X, such that U* is locally finite in X and a refinement of U.

(b) Let an fsts X is (1 - α)-supra compact, if for every $U \subset t^*$ such that Sup U>(1 - α), so by first part is α-supra Paracompact.

Remark: We know the concept of α - compactness may not imply the concept of α - paracompactness. By a Counter example we can show that if X is not α - supra paracompact, although it is α - supra compact

Counter example: Let $\alpha = \frac{3}{4}$, consider the fuzzy supra topology σ on I generated by the family $\{0, 1\} \cup \{B_y : y \in \{0, \frac{1}{2}\}$, where By is defined as follows:

$B_y(x) = 1, \qquad x \in (\frac{1}{2}, 1]$

$B_y(x) = \frac{1}{2} \qquad x = y$

$B_y(x) = 0, \qquad x \in [0, \frac{1}{2}] - \{y\}$

The family U= $\{B_y: y \in \{0, \frac{1}{2}\}$, is an supra open (1-$\alpha$) shading . There is no supra-open shading U* which is both locally finite and a refinement of U. In fact U is not locally finite. Thus X is not α- supra paracompact, although it is α- supra compact.

8.3.4. Lemma: The following statements are true.

(1) Fuzzy supra compactness ⇒ * - fuzzy supra paracompact.

(2) S^*- supra paracompact ⇒ * -fuzzy supra paracompact.

(3) S^*- supra paracompact ⇒ S- supra paracompact. **[37]**

Proof: **(1)** follows directly from definition (8.1. 2) and (8.3.7) replacing cover by refinement and finite by locally finite.

(2) The proof is straight forward from definition (8.3.10) and (8.3.7).

(3) From definition (8.3.10) it is clear that let *A* be a set in fsts. then *A* is said to be S^*- supra paracompact if for every $\alpha \in (0, 1]$, *A* is α-supra Paracompact (resp α^*- supra paracompact). i.e. for any x in X, there exists some neighborhood V(x) of x such that U_α is quasi-coincident {resp $\alpha \in A : U_\alpha \wedge v(x) \neq 0$} with at most a finite number of fuzzy sets of A. Hence *A* is S- supra paracompact.

8.3.5. Theorem: If A be an α – supra paracompact (resp α^*-supra paracompact) set in fsts. (X, t^*), then for each closed set B in (X, t^*), each α – open Q-cover of set $B \cap A$ has an supra open refinement which is both an α – Q- cover of $B \cap A$ and locally finite (resp,*- locally finite) in A.

Proof: - Clearly we prove the $\alpha-$ supra paracompactness of C = B$\cap$A. Let $\mathcal{U}$ is the α- supra open Q-cover of C, then $\mathcal{U} \cup \{B\}$ is an supra open Q-cover of $A_{<\alpha>}$ and it has on open refinement say $\mathcal{V}$ which is both locally finite in A and supra open Q-cover of A<α>. Let $\mathcal{V}_0 = \{v\in : \exists u \in\mathcal{U}, v\subset u\}$ then $\mathcal{V}_0$ is a supra open Q-cover of c<α>. If not then there exists an $x\in$ supp($c_{<\alpha>}$) such that $\cup\mathcal{V}_0(x) \leq 1-\alpha$.But $\mathcal{V}$ is the supra open Q-cover of $A_{<\alpha>}$ and $c_{<\alpha>}\subset A_{<\alpha>}$. Hence there exist v$\in\mathcal{V}$ such that $v(x) > 1-\alpha$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}\cup\{B\}$, so $v \subset B$; Again $C(x)\geq \alpha$, by given condition B is closed we have $B(x)\geq \alpha$ therefore, $1-\alpha < v(x) \leq B(x) \leq 1-\alpha$. This is a contradiction of theorem. This completes the proof

8.3.6. Theorem: Fuzzy supra paracompactness is weakly hereditary.

or, Every closed subspace of a fuzzy supra paracompact space is paracompact.

Proof: Let μ be a closed subspace in a fuzzy supra paracompact space X. Let $\{\mu_i\}$ be the covering of μ by the sets supraopen in μ. For each $\mu_i \in \{\mu_i\}$,choosing an supra open set μ_i' of X such that $\mu_i' \wedge \mu = \mu_i$ cover X by the supra open sets μ_i', along with the supra open set X-μ. Let $\mathcal{B}$ be a locally finite supraopen refinement of this covering that covers X. The collection $\mathcal{C}=\{\mu_i' \wedge \mu = \mu_i' \in \mathcal{B}\}$ is the required locally finite supra open refinement of $\{\mu_i\}$.Hence the theorem.

Remark: Fuzzy supra paracompactness is not hereditary.

Remark: Fuzzy supra paracompactness is a good extension of supra paracompactness.

8.3.7. Theorem: Let (X, t^*) is a regular weakly induced FSTS. Then we have the following equivalent conditions **[37]**.

(i) (X, t^*) is S- supra paracompact.

(ii) If $\alpha \in I_0$ then $\alpha-$ supra open Q- cover of X has a locally finite supra open refinement which is an $\alpha-$ Q- cover of X also.

(iii) $\exists$ an $\alpha \in I_{0,1}$ such that every $\alpha-$ supra open Q- cover of X has a locally finite supra open refinement which is an $\alpha-$ Q- cover of X also.

(iv) If $\alpha \in I_1$ then $\alpha -$supra open Q- cover of X has a locally finite supra open refinement which is an $\alpha -$Q- cover of X also.

Proof: (i) ⇒ (ii)

From the definitions 8.3.9 and 8.3.10 if (X, t^*) is S- supra paracompact and if A be an fuzzy set in (X, t^*) then for every $\alpha -$open Q-cover of A, there exists an supra open refinement of it which is locally finite.

(ii) ⇒ (iii) is straight forward.

(i) ⇒ (iv) Also from the definitions 8.3.9 and 8.3.10, it is clear that (i)⇒ (iv) .■

References

1. **Abd El-Monsef, M.E. and Ramadan, A.E.,** On Fuzzy Supra Topological Spaces, Indian J. pure and appl. Math., 18(1987), No.4, 322-329.
2. **Abd El-Monsef, M.E., Zeyada, F.M., El-Deen, S.N. and Hanafy, I. M.,** Good extensions of Paracompactness, Math. Japonica 37(1992), No. 1, 195-200.
3. **Adnadjevic, D.,** Separation properties in F-spaces, Mat. Vesnik. 6(1982). 1-8.
4. **Ahmed, B. and Kharal,** A., Fuzzy S-Open and S- Closed mappings, Hindi. Pub. Cor., Advances in Fuzzy Systems, V. 1(2009), Art. ID 303042, 5 pages.
5. **Ali, D.M. and Srivastava, A.K.,** A comparison of some FT_2- concepts, Fuzzy set and systems, 23 (1987), 289- 294.
6. **Ali, D.M.,** A note on fuzzy regularity concepts, Fuzzy sets and systems 35(1990), 101-104.
7. **Ali, D.M.,** Some weaker separation axioms in Fuzzy topological spaces, app. Math. 3 (1988), 1-7.
8. **Ali, D.M., Wuyts, P. and Srivastava, A.K.,** On the R_0 property in fuzzy topology, Fuzzy sets and systems 35 (1990), 101–104.
9. **Ali, D.M.,** On the R_1 Property in fuzzy topology, Fuzzy sets and systems. 50(1992), 97-101.
10. **Ali, D.M.,** On Certain Separation and Connectedness Concepts in Fuzzy Topology, Ph.D. Thesis, B. H. U. Varnasi (1990).
11. **Ali, D.M. and Faqruddin A. A.,** On Some Fuzzy R_1-Topological Spaces, 16 Math. Conference of Bangladesh Math. Society, 2009, BUET, Bangladesh.
12. **Ali, D.M. and Faqruddin A. A.,** Relation between Fuzzy R_0 and R_1-Topological Spaces, 16 Math. Con. of Bangladesh Math. Society, 2009, BUET, Bangladesh.
13. **Ali, D.M,** Some remarks on regularity in Fuzzy Topological spaces, the Raj. Uni. Stu. (B) 18(1990), 181-190.
14. **Ali, D.M. and Hossain, M.S.,** On T_1-Fuzzy Topological Spaces, Gnit. J. Math, soc. of Bangladesh 24(2004), 99-106.
15. **Ali, D.M. and Hossain, M.S.,** On T_2-Fuzzy Topological Spaces, J. Bangladesh Academy Science, vol. 29(2005), No. 2, 201-208.

31. **Kelly, J.L.,** General Topology, D. Van Nostrand company, Inc. Princeton, New Jersey, New York.

32. **Klien, A.J.,** α- Closure in fuzzy topology, Rocky Mount. J. of Math., 11(1981), 553-560.

33. **Lowen, R.,** Fuzzy Topological Spaces and Fuzzy Compactness, J. Math. Anal. and Appl. 56 (1976), 621-633.

34. **Lowen, R.,** A comparison of Different compactness notations in Fuzzy topological spaces J.Math. Anal. and Appl. 64(1978), 446–454.

35. Lowen, R., Initial and final topologies and fuzzy Tychonoff Theorem,. Math. Anal. and Appl. 58(1977), 11 -21.

36. **Lowen, R. and Wuyts, P.,** On Separation axioms in fuzzy topological spaces, fuzzy nbd. spaces and fuzzy uniform spaces, J. Math. Anal. and Appl. 93(1983), 27-41.

37. **Mao-Kang, L. Paracompactness,** in fuzzy topological spaces, J. Math. Anal. And Appl., 130(1988), 55-77.

38. **Mashour, A.S., Allam, A.A. and Khedr, F.S.,** On Supra topological spaces, Indian J. pure appl. Math., 14(1983), No.4 502-510.

39. **Mashhour,A.S. and Ghanim, M.H.,** Fuzzy Closure Spaces J. Math. Anal. Appl. 106 (1985), 154-170

40. **Min, W.K.,** On Fuzzy S-open maps Kangweon –Kyungki Math. J. 4(1996), No. 2, 135-140.

41. **Min, W.K.,** On Fuzzy s-continuous functions Kangweon –Kyungki Math. J. 4(1996), No. 1, 77-82.

42. **Ming L.Y., and Mao-Kao-Kang, L.,** Fuzzy topology, World Scientific Pub.co, Singapore, 1997.

43 **Mukherjee, A. and Bhattacharya B.,** Pre-Induced L-Supra Topological Spaces, Indian J. pure and appl. Math., 34(2003.), No. 10, 1487-1493

44. **Nanda, S.,** On fuzzy topological spaces, Fuzzy sets and systems, 19(1986), 193-197.

45. **Nanda, S.,** Strongly Compact Fuzzy topological spaces, 42(1992), No. 2,259-262.

46. **Palanippn, N.,** Fuzzy topology Norosa Publishing. House, New Delhi Chenna, Mumbai, Kolkata 5.

47. **Pao. M. P. and Ying, Fuzzy M. L.,** Topology ii, Product and Quotient Spaces, J. Math. Anal. Appl. 77(1980), 20-37.

48. **Rodabaugh, S.E.,** The Hausdorff separation axiom for Fuzzy topological spaces, Topology and its applications 11(1980), 319-334.

49. **Rodabaugh, S.E.,** A Categorical accommodation of various notations of Fuzzy Topology, Fuzzy sets and systems, 9(1983), 241- 265.

50. **Saha,S., and Ganguly, S.,** On separation axioms and separations of connected sets in fuzzy topological spaces; Bull cal. Math.Soc. 79(1987), 215-225.

51. **Saif A.,and Klic man,** A., On Lower separation and regularity axioms in Fuzzy topological spaces Hindawi Pub. Corporation. Advances in Fuzzy Systems, Vol., 2011. Article ID 941982, 1-6.

52 **Sarkar , M.,** On Fuzzy Topological Spaces, J. Math. Anal. and Appl. Jadavpur, university, Calcutta 700032, India, 79 (1981) , 384-394.

53. **Shanin, N.A.,** "On separation in topological spaces" C.R (Doklady) Acad, Sci. U.R.S.S (N.S.). 38(1943), 110-113,

54 **Shani, S.A.,** "On separation in topological spaces" Doldady Akad, nauk. U.S.S.R. revised institute of Math. Aca. Sinica Peking, China. 38(1982), 110-113

55 **Shi, F.G.,** A new form of Fuzzy α – compactness, Math. Bohmica, 131(2006), No. 1, 15-28.

56. **Shostak; A.P.,** Two decades of fuzzy topology, basic ideas, notions, and results Russian Math Surveys 6(1989), No.44, 125-186.

57. **Srivastava,** A.K, R_1-Fuzzy Topological Spaces, J. Math. Anal. & Appl. 127 (1987) 151 – 154.

58. **Srivastava, R. Lal S.N. and Srivastava, A.K.,** Fuzzy Hausdorff topological spaces, J. Math. Anal. Appl.81(1981), 497-506.

59. **Srivastava, R. Lal S.N. and Srivastava, A.K.,** On Fuzzy T_0 and R_0 topological spaces, J. Math. Anal. Appl.136 (1988), 66-73.

60. **Wang, G.J.,** A new fuzzy compactness defined by fuzzy nets, J. Math. Anal. Appl. 94 (1983),1-23.

61. **Warren, R.H.,** Neighbourhood, bases and continuity in Fuzzy topological spaces, Rocky Mount. J. Math. 8(1978), No.3, 459-470.

62. **Wikipedia,** Paracompact spaces, the free Encyclopedia.

63. **Wong, C.K.**, Fuzzy points and Local properties of Fuzzy topology, J. Math. Anal. Appl. 46(1974), 316-328.

64. **Wong, C.K.,** Fuzzy topology, Fuzzy sets and Their Appl. to Cognitive and Decision processes, Edited by L.A. Zadeh, K.s Fu, Acda. Press (1975) 171-190.

65. **Yahia, M.,** Fuzzy supra Topological Spaces, M. Phil. Thesis, KUET, Bangladesh (2007)

66. **Zadeh, L.A.,** Fuzzy sets, Information and Control 8(1965), 338-353.