

**A Case Study of the Solution of Laminar Convective
Boundary Layer Flow around a Vertical Slender Body with Transpirations**

by
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**A thesis submitted in partial fulfillment of the requirements for the degree of
Master of Science in Mathematics**

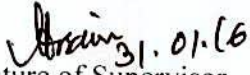


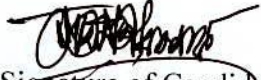
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January 2016

Declaration

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List of Publication

A paper entitled “Numerical Investigation of the Solution of Laminar Convective Boundary Layer Flow around a Vertical Slender Body with Transpirations” has been extracted from this thesis works and published in the proceedings of *1st International Conference on Mathematics and Its Applications*, December 23, 2015 organized by Mathematics Discipline, Khulna University, Khulna-9208, Bangladesh.

Acknowledgement

It is my great pleasure to thank my reverend supervisor Professor Dr. M. M. Touhid Hossain, Department of Mathematics, Khulna University of Engineering & Technology, Khulna-9203, Bangladesh, whose critical thought and efficient supervision helps me to complete the valuable thesis paper without any obstacle.

I express my deepest sense of gratitude to my estimable teacher Professor Dr. M. M. Touhid Hossain, Head, Department of Mathematics, Khulna University of Engineering & Technology, Khulna, for providing me all kinds of support and help from the department. I would like to extend my thanks to all of my respected teachers of the Department of Mathematics, KUET, especially to Professor Dr. Mohammad Arif Hossain, Professor Dr. Md. Abul Kalam Azad and Associate Professor Dr. B. M. Ikramul Haque for their cordial help and suggestions.

I am thankful to my parents and all of my family members. Finally, I would like to thank all of my friends.

The author

ABSTRACT

In this thesis, a case of the similarity solution of laminar convective boundary layer flow around a vertical slender body with transpirations has been investigated. Firstly, the governing boundary layer partial differential equations have been made dimensionless and then simplified by using Boussinesq approximation. Secondly, similarity transformations are introduced on the basis of detailed analysis in order to transform the simplified coupled partial differential equations into a set of ordinary differential equations. In the present thesis one of the important similarity case, out of four cases has been studied. Under the considered case, the transformed complete similarity equations are solved numerically by using computer software MATLAB. Further, the flow phenomena have been characterized with the help of obtained flow controlling parameters such as suction/blowing parameter, buoyancy parameter, Prandtl number, body-radius parameter and other driving parameter. Finally, the effects of these involved parameters on the velocity and temperature fields are presented graphically. It is observed that a small suction or blowing played a significant role on the patterns of flow and temperature fields.

Contents

	PAGE
Title Page	<i>i</i>
Declaration	<i>ii</i>
Approval	<i>iii</i>
List of Publication	<i>iv</i>
Acknowledgement	<i>v</i>
Abstract	<i>vi</i>
Contents	<i>vii</i>
List of Figure	<i>viii</i>
Nomenclature	<i>x</i>
CHAPTER I	Introduction and Literature Review
	1
CHAPTER II	Mathematical Formulations
	2.1 Basic Equations
	11
	2.2 Transformations
	15
CHAPTER III	Similarity Solution: Study of Case A
	3.1 Analysis for combined convection $(u_e \infty (x+x_0)^\beta)$
	26
	3.2 Flow Parameters
	34
	3.3 Numerical Scheme and Procedure
	36
CHAPTER IV	4.1 Numerical Results and Discussion
	37
CHAPTER V	Conclusions and Recommendations
	48
	REFERENCES
	50

LIST OF FIGURE

Figure No	Description	Page
2.1	Physical configuration and coordinate system.	11
4.1	Velocity profiles for different values of F_w (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $R_0 = 0.1$, $Pr = 0.71$ and $\beta = 0.5$).	38
4.2	Temperature profiles for different values of F_w (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $R_0 = 0.1$, $Pr = 0.71$ and $\beta = 0.5$).	39
4.3	Velocity profiles for different values of Pr (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $\beta = 0.5$, $F_w = 0.5$ and $R_0 = 0.1$).	40
4.4	Temperature profiles for different values of Pr (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $\beta = 0.5$, $F_w = 0.5$ and $R_0 = 0.1$).	41
4.5	Velocity profiles for different values of R_0 (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $F_w = 0.5$, $Pr = 0.71$ and $\beta = 0.5$).	42
4.6	Temperature profiles for different values of R_0 (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $F_w = 0.5$, $Pr = 0.71$ and $\beta = 0.5$).	43
4.7	Velocity profiles for different values of $\frac{U_F^2}{u_e^2}$ (with fixed values of $\beta = 0.5$, $Pr = 0.71$, $F_w = 0.5$ and $R_0 = 0.1$).	44

- 4.8 Temperature profiles for different values of $\frac{U_F^2}{u_e^2}$ (with fixed values of $\beta = 0.5$, $Pr=0.71$, $F_w = 0.5$ and $R_0 = 0.1$). 45
- 4.9 Velocity profiles for different values of β (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $Pr=0.71$, $F_w = 0.5$ and $R_0 = 0.1$). 46
- 4.10 Temperature profiles for different values of β (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $Pr=0.71$, $F_w = 0.5$ and $R_0 = 0.1$). 47

Nomenclature

x, y	Cartesian Coordinates
t	Time
F_w	Suction parameter
u	Fluid velocity in the x direction
v	Fluid velocity in the y direction
v_w	Suction/blowing velocity perpendicular to the surface of the body
k	Thermal conductivity
μ	Dynamic coefficient of viscosity
η	Similarity variable
ν	Kinematic Coefficient of viscosity
ψ	Stream function
ρ	Fluid density
β	The Exponent of external velocity (driving parameter)
θ	Dimensionless temperature
γ	Boundary layer thickness
λ	The second coefficient of viscosity
τ_w	Shearing stress at the wall
ϕ	Velocity potential
ϕ_0	Value of ϕ on the body
p	Pressure
m	Molecular weight of the gas
$f(\eta)$	Similarity function
β_T	Coefficient of thermal expansion
q_w	Heat transfer coefficient

u_e	External velocity
L_c/L	Suitable Characteristics length
T_e	Ambient temperature
g_x	x Component body force
R_e	Reynolds number
C_p	Specific heat at constant temperature
Pr	Prandtl number
E_c	Eckert number
$\frac{U_F^2}{u_e^2}$	Buoyancy parameter
R_0	Body-radius parameter

CHAPTER I

Introduction and Literature Review

Fluid dynamics is a subject of widespread interest to researcher and it become an obvious challenge for the scientists, engineers as well as users to understand more about fluid motion. An important contribution to the fluid dynamics is the concept of boundary layer flow introduced first by L. Prandtl [1]. The concept of the boundary layer is the consequence of the fact that flows at high Reynolds numbers can be divided into unequally spaced regions. A very thin layer (called boundary layer) in the vicinity (of the object) in which the viscous effects dominate, must be taken into account, and for the bulk of the flow region, the viscosity can be neglected and the flow corresponds to the inviscid outer flow. Although the boundary layer is very thin, it plays a vital role in the fluid dynamics. Boundary layer theory has become an essential study now-a-days in analyzing the complex behaviors of real fluids. The concept of boundary layer can be used to simplify the Navier-Stocks' equations to such an extent that the viscous effects of flow parameters are evaluated, and these are useable in many practical problems (viz. the drag on ships and missiles, the efficiency of compressors and turbines in jet engines, the effectiveness of air intakes for ram and turbojets and so on).

Further the boundary layer effects caused by free convection are frequently observed in our environmental happenings and engineering devices. We know that if externally induced flow is provided and flows arising naturally solely due to the effect of the differences in density caused by temperature or concentration differences in the body force field (such as gravitational field), this type of flow is called 'free convection' or 'natural convection' flow. The density difference causes buoyancy effects and these effects act as 'driving forces' due to which the flow is generated. Hence free convection is the process of heat transfer which occurs due to movement of the fluid particles by density differences associated with temperature differences in a fluid. In such case, the free stream velocity falls away, in deed, no reference velocity does a priori exist. If the density in the vicinity of the surface is kept constant, natural convection flow can not be formed. Thus, the natural convection is an effect

of variable properties, where there is a mutual coupling between momentum and heat transport. The direct origin of the formation of natural convection flow is a heat transfer via conduction through the fixed surfaces surrounding the fluid. If the surface temperature is greater than that of ambient fluid, heat is transferred from the surface to the fluid leads to an increase in temperature of the fluid close the surfaces and to a change in the density, because it is temperature dependent. If the density decreases with increasing temperature, buoyancy forces arise close to the surface and warmer fluid moves upwards. Such buoyancy forces are proportional to the coefficient of thermal expansion β_T , defined as $\beta_T = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_{p=\text{constant}}$,

where ρ , T and p are density, temperature and pressure respectively. It is observed that

$\beta_T = \frac{1}{T}$ for a perfect gas, and we see that stream is well approximated by the perfect-gas

result $\beta_T T = 1$ at low pressure and high temperature. Also $\beta_T < \frac{1}{T}$ for a liquid and may even

be negative, and $\beta_T > \frac{1}{T}$ for imperfect gas, particularly at high pressure. β_T is also useful

in estimating the dependence of enthalpy 'h' on pressure, from the thermodynamic relation

$dh = C_p dT + (1 - \beta_T T) \frac{dp}{\rho}$, where T is the absolute temperature and C_p is the specific heat at

constant temperature. For the perfect gas, the second term vanishes, so that $h = h(T)$ only.

The natural convection studies begun in the year 1881 with Lorentz and continued at a relatively constant rate until recently. This mode of heat transfer occurs very commonly, the cooling of transmission lines, electric transformers and rectifiers, the heating of rooms by use of radiators, the heat transfer from hot pipes and ovens surrounded by cooled air, cooling the reactor core (in nuclear power plant) and carry out the heat generated by nuclear fission etc. The Mixed convection flows, combined forced and free convection flows, arise in many transport processes in engineering devices and in nature. This follows are characterized by the buoyancy parameter (measure of the influence of the free convection in comparison with that of forced convection on the fluid flow) which depends on the flow configuration and the surface heating conditions. Bulks of information are now available in literature about the boundary layer form of natural convection flows over bodies of different shapes. The

theoretical, experimental and numerical analysis for the natural and the mixed convection boundary layer flow about isothermal, vertical porous flat plates have been carried out widely by many authors in view of its applications in many engineering and geographical problems. Ramanaiah et al. [2] considered the problem of mixed convection over a horizontal plate subjected to a temperature or surface heat flux varying as a power of x .

Schmidt [3] was apparently the first researcher who investigated experimentally the behavior of the flow near the leading edge above a flat horizontal surface. The theoretical analysis of the laminar, two-dimensional, steady natural convection boundary layer flow on a semi-infinite horizontal flat plate was first considered by Stewartson [4] (later corrected by Gill et al. [5]). In that analysis he used the Boussinesq approximation to show how the boundary layer analysis could be incorporated with the natural convection on rectangular plates, which are of high plane form aspect ratio.

Rotem and Claassen [6] investigated the boundary layer equation over a semi-infinite horizontal surface of uniform temperature and results were presented for some specific values of Prandtl number with its limits from zero to infinity. The effect of buoyancy force that exist in boundary layer flow, over a horizontal surface, where the surface temperature differs from that of ambient fluid, was studied by Sparrow and Minkowycz [7]. The free convection above a heated and almost horizontal plate has been treated by Jones [8].

The problem of mixed convection due to a heated or cooled vertical flat plate provides one of the most basic scenarios for heat transfer theory and thus is of considerable theoretical and practical interest and has been extensively studied by Sparrow et al. [9], Wilks [10], Afzal and Banthiya [11], Hunt and Wilks [12], Lin and Chen [13], Hussain and Afzal [14], Merkin et al. [15] and many others. However, the problem of forced, free and mixed convection flows past a heated or cooled body with porous wall is of interest in relation to the boundary layer control on airfoil, lubrication of ceramic machine parts and food processing. Watanabe [16] has considered the mixed convection boundary layer flow past an isothermal vertical porous flat plate with uniform suction or injection. Satter [17] made analytical studies on the combined forced and free convection flow in a porous medium.

A vast literature of similarity solution has appeared in the area of fluid mechanics, heat transfer, and mass transfer, etc. as it is one of the important means for the reduction of a number of independent variable with simplifying assumptions. It is revealed that the similarity solution, which being attained for some suitable values of different parameters, might be thought of being the solution of the convective boundary-layer context either near the leading edge or far away in the downstream. Deswita et al. [18] obtained a similarity solution for the steady laminar free convection boundary layer flow on a horizontal plate with variable wall temperature.

The boundary layer type of the natural convection flow, which occurs on the upper surface of heated horizontal surface has been investigated theoretically and experimentally by many others Rotem and Claassen [6], Pera and Gebhart [19, 20] and Goldstein et al. [21]. It is seen from their experiments and also from the flow visualization of Husar and Sparrow [22] that a boundary layer starts from each edge of a plate edge, each boundary layer having its leading at a straight-side plate edge. The boundary layer development occurs normal to the corresponding edge so that collisions between opposing boundary layer flows occur on the plate surface. After collision, the fluid cheeked in the boundary layer forms a rising buoyant plume.

The solution of a system of coupled partial differential equations with boundary conditions is often difficult and even impossible to solve with the usual classical method. Thus, it is imperative to reduce the number of variables from the system which reached in a stage of great extent. Similarity solution is one of the important means for the reduction of a number of independent variables with simplifying assumptions and finally the system of partial differential equations reduces to a set of ordinary differential equations successfully. The study of complete similarity solutions of the unsteady laminar natural convection boundary layer flow above a heated horizontal semi-infinite porous plate have been considered by Hossain and Mojumder [23] and Hossain et al. [24]. The similarity solutions in the context of mixed convection boundary layer flow of steady viscous incompressible fluid over a porous vertical flat plate were discussed by Ishak et al. [25]. Ramanaiah and Malarvizhi [26] studied the similarity solutions of free, mixed and forced convection problems in a saturated porous media.

In 1978, Johnson and Cheng [27] examined the necessary and sufficient conditions under which similarity solutions exist for free convection boundary layers adjacent to flat plates in porous media. The solutions obtained in their work were more general than those appearing in the previous studies. With a parameter associated with the body shapes a similarity solution on the natural convection flow has also been studied by Pop and Takhar [28]. Ferdows et al. [29] have been made a similarity analysis for the forced as well as free convection boundary layer flow of an electrically conducting viscous incompressible fluid past a semi-infinite non-conducting vertical porous plate by introducing a time dependent suction.

Most of the above analysis were based on the Boussinesq approximation where density, viscosity, thermal conductivity and specific heat variations are ignored except for the necessary inclusion of the density-variation in the body force term and have been concerned with the seeking of similarity solutions in which the surface temperature varies with the distance from surface leading edge.

An analysis is performed by Chen et al. [30] to study the flow and heat transfer characteristic of laminar natural convection in boundary layer flows from horizontal, inclined and vertical plates with power law variation of the wall temperature.

In most of the above analysis the boundary layer of the natural convection flows were considered over heated or uniformly heated horizontal vertical, horizontal or near horizontal, semi-infinite, rectangular porous plates. The surface is impermeable to the fluid, so that there is no transpiration i.e., suction or blowing velocity normal to the surface. This led to the kinematic boundary condition $v_w = 0$.

The problem of boundary layer control has become very important factor; in actual application it is often necessary to prevent separation. The separation of the boundary layer is generally undesirable, since separated flow causes a great increase in the drag experienced by the body. So it is often necessary to prevent separation in order to reduce pressure drag.

Suction (or blowing) is one of the useful means in preventing boundary layer separation. The effect of suction consists in the removal of decelerated particles from the boundary layer before they are given a chance to cause separation. The surface is considered to be permeable

to the fluid, so that the surface will allow a non-zero normal velocity and fluid is either sucked or blown through it. In doing this however, no-slip condition $u_w = 0$ at the surface (non-moving) shall continue to remain valid.

In deriving the boundary layer equation, it is anticipated that the v -component of the velocity is small quantity of the order of magnitude $O\left(\text{Re}^{\frac{1}{2}}\right)$ and it is assumed that the suction (or blowing) velocity $v_w = 0$ normal to the surface has its magnitude of order (characteristic Reynolds number) $^{-1/2}$. The consequence of this is that outer flow is independent of v_w and the boundary condition at the surface is given by $y = 0 ; u = 0, v = v_w(x)$.

For ordinary boundary layer flows of adverse pressure gradients, the boundary layer flow will eventually separate from the surface. Separation of the flow causes many undesirable features over the whole field; for instance if separation occurs on the surface of an airfoil, the lift of the airfoil will decrease and the drag will enormously increase. In some problems we wish to maintain laminar flow without separation. Various means have been proposed to prevent the separation of boundary layer flows; suction and blowing are two of them.

Many authors have made mathematical studies on these problems, especially in the case of steady flow. Among them the name of Cobble [31] may be cited who obtained the conditions under which similarity solutions exist for hydromagnetic boundary layer flow past a semi-infinite flat plate with or without suction. Following this, Soundalgekar and Ramanamurthy [32] analyzed the thermal boundary layer. Then Singh [33] studied this problem for large values of suction velocity employing asymptotic analysis in the spirit of Nanbu et al. [34]. Singh and Dikshit [35] have again adopted the asymptotic method to study the hydromagnetic effect on the boundary layer development over a continuously moving plate. In a similar way Bestman [36] studied the boundary layer flow past a semi infinite heated porous plate for two component plasma.

On the other hand, engineers engaged in high speed flow are facing an important problem regarding the cooling of the surface to avoid the structural failures as a result of frictional heating and other factors. In these respect the possibility of using blowing at the surface is a

measure to cool the body in the high temperature fluid. Blowing of secondary fluid through porous walls is of practical importance in film cooling of turbine blades combustion chambers. In such applications blowing usually occurs normal to the surface and the blowed fluid may be similar to or different from the primary fluid. In some recent applications, however, it has been recognized that the cooling efficiency can be enhanced by vectored injection at an angle other than 90° to the surface. In addition, most previous calculations have been limited to blowing rates ranging from small to moderate. Suction or blowing causes double effects with respect to the heat transfer. On the one hand, the temperature profile is influenced by the changed velocity field in the boundary-layer, leading to a change in the heat conduction at the surface. On the other hand, convective heat transfer occurs at the surface along with the heat conduction for $v_w \neq 0$. A summary of flow separation and its control can be found in Chang [37, 38].

The study of natural convection on a horizontal plate with suction and blowing is of huge interest in many engineering applications, for instance, transpiration cooling, boundary layer control and other diffusion operations. The effects of blowing and suction on forced or free convection flow over vertical as well as horizontal plates were analyzed in a systematic way by Gortler [39], Sparrow and Cess [40], Koh and Hartnett [41], Gersten and Gross [42], Merkin [43, 44, 45], Vedanayagam et al. [46], Hsiao-Tsung and Wen-Shing [47] and Acharya et al. [48] etc.

Using the usual asymptotic approach to obtain the similar solutions of the steady natural convection boundary layer for a non-similar flow situation on a horizontal plate with large suction approximation has been developed by Afzal and Hussain [49]. A detailed study on similarity solutions for free convection boundary layer flow over a permeable wall in a fluid saturated porous media was carried out by Chaudhary et al. [50]. They have shown that the system depends on the power law exponent and the dimensionless surface mass transfer rate. They also examined the range of exponent under which the solution exists. With constant plate temperature and particular distribution of blowing rate Clarke and Riley [51] obtained a special case of similarity solution, allowing variable fluid density. But there is still a shortage of accurate data for a wide range of both suction and blowing rate. Lin and Yu [52] presented a non-similar solution for the laminar free convection flow over a semi-infinite heated

upward-facing horizontal porous plate with suitable transpiration rate as a power-law variation. Emphasis was given for an isothermal plate under the condition of uniform blowing or suction. Lately, using a parameter concerned pseudo-similarity technique of time and position coordinates, Cheng and Huang [53] studied the unsteady laminar boundary layer flow and heat transfer in the presence and absence of heat source or sink on a continuous moving and stretching isothermal surface with suction and blowing. In their analysis they paid attention on the temporal developments of the hydrodynamic and thermal characteristics after the sudden simultaneous changes in momentum and heat transfer. Recently, an analysis is performed by Aydin and Kayato [54] for the laminar boundary layer flow over a porous horizontal flat plate, particularly, to study the effect of uniform suction/injection on the heat transfer. Using the constant surface temperature as thermal boundary condition they also investigated the effect of Prandtl number on heat transfer.

Recently, Hossain and Mojumder [55] presented the similarity solution for the steady laminar free convection boundary layer flow generated above a heated horizontal rectangular surface. They investigated the effect of suction and blowing on fluid flow and heat transfer as well as skin friction coefficients. They also found that suction increased skin-friction and heat transfer coefficients whereas injection caused a decrease in both.

Hossain et al. [56] obtained a complete similarity solution of the unsteady laminar combined free and forced convection boundary layer flow about a heated vertical porous plate in viscous incompressible fluid and investigated the effects of several involved parameters on the velocity and temperature fields and other flow parameters like skin friction, heat transfer coefficients across the boundary layer. The combined free and force convective laminar fluid motion caused by a heated (or cooled) vertical slender body moving through a viscous fluid has not so far been widely studied. Van Dyke [57] successfully analyzed a natural convection flow near a vertical thin needle for the case of a constant surface temperature. Kuiken [58] has studied the axi-symmetric free convective boundary layer along an isothermal vertical cylinder of constant thickness.

The problem of forced laminar flow over thin needles, such that the boundary layer thickness is comparable to the local needle thickness, has been investigated by Lee [59] and Narain and Uberoi [60]. The combined free and force convective laminar fluid flow for the steady case

has been studied also by Narain and Uberoi [61] for slender needles for the cases of isothermal wall and uniform wall heat flux. Furthermore, less attention has been paid to the corresponding unsteady problems of needle flows which may have applications in the field of aeronautics, atomic power, chemical engineering and electrical engineering etc.

Recently, Hasanuzzaman et al. [62, 63] presented similarity solution of convective laminar boundary layer flow around a vertical slender body with suction and blowing. In their analysis, four different similarity cases have been arised (viz. Case A, Case B, Case C and Case D). But they considered two similarity cases (Case B and Case D) out of those four cases. In this present study, we have investigated another similarity case for the complete similarity solution of combined convection laminar boundary layer flow around a vertical slender body with suction and blowing. We have considered the effects of several involved parameters on the velocity and temperature fields and other flow parameters like skin friction and heat transfer coefficients across the boundary layer. The thermal distributions on the outer surface of the body as well as the motion of the body itself are assumed to be unsteady. Furthermore, throughout the investigation, the effect of suction or blowing has been taken into consideration and we have investigated small suction or blowing velocity on these parameters as well.

In attacking this problem the equations expressing conservation of mass, momentum and energy will be formulated in a manner which readily admits the variation of thermodynamic and transport properties of fluid with temperature and pressure. The governing non-dimensional boundary layer partial differential equations are simplified first based on the Boussinesq approximation. The similarity transformations are then introduced on the basis of detailed analysis in order to transform the simplified coupled partial differential equations into a set of ordinary differential equations. The transformed complete similarity equations are then solved numerically by using computer software. The flow phenomenon has been characterized with the help of obtained flow controlling parameters such as suction parameter, buoyancy parameter, Prandtl number and the other driving parameter.

The numerical solutions including the velocity and temperature fields are to be presented for different selected values of the appeared dimensionless parameters. The influences of these various parameters on the velocity and temperature profiles will be exhibited in the present

analysis. It may be expected that the effects of suction and blowing can play an important role on the velocity and temperature fields, so that their effects should be taken into account with other useful parameters associated.

Here we adopt the method of classical ‘separation of variables’ which is of the simplest and most straightforward method of determining similarity solutions. This method was first initiated by Abbott and Kline [64], (1960). In this method, a form of similarity variable is chosen, the given PDE is transformed under the selected co-ordinate system. The dependent variables are to be expressed in terms of the product of separable functions of the new independent variables where each function is dependent on the single variable. Substitution of the product form of the dependent variables into the original partial differential equation generally leads to an equation in which no functions of single variable can be isolated on the two sides of the equation unless certain parameters are to be specified. Usually, these parameters can be specified quite readily and “separation of the variables” is achieved. On this way the separation proceeds until the one side becomes an ordinary differential equation. This thesis is composed of Five Chapters. An introduction of basic principles of boundary layer theory, natural convection flows, suction and blowing phenomena with historical review of earlier researches and background of our problem are presented in CHAPTER I.

Basic equations governing the problem, dimensional analysis with simplifying assumptions and similarity transformations with possible similarity cases are given in CHAPTER II. In CHAPTER III, a detailed discussion of one of the four similarity cases, namely, Case A has been given. Under the considered case, analytical approach for combined convection have also been discussed there. The numerical solutions with the graphs for some selected values of the appeared parameters are presented in CHAPTER IV. In CHAPTER V, the conclusions gained from this work and brief descriptions for further works related to our present research have been discussed.

CHAPTER-II

Mathematical Formulations

2.1 Basic Equations

Let an axi-symmetric heated (or cooled) slender body of finite axial length is immersed vertically in a viscous fluid of variable properties. The surface temperature ($=T_w$), the velocity and the temperature of the undisturbed fluid (u_e and T_e) close to the body surface but outside the boundary-layer are all general functions of x and t . r_w is the radial distance from the axis of symmetry to the surface of the body, x is the distance measured along the axis of symmetry of the body and t is the time. The physical configuration and the coordinate system of the problem are shown in Figure 2.1.

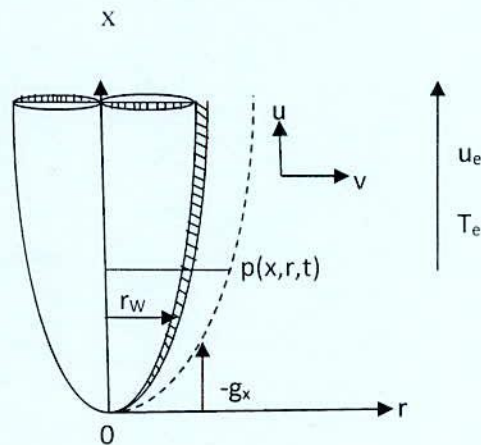


Figure 2.1: Physical configuration and coordinate system.

The influence of body force generated by buoyancy effects on the flow field near the surface is significant if the Froude number of such a flow field is of order unity. That is, the non dimensional form of the buoyancy force is

$$\frac{T_w - T_e}{T_e} \frac{g_x L_c}{U^2} \cong 0(1) \quad (2.1)$$

where g_x is the gravity component in the x -direction, L_c is a suitable characteristic length and U is a suitable characteristic velocity. In attacking this problem the equations expressing conservation of mass, momentum and energy will be formulated in a manner which readily admits the variation of thermo-dynamic and transport properties of the fluid with temperature and pressure.

If u , v are the components of the velocity in the x and r directions respectively then the equation governing the motion of the fluid as shown in Figure 2.1 may be written as follows:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \left[\frac{\partial}{\partial x} (\rho r u) + \frac{\partial}{\partial r} (\rho r v) \right] = 0 \quad (2.2)$$

$$\begin{aligned} \frac{Du}{Dt} = \frac{\rho}{\nu} g_x - \frac{1}{\nu} \frac{\partial p}{\partial x} + \frac{1}{\nu r} \frac{\partial}{\partial r} \left[\mu r \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x} \right) \right] \\ + \frac{1}{\nu} \frac{\partial}{\partial x} \left[2\mu \frac{\partial u}{\partial x} + \lambda \left(\frac{1}{r} \frac{\partial}{\partial r} (rv) + \frac{\partial u}{\partial x} \right) \right] \end{aligned} \quad (2.3)$$

$$\begin{aligned} \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{1}{\rho} \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \right) \right] + \frac{2\mu}{\rho r} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \\ + \frac{1}{\rho} \frac{\partial}{\partial r} \left[2\mu \frac{\partial v}{\partial r} + \lambda \left(\frac{1}{r} \frac{\partial}{\partial r} (rv) + \frac{\partial v}{\partial x} \right) \right] \end{aligned} \quad (2.4)$$

$$\begin{aligned} \frac{DT}{Dt} = \frac{1}{C_p \rho r} \frac{\partial}{\partial r} \left(rk \frac{\partial T}{\partial r} \right) + \frac{1}{C_p \rho} \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) \\ + T \beta_r \frac{1}{C_p \rho} \left\{ u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial r} \right\} + \frac{\phi}{C_p \rho} \end{aligned} \quad (2.5)$$

where

$$\phi = \mu \left[2 \left\{ \left(\frac{\partial v}{\partial r} \right)^2 + \left(\frac{v}{r} \right)^2 + \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \right)^2 \right\} \right] + \lambda \left[\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial u}{\partial x} \right]^2$$

and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial r}$$

Also ρ is the density, p is the pressure, μ and λ are the dynamic and second coefficients of viscosity, T the temperature, k the thermal conductivity, C_p the specific heat at constant pressure and $\beta_T = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p$ is the coefficient of thermal expansion of the fluid. As usual in the boundary-layer approximation, the equations (2.2)-(2.5) are reduced to (on the basis of dimensional analysis in boundary-layer theory):

$$r \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho r u) + \frac{\partial}{\partial r}(\rho r v) = 0 \quad (2.6)$$

$$\frac{Du}{Dt} = g_x - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho r} \frac{\partial}{\partial r} \left(\mu r \frac{\partial u}{\partial r} \right) \quad (2.7)$$

$$0 = \frac{\partial p}{\partial r} \quad (2.8)$$

$$\frac{DT}{Dt} = \frac{1}{\rho C_p r} \frac{\partial}{\partial r} \left(r k \frac{\partial T}{\partial r} \right) \quad (2.9)$$

The boundary conditions to be specified here are as follows:

(i) For a solid surface the fluid must adhere to the surface of body (i.e. the non-slip condition of the viscous flows) and the surface must be a stream line

$$\text{i.e. } u(x, r, t) = 0 \quad \text{when } r = 0$$

$$u(x, r, t) = 0, \quad v(x, r, t) = v_0 \quad \text{when } r = r_w$$

(ii) The fluid at a large distance from the body surface must be undisturbed by the boundary layer, i.e.

$$u(x, r, t) = \frac{u_e}{U} \quad \text{as } r \rightarrow \infty$$

$$\text{or, } u(x, \infty, t) = \frac{u_e}{U}, \quad u_e \text{ being in general a function of both } x \text{ and } t.$$

(iii) The temperature of the fluid at the surface of the body must be equal to the body surface temperature. Hence $T = T_w$ when $r = 0$ or, $r = r_w$

Hence $\theta(x, r, t) = \theta(x, r_w, t) = 1$

i.e. $\theta(x, 0, t) = 1$

as $\theta_w = 1$ i.e. $\theta(x, r_w, t) = 1$

(iv) The temperature at a large distance from the body surface must be equal to the undisturbed fluid temperature T_e i.e. $\theta(x, r_w, t) = 0$ when $r \rightarrow \infty$

or, $\theta(x, \infty, t) = 0$

In the non-dimensional form of the boundary-layer momentum equations (2.7) and (2.8), the terms of order Re^{-1} are ignored whilst in the energy equation (2.9), terms of order Re^{-1} and E_t (for the stress work terms) are also ignored. Here $Re = \frac{UL_e}{u_e}$ and

$E_t = \frac{U^2}{C_p(T_w - T_e)}$ are the Reynolds and Eckert numbers of the flow respectively. We are

concerned with those types of boundary-layer flows where $Re \rightarrow \infty$ and $E_t \ll 1$.

Imposing the boundary conditions at the outer edge of the boundary layer (as $\rho \rightarrow \rho_e$, $u \rightarrow u_e$, $T \rightarrow T_e$, $p \rightarrow p_e$ and $\frac{\partial p}{\partial r} \rightarrow 0$) equations (2.7)-(2.9) are now transformed to:

$$\rho_e \frac{\partial u_e}{\partial t} + \rho_e u_e \frac{\partial u_e}{\partial x} = \rho_e g_x - \frac{\partial p_e}{\partial x} \quad (2.10)$$

$$p = p(x, t) \quad (2.11)$$

$$\frac{\partial T_e}{\partial t} + u_e \frac{\partial T_e}{\partial x} = 0 \quad (2.12)$$

Here subscript 'e' refers to conditions in the outer or external flow. The values with the subscript are the values to which the solutions in the boundary-layer must be matched as the boundary-layer co-ordinate normal to the surface tends to infinity. In general

u_e, T_e, ρ_e are functions of x and t . However, equation (2.12) has the solution $T_e = \text{constant}$ for a given fluid element. In such follows, it will be assumed that $T_e = \text{constant}$ throughout the flow field. By virtue of (2.6)-(2.9) and (2.10)-(2.12), the transformed boundary-layer equations which are to be studied in this chapter are written in the following forms:

$$r \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho r u) + \frac{\partial}{\partial r}(\rho r v) = 0 \quad (2.13)$$

$$\rho \frac{Du}{Dt} = (\rho - \rho_e)g_x + \rho_e \left\{ \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} \right\} + \frac{1}{r} \frac{\partial}{\partial r} \left(\mu r \frac{\partial u}{\partial r} \right) \quad (2.14)$$

$$\rho C_p \frac{DT}{Dt} = \frac{1}{r} \frac{\partial}{\partial r} \left(r k \frac{\partial T}{\partial r} \right) \quad (2.15)$$

2.2. Transformations

The equations (2.13)-(2.15) take the following forms for a Boussinesq fluid:

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) = 0 \quad (2.16)$$

$$\frac{Du}{Dt} = -g_x \beta_T \Delta T \theta + \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} + \frac{v}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \quad (2.17)$$

$$\frac{D\theta}{Dt} = - \left\{ u \frac{\partial}{\partial x} (\log \Delta T) + \frac{\partial}{\partial t} (\log \Delta T) \right\} \theta + \frac{1}{\text{Pr}} \frac{v}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r} \right) \quad (2.18)$$

where $\nu = \frac{\mu}{\rho}$, $\text{Pr} = \frac{\mu C_p}{k}$, $T - T_0 = \Delta T_0$, $\Delta T = T_w - T_0$, $\theta = \frac{T - T_e}{T_w - T_e}$, and $T_e = T_0$ is

treated here as constant temperature for the ambient fluid. Since the state relations is $\rho = \rho(T)$, it follows that $\rho_e = \rho_0$ (constant). T and T_w in general depend on both x and t . A solution of the equations (2.16)-(2.18) is now sought, these equations being valid in the limit $Re \rightarrow \infty$ and $E_c \rightarrow 0$. In our present investigation we have considered only the first order boundary-layer approximations and the higher order effects are ignored.

The complexity of the above governing differential equations makes the use of simplifying approximations desirable so that tractable solutions may be obtained. The

method of similarity provides a convenient and accurate procedure for computing heat transfer, skin friction and other laminar boundary-layer characteristics. Guided by the idea of similarity solution the independent variables (x, r, t) are changed to a new set of variables (ξ, ϕ, τ) where the relations between the two sets are

$$\xi = x, \quad \tau = t, \quad \phi = \frac{r^2}{2\gamma(x, t)} \quad (2.19)$$

Here $\gamma(x, t)$ is thought to be proportional to the square of the local boundary-layer thickness. This definition of ϕ arises purely for reasons of convenience in using the axisymmetric form of the equations. From (2.19), we get

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial \tau} - \frac{\phi}{\gamma} \gamma_{\tau} \frac{\partial}{\partial \phi} \quad (2.20)$$

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \xi} - \frac{\phi}{\gamma} \gamma_{\xi} \frac{\partial}{\partial \phi} \quad (2.21)$$

$$\frac{\partial}{\partial r} \equiv \frac{r}{\gamma} \frac{\partial}{\partial \phi} \quad (2.22)$$

Here suffices denote partial differentiation with respect to corresponding arguments. The continuity equation (2.16) is identically satisfied by introducing a stream function $\psi(x, r, t)$ defined by

$$ru = \frac{\partial \psi}{\partial r} \quad (2.23)$$

$$-rv = \frac{\partial \psi}{\partial x} \quad (2.24)$$

By virtue of (2.22) and (2.23) we have

$$u = 2 \frac{\partial \psi}{\partial (r^2)} = \frac{\partial}{\partial \phi} \left\{ \frac{\psi}{\gamma(x, t)} \right\} \quad (2.25)$$

Using a non-dimensional scaling factor U for the velocity component u we can write

$$\frac{u}{U} = \frac{\partial}{\partial \phi} \left\{ \frac{\psi}{\gamma U} \right\}$$

$$\begin{aligned} \text{or, } \psi(\xi, \phi, \tau) - \psi(\xi, \phi_0, \tau) &= \gamma U \int_{\phi_0}^{\phi} \frac{u}{U} d\phi \\ &= \gamma U F(\xi, \phi, \tau) \quad (\text{say}) \end{aligned} \quad (2.26)$$

where $F(\xi, \phi, \tau) = \int_{\phi_0}^{\phi} \frac{u}{U} d\phi$ and ϕ_0 is the value of ϕ on the body. That is,

$$\phi_0 = \frac{r_w^2}{2\gamma(x, t)} \quad (2.27)$$

In view of (2.21), (2.24) and (2.26) we can write

$$\begin{aligned} -rv &= \left(\frac{\partial}{\partial \xi} - \frac{\phi}{\gamma} \gamma_{\xi} \frac{\partial}{\partial \phi} \right) \psi(\xi, \phi, \tau) \\ &= \frac{\partial \psi(\xi, \phi, \tau)}{\partial \xi} - \frac{\phi}{\gamma} \gamma_{\xi} \frac{\partial \psi(\xi, \phi, \tau)}{\partial \phi} \\ &= \frac{\partial}{\partial \xi} \{ \gamma U F(\xi, \phi, \tau) + \psi(\xi, \phi_0, \tau) \} - \frac{\phi}{\gamma} \gamma_{\xi} \frac{\partial}{\partial \phi} \{ \gamma U F(\xi, \phi, \tau) + \psi(\xi, \phi_0, \tau) \} \end{aligned}$$

$$\text{or, } -rv = (\gamma U F)_{\xi} - \phi \gamma_{\xi} U F_{\phi} - r_w v_w \quad (2.28)$$

$$\text{where } -r_w v_w = \psi_{\xi}(\xi, \phi_0, \tau) \quad (2.29)$$

We assumed that the surface of the body is porous, therefore $v_w \neq 0$ represents the suction or blowing effects and since $r_w = r_w(x)$ only we take $\psi_{\xi}(\xi, \phi_0, \tau) \neq 0$. In this

situation the convective operator $\frac{D}{Dt}$ becomes

$$\begin{aligned} \frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial r} \\ &= \frac{\partial}{\partial \tau} - \frac{\phi}{\gamma} \gamma_{\tau} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi} \left(\frac{\psi}{\gamma} \right) \left(\frac{\partial}{\partial \xi} - \frac{\phi}{\gamma} \gamma_{\xi} \frac{\partial}{\partial \phi} \right) - \frac{1}{r} \{ (\gamma U F)_{\xi} - \phi \gamma_{\xi} U F_{\phi} - r_w v_w \} \frac{r}{\gamma} \frac{\partial}{\partial \phi} \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial \tau} - \frac{\phi}{\gamma} \gamma_{\tau} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi} \left\{ \frac{\gamma U F - \psi(\xi, \phi_0, \tau)}{\gamma} \right\} \left(\frac{\partial}{\partial \xi} - \frac{\phi}{\gamma} \gamma_{\xi} \frac{\partial}{\partial \phi} \right) - \frac{(\gamma U F)_{\xi}}{\gamma} \frac{\partial}{\partial \phi} \\
&\quad + \frac{1}{\gamma} \phi \gamma_{\xi} U F_{\phi} \frac{\partial}{\partial \phi} + \frac{r_w v_w}{\gamma} \frac{\partial}{\partial \phi} \\
&= \frac{\partial}{\partial \tau} - \frac{\phi}{\gamma} \gamma_{\tau} \frac{\partial}{\partial \phi} + U F_{\phi} \frac{\partial}{\partial \xi} - \frac{1}{\gamma} \phi \gamma_{\xi} U F_{\phi} \frac{\partial}{\partial \phi} - \frac{(\gamma U F)_{\xi}}{\gamma} \frac{\partial}{\partial \phi} \\
&\quad + \frac{1}{\gamma} \phi \gamma_{\xi} U F_{\phi} \frac{\partial}{\partial \phi} + \frac{r_w v_w}{\gamma} \frac{\partial}{\partial \phi} \\
&= \frac{\partial}{\partial \tau} - \left\{ \frac{\phi}{\gamma} \gamma_{\tau} + \frac{(\gamma U F)_{\xi}}{\gamma} - \frac{r_w v_w}{\gamma} \right\} \frac{\partial}{\partial \phi} + U F_{\phi} \frac{\partial}{\partial \xi} \tag{2.30}
\end{aligned}$$

At this stage the boundary conditions are simplified as

$$F(\xi, 0, \tau) = F(\xi, \phi_0, \tau) = 0$$

$$F_{\phi}(\xi, 0, \tau) = 0, \quad F_{\phi}(\xi, \infty, \tau) = 1$$

$$\theta(\xi, 0, \tau) = 1, \quad \theta(\xi, \phi_0, \tau) = 1$$

$$\theta(\xi, \infty, \tau) = 0$$

In attempting separation of variables for $F(\xi, \phi, \tau)$ and $\theta(\xi, \phi, \tau)$ we write

$$F(\xi, \phi, \tau) = L(\xi, \tau) \tilde{f}(\phi), \quad \theta(\xi, \phi, \tau) = m(\xi, \tau) \mathcal{G}(\phi) \tag{2.31}$$

Since $\theta(\xi, \phi_0, \tau) = 1$, we may put without loss of generality

$$m(\xi, \tau) = 1 \text{ and then } \mathcal{G}(\phi_0) = 1 \tag{2.32}$$

Now,

$$\begin{aligned}
\frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) &= \frac{\nu}{r} \cdot \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left[r \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(\frac{\psi}{\gamma} \right) \right\} \right] \\
&= \frac{\nu}{\gamma} \frac{\partial}{\partial \phi} \left[2\phi \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(\frac{\gamma U F}{\gamma} \right) \right\} \right] \\
&= \frac{\nu}{\gamma} \frac{\partial}{\partial \phi} \left[2\phi \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} (UL\tilde{f}) \right\} \right] \\
&= \frac{2\nu}{\gamma} UL \frac{\partial}{\partial \phi} \left[\phi \frac{\partial}{\partial \phi} \left\{ \frac{\partial \tilde{f}}{\partial \phi} \right\} \right] \\
&= \frac{2\nu}{\gamma} UL \frac{\partial}{\partial \phi} \left[\phi \frac{\partial}{\partial \phi} (\tilde{f}_\phi) \right] \\
&= \frac{2\nu}{\gamma} UL \frac{\partial}{\partial \phi} [\phi \tilde{f}_{\phi\phi}] \\
&= \frac{2\nu}{\gamma} \phi UL \tilde{f}_{\phi\phi\phi} + \frac{2\nu}{\gamma} UL \tilde{f}_{\phi\phi}
\end{aligned} \tag{2.33}$$

$$\frac{\partial u_e}{\partial t} = \left(\frac{\partial}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial}{\partial \phi} \right) u_e$$

$$\text{or, } \frac{\partial u_e}{\partial t} = \frac{\partial u_e}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial u_e}{\partial \phi} \tag{2.34}$$

$$\text{Hence } u_e \frac{\partial u_e}{\partial x} = u_e \left(\frac{\partial}{\partial \xi} - \frac{\phi}{\gamma} \gamma_\xi \frac{\partial}{\partial \phi} \right) u_e$$

$$\text{or, } u_e \frac{\partial u_e}{\partial x} = u_e \frac{\partial u_e}{\partial \xi} - u_e \frac{\phi}{\gamma} \gamma_\xi \frac{\partial u_e}{\partial \phi} \tag{2.35}$$

By virtue of the above equations from (2.19) to (2.35) the momentum equation (2.17) reduce to

$$\left(\frac{\partial}{\partial \tau} - \left\{ \frac{\phi}{\gamma} \gamma_\tau + \frac{(\gamma UF)_\xi}{\gamma} - \frac{r_w V_w}{\gamma} \right\} \frac{\partial}{\partial \phi} + UF_\phi \frac{\partial}{\partial \xi} \right) \frac{\partial}{\partial \phi} \left(\frac{\psi}{\gamma} \right) = -g_x \beta_T \Delta T \theta + \frac{\partial u_e}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial u_e}{\partial \phi} + u_e \frac{\partial u_e}{\partial \xi} - u_e \frac{\phi}{\gamma} \gamma_\xi \frac{\partial u_e}{\partial \phi} + \frac{2\nu}{\gamma} \phi UL \tilde{f}_{\phi\phi\phi} + \frac{2\nu}{\gamma} UL \tilde{f}_{\phi\phi}$$

$$\text{or, } \left(\frac{\partial}{\partial \tau} - \left\{ \frac{\phi}{\gamma} \gamma_\tau + \frac{(\gamma UF)_\xi}{\gamma} - \frac{r_w V_w}{\gamma} \right\} \frac{\partial}{\partial \phi} + UF_\phi \frac{\partial}{\partial \xi} \right) \frac{\partial}{\partial \phi} \left(\frac{\gamma UF + \psi(\xi, \phi_0, \tau)}{\gamma} \right) = -g_x \beta_T \Delta T \theta + \frac{\partial u_e}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial u_e}{\partial \phi} + u_e \frac{\partial u_e}{\partial \xi} - u_e \frac{\phi}{\gamma} \gamma_\xi \frac{\partial u_e}{\partial \phi} + \frac{2\nu}{\gamma} \phi UL \tilde{f}_{\phi\phi\phi} + \frac{2\nu}{\gamma} UL \tilde{f}_{\phi\phi}$$

$$\text{or, } \left(\frac{\partial}{\partial \tau} - \left\{ \frac{\phi}{\gamma} \gamma_\tau + \frac{(\gamma UF)_\xi}{\gamma} - \frac{r_w V_w}{\gamma} \right\} \frac{\partial}{\partial \phi} + UF_\phi \frac{\partial}{\partial \xi} \right) \frac{\partial}{\partial \phi} \left(\frac{\gamma UF + \psi(\xi, \phi_0, \tau)}{\gamma} \right) = -g_x \beta_T \Delta T \theta + \frac{\partial u_e}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial u_e}{\partial \phi} + u_e \frac{\partial u_e}{\partial \xi} - u_e \frac{\phi}{\gamma} \gamma_\xi \frac{\partial u_e}{\partial \phi} + \frac{2\nu}{\gamma} \phi UL \tilde{f}_{\phi\phi\phi} + \frac{2\nu}{\gamma} UL \tilde{f}_{\phi\phi}$$

$$\text{or, } \left(\frac{\partial}{\partial \tau} - \left\{ \frac{\phi}{\gamma} \gamma_\tau + \frac{(\gamma UF)_\xi}{\gamma} - \frac{r_w V_w}{\gamma} \right\} \frac{\partial}{\partial \phi} + UF_\phi \frac{\partial}{\partial \xi} \right) \frac{\partial}{\partial \phi} (UL \tilde{f}) = -g_x \beta_T \Delta T \theta + \frac{\partial u_e}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial u_e}{\partial \phi} + u_e \frac{\partial u_e}{\partial \xi} - u_e \frac{\phi}{\gamma} \gamma_\xi \frac{\partial u_e}{\partial \phi} + \frac{2\nu}{\gamma} \phi UL \tilde{f}_{\phi\phi\phi} + \frac{2\nu}{\gamma} UL \tilde{f}_{\phi\phi}$$

$$\text{or, } \left(\frac{\partial}{\partial \tau} - \left\{ \frac{\phi}{\gamma} \gamma_\tau + \frac{(\gamma UF)_\xi}{\gamma} - \frac{r_w V_w}{\gamma} \right\} \frac{\partial}{\partial \phi} + UF_\phi \frac{\partial}{\partial \xi} \right) (UL \tilde{f}_\phi) = -g_x \beta_T \Delta T \theta + \frac{\partial u_e}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial u_e}{\partial \phi} + u_e \frac{\partial u_e}{\partial \xi} - u_e \frac{\phi}{\gamma} \gamma_\xi \frac{\partial u_e}{\partial \phi} + \frac{2\nu}{\gamma} \phi UL \tilde{f}_{\phi\phi\phi} + \frac{2\nu}{\gamma} UL \tilde{f}_{\phi\phi}$$

$$\text{or, } \frac{\partial}{\partial \tau} (UL \tilde{f}_\phi) - \left\{ \frac{\phi}{\gamma} \gamma_\tau + \frac{(\gamma UF)_\xi}{\gamma} - \frac{r_w V_w}{\gamma} \right\} \frac{\partial}{\partial \phi} (UL \tilde{f}_\phi) + UF_\phi \frac{\partial}{\partial \xi} (UL \tilde{f}_\phi) = -g_x \beta_T \Delta T \theta + \frac{\partial u_e}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial u_e}{\partial \phi} + u_e \frac{\partial u_e}{\partial \xi} - u_e \frac{\phi}{\gamma} \gamma_\xi \frac{\partial u_e}{\partial \phi} + \frac{2\nu}{\gamma} \phi UL \tilde{f}_{\phi\phi\phi} + \frac{2\nu}{\gamma} UL \tilde{f}_{\phi\phi}$$

$$\begin{aligned} \text{or, } (UL)_\tau \tilde{f}_\phi - \left\{ \frac{\phi}{\gamma} \gamma_\tau + \frac{(\gamma UL \tilde{f})_\xi}{\gamma} - \frac{r_w V_w}{\gamma} \right\} UL \tilde{f}_{\phi\phi} + U(L\tilde{f})_\phi \frac{\partial}{\partial \xi} (UL \tilde{f}_\phi) &= -g_x \beta_T \Delta T \theta \\ &+ \frac{\partial u_c}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial u_c}{\partial \phi} + u_c \frac{\partial u_c}{\partial \xi} - u_c \frac{\phi}{\gamma} \gamma_\xi \frac{\partial u_c}{\partial \phi} + \frac{2\nu}{\gamma} \phi UL \tilde{f}_{\phi\phi\phi} + \frac{2\nu}{\gamma} UL \tilde{f}_{\phi\phi} \end{aligned}$$

$$\begin{aligned} \text{or, } (UL)_\tau \tilde{f}_\phi - \frac{\phi}{\gamma} \gamma_\tau UL \tilde{f}_{\phi\phi} - \frac{(\gamma UL)_\xi \tilde{f}}{\gamma} UL \tilde{f}_{\phi\phi} + \frac{r_w V_w}{\gamma} UL \tilde{f}_{\phi\phi} + UL \tilde{f}_\phi (UL)_\xi \tilde{f}_\phi &= -g_x \beta_T \Delta T \theta \\ &+ \frac{\partial u_c}{\partial \tau} - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial u_c}{\partial \phi} + u_c \frac{\partial u_c}{\partial \xi} - u_c \frac{\phi}{\gamma} \gamma_\xi \frac{\partial u_c}{\partial \phi} + \frac{2\nu}{\gamma} \phi UL \tilde{f}_{\phi\phi\phi} + \frac{2\nu}{\gamma} UL \tilde{f}_{\phi\phi} \end{aligned}$$

$$\begin{aligned} \text{or, } \frac{\gamma}{UL} (UL)_\tau \tilde{f}_\phi - \frac{\gamma}{UL} \frac{\phi}{\gamma} \gamma_\tau UL \tilde{f}_{\phi\phi} - \frac{\gamma}{UL} \frac{(\gamma UL)_\xi \tilde{f}}{\gamma} UL \tilde{f}_{\phi\phi} + \frac{\gamma}{UL} \frac{r_w V_w}{\gamma} UL \tilde{f}_{\phi\phi} + \frac{\gamma}{UL} UL \tilde{f}_\phi (UL)_\xi \tilde{f}_\phi \\ = -\frac{\gamma}{UL} g_x \beta_T \Delta T \theta + \frac{\gamma}{UL} \left(\frac{\partial u_c}{\partial \tau} + u_c \frac{\partial u_c}{\partial \xi} \right) + 2\nu \phi \tilde{f}_{\phi\phi\phi} + 2\nu \tilde{f}_{\phi\phi} \end{aligned}$$

$$\begin{aligned} \text{or, } \frac{\gamma}{UL} (UL)_\tau \tilde{f}_\phi - \phi \gamma_\tau \tilde{f}_{\phi\phi} - (\gamma UL)_\xi \tilde{f} \tilde{f}_{\phi\phi} + r_w V_w \tilde{f}_{\phi\phi} + \gamma (UL)_\xi \tilde{f}_\phi^2 \\ = -\frac{\gamma}{UL} g_x \beta_T \Delta T \theta + \frac{\gamma}{UL} \left(\frac{\partial u_c}{\partial \tau} + u_c \frac{\partial u_c}{\partial \xi} \right) + 2\nu \phi \tilde{f}_{\phi\phi\phi} + 2\nu \tilde{f}_{\phi\phi} \end{aligned}$$

$$\text{or, } a_4 \tilde{f}_\phi - \left(\bar{\phi} + a_3 \right) a_0 \tilde{f}_{\phi\phi} - (a_1 + a_2) \tilde{f} \tilde{f}_{\phi\phi} - a_0 \tilde{f}_{\phi\phi} + a_2 \tilde{f}_\phi^2 = a_5 \theta + a_6 + 2\nu \left\{ (\bar{\phi} + a_3) \tilde{f}_{\phi\phi\phi} \right\}_\phi$$

[Using equation 2.39]

$$\begin{aligned} \text{or, } 2\nu \left\{ (\bar{\phi} + a_3) \tilde{f}_{\phi\phi\phi} \right\}_\phi + \left\{ a_0 (\bar{\phi} + a_3) + a_0 \right\} \tilde{f}_{\phi\phi} + (a_1 + a_2) \tilde{f} \tilde{f}_{\phi\phi} \\ - a_2 \tilde{f}_\phi^2 - a_4 \tilde{f}_\phi + a_5 \theta + a_6 = 0 \end{aligned} \quad (2.36)$$

and energy equation (2.18) reduces to

$$\begin{aligned} \text{or, } \frac{\partial}{\partial \tau} (m\mathcal{G}) - \frac{\phi}{\gamma} \gamma_\tau \frac{\partial}{\partial \phi} (m\mathcal{G}) - \frac{(\gamma UL \tilde{f})_\xi}{\gamma} \frac{\partial}{\partial \phi} (m\mathcal{G}) + \frac{r_w V_w}{\gamma} \frac{\partial}{\partial \phi} (m\mathcal{G}) \\ = - \left\{ u \frac{\partial}{\partial x} (\log \Delta T) + \frac{\partial}{\partial t} (\log \Delta T) \right\} (m\mathcal{G}) + \frac{1}{P_r} \frac{\nu}{\gamma} \frac{\partial}{\partial \phi} \left\{ 2\phi \frac{\partial}{\partial \phi} (m\mathcal{G}) \right\} \end{aligned}$$

$$\begin{aligned} \text{or, } -\frac{\phi}{\gamma} \gamma_\tau \mathcal{G}_\phi - \frac{(\gamma UL)_\xi}{\gamma} \tilde{f} \mathcal{G}_\phi + \frac{r_w V_w}{\gamma} \mathcal{G}_\phi \\ = - \left\{ u \frac{\partial}{\partial x} (\log \Delta T) + \frac{\partial}{\partial t} (\log \Delta T) \right\} (\mathcal{G}) + \frac{1}{P_r} \frac{\nu}{\gamma} \frac{\partial}{\partial \phi} \left\{ 2\phi \frac{\partial}{\partial \phi} (\mathcal{G}) \right\} \end{aligned}$$

$$\text{or, } -\frac{\phi}{\gamma} \gamma_{\tau} \vartheta_{\phi} - \frac{(\gamma UL)_{\xi}}{\gamma} \tilde{f} \vartheta_{\phi} + \frac{r_w V_w}{\gamma} \vartheta_{\phi} \\ = -\left\{ u \frac{\partial}{\partial x} (\log \Delta T) + \frac{\partial}{\partial t} (\log \Delta T) \right\} (\vartheta) + \frac{1}{P_r} \frac{\nu}{\gamma} \frac{\partial}{\partial \phi} \{ 2\phi \vartheta_{\phi} \}$$

$$\text{or, } -(a_1 + a_2) \tilde{f} \vartheta_{\phi} - a_9 \vartheta_{\phi} - \left(\bar{\phi} + \frac{r_w^2}{2\gamma} \right) a_0 \vartheta_{\phi} = \\ -\gamma \left\{ u \left[\frac{\partial}{\partial \xi} - \frac{\phi}{\gamma} \gamma_{\xi} \frac{\partial}{\partial \phi} \right] (\log \Delta T) + \left[\frac{\partial}{\partial \tau} - \frac{\phi}{\gamma} \gamma_{\tau} \frac{\partial}{\partial \phi} \right] (\log \Delta T) \right\} \vartheta + \frac{2\nu}{P_r} \frac{r^2}{2\gamma} \vartheta_{\phi\phi}$$

[Using equation 2.39 and multiplying by γ]

$$\text{or, } -(a_1 + a_2) \tilde{f} \vartheta_{\phi} - a_9 \vartheta_{\phi} - (\bar{\phi} + a_3) a_0 \vartheta_{\phi} = -\gamma \left\{ u (\log \Delta T)_{\xi} + (\log \Delta T)_{\tau} \right\} \vartheta + \frac{2\nu}{P_r} (\bar{\phi} + a_3) \vartheta_{\phi\phi}$$

$$\text{or, } -(a_1 + a_2) \tilde{f} \vartheta_{\phi} - \{ a_0 (\bar{\phi} + a_3) + a_9 \} \vartheta_{\phi} = -\gamma \left\{ \frac{\partial}{\partial \phi} \left(\frac{\psi}{\gamma} \right) (\log \Delta T)_{\xi} + (\log \Delta T)_{\tau} \right\} \vartheta + \frac{2\nu}{P_r} \left\{ (\bar{\phi} + a_3) \vartheta_{\phi} \right\}_{\phi}$$

$$\text{or, } -(a_1 + a_2) \tilde{f} \vartheta_{\phi} - \{ a_0 (\bar{\phi} + a_3) + a_9 \} \vartheta_{\phi} = -\gamma \{ UL \tilde{f}_{\phi} (\log \Delta T)_{\xi} + (\log \Delta T)_{\tau} \} \vartheta \\ + \frac{2\nu}{P_r} \left\{ (\bar{\phi} + a_3) \vartheta_{\phi} \right\}_{\phi}$$

$$\text{or, } -(a_1 + a_2) \tilde{f} \vartheta_{\phi} - \{ a_0 (\bar{\phi} + a_3) + a_9 \} \vartheta_{\phi} = - (a_8 \tilde{f}_{\phi} + a_7) + \frac{2\nu}{P_r} \left\{ (\bar{\phi} + a_3) \vartheta_{\phi} \right\}_{\phi}$$

$$\text{or, } \frac{2\nu}{P_r} \left\{ (\bar{\phi} + a_3) \vartheta_{\phi} \right\}_{\phi} + \{ a_0 (\bar{\phi} + a_3) + a_9 \} \vartheta_{\phi} + (a_1 + a_2) \tilde{f} \vartheta_{\phi} - (a_7 + a_8 \tilde{f}_{\phi}) \vartheta = 0 \quad (2.37)$$

The transformed boundary conditions are

$$\tilde{f}_{\phi}(0) = 0, \quad \tilde{f}_{\phi}(\infty) = 1, \quad \theta(0) = 1, \quad \theta(\infty) = 0 \quad (2.38)$$

where

$$\begin{aligned}
 (i) \quad & \gamma_\tau = a_0 \\
 (ii) \quad & (\gamma UL)_\xi = \gamma_\xi UL + \gamma(UL)_\xi = a_1 + a_2 \\
 (iii) \quad & \frac{r_w^2}{2\gamma} = a_3 = \phi_0 \\
 (iv) \quad & \frac{\gamma(UL)_\tau}{UL} = a_4 \\
 (v) \quad & \gamma(UL)_\xi = a_2
 \end{aligned} \tag{2.39}$$

$$\begin{aligned}
 (vi) \quad & \gamma_\xi(UL) = a_1 \\
 (vii) \quad & -\frac{\gamma}{UL} g_x \beta_T \Delta T = a_5 \\
 (viii) \quad & \frac{\gamma}{UL} \{ (u_e)_\tau + u_e (u_e)_\xi \} = a_6 \\
 (ix) \quad & \gamma(\log \Delta T)_\tau = a_7 \\
 (x) \quad & \gamma UL \{ \log \Delta T \}_\xi = a_8 \\
 (xi) \quad & -r_w V_w = a_9
 \end{aligned}$$

$$\text{and } \bar{\phi} = \frac{r^2 - r_w^2}{2\gamma(x,t)} = \phi - \phi_0, \quad \frac{\partial}{\partial \phi} \equiv \frac{\partial}{\partial \phi} \cdot \frac{\partial \bar{\phi}}{\partial \phi} \equiv \frac{\partial}{\partial \bar{\phi}}$$

Similar solutions of (2.36) and (2.37) exist only when all a 's defined in equations (2.38) are finite and independent of ξ and τ . Thus, all a 's must be treated as constants. In such cases, the equations (2.36) and (2.37) lead to a set of non-linear ordinary differential equations (i.e. equations containing one independent variable). If $u_e = 0$, UL is treated then as the non-dimensional characteristic velocity (e.g. maximum velocity) induced by buoyancy effects at a particular station (x,r) . On the other hand, if $u_e \neq 0$ without loss of generality we may put $UL = u_e$ to simplify the outer boundary condition. This then asserts that the velocity component u is non-dimensionalised by the external forcing velocity. Thus the equations (2.34) give us the relations for UL or consequently u_e and γ separately. The latter are the scale factors for the velocity component u and the square of the ordinate r . Hence the contractions of ξ and τ expressed in terms of the a 's in equations (2.38) become the conditions to transform the boundary-layer equations into similarity equations.

Integrating (i) of (2.39), we get

$$\gamma = a_0\tau + B(\xi) \quad (2.40)$$

Again integrating (ii) of (2.39), we get

$$\gamma UL = (a_1 + a_2)\xi + A(\tau) \quad (2.41)$$

Differentiating (2.40) with respect to ξ and using (vi) of (2.39) we get

$$\gamma_\xi = \frac{d}{d\xi} \{B(\xi)\} = \frac{a_1}{UL} \quad (2.42)$$

Similarly, differentiating (2.41) with respect to τ and making use of conditions (ii) and

(iv) of (2.39) we get $(\gamma UL)_\tau = \frac{dA(\tau)}{d\tau}$

$$\text{or, } \gamma(UL)_\tau + \gamma_\tau UL = \frac{dA(\tau)}{d\tau}$$

$$\text{or, } UL a_4 + a_0 UL = \frac{dA(\tau)}{d\tau}$$

$$\text{or, } UL(a_0 + a_4) = \frac{d}{d\tau} \{A(\tau)\} \quad (2.43)$$

As result of (2.42) and (2.43) it may be obtained that

$$\frac{d}{d\tau} \{A(\tau)\} \cdot \frac{d}{d\xi} \{B(\xi)\} = a_1(a_0 + a_4) \quad (2.44)$$

Since γ and UL depend solely on the choice of $A(\tau)$ and $B(\xi)$, the equation (2.44) plays a significant role in determining the possible cases of similarity solutions.

These possible cases are:

$$\text{Case A: Both } \frac{d}{d\tau}\{A(\tau)\} \text{ and } \frac{d}{d\xi}\{B(\xi)\} \text{ are zero} \quad (2.45)$$

$$\text{Case B: Both } \frac{d}{d\tau}\{A(\tau)\} \text{ and } \frac{d}{d\xi}\{B(\xi)\} \text{ are constants} \quad (2.46)$$

$$\text{Case C: } \frac{d}{d\tau}\{A(\tau)\} = 0 \text{ and } \frac{d}{d\xi}\{B(\xi)\} \neq 0 \quad (2.47)$$

$$\text{Case D: } \frac{d}{d\tau}\{A(\tau)\} \neq 0 \text{ and } \frac{d}{d\xi}\{B(\xi)\} = 0 \quad (2.48)$$

It is seen that the similarity conditions (2.39) lead to four possible classes of similarity listed in equations (2.45)-(2.48). However, in the present situation an additional condition for similarity arises because ϕ_0 (equation (2.27)) must be constant also. Since r_w is a function of x only, for ϕ_0 to be constant therefore requires that γ is a function of x only. Thus similarity is achieved in the present problem in those circumstances. With this context, the various similarity cases (case A, case B, case C, case D) arised above, an important similarity case (case A) for combined convection has been discussed in Chapter III.

CHAPTER III

Similarity Solution: Study of Case A

3.1 Analysis for combined convection $(u_e \propto (x + x_0)^\beta)$

We shall now discuss in detail the restrictions on the general equations for the case A:

$\frac{dA(\tau)}{d\tau} = 0$ and $\frac{dB(\xi)}{d\xi} = 0$. These restrictions are illustrated in equations (2.44). In view

of equations (2.41) and (2.42), the equation (2.44) can be written as

$$\frac{dA(\tau)}{d\tau} = UL(a_4 + a_0) = 0 \Rightarrow a_0 = -a_4, \text{ provided } UL \neq 0$$

and

$$\frac{dB(\xi)}{d\xi} = \frac{a_1}{UL} = 0 \Rightarrow a_1 = 0$$

UL is a function of ξ only. Now we get from equation (2.40),

$$\gamma UL = (a_1 + a_2)\xi + A, \quad A \text{ is constant.}$$

Differentiating with respect to ξ , we obtain

$$(\gamma UL)_\xi = (a_1 + a_2),$$

as $a_1 = 0$, we have

$$(\gamma UL)_\xi = a_2 \tag{3.1}$$

Since UL is a function of ξ only, therefore γ might either be a function of ξ or constant. Again from (vi) of equation (2.38), we get

$$a_1 = \gamma_\xi UL$$

$$\text{or, } \gamma_\xi UL = 0 \quad \left[\because a_1 = 0 \right]$$

or, $\gamma_\xi = 0$ where, $UL \neq 0$.

Integrating, we obtain

$$\gamma = K_1 \tag{3.2}$$

where, K_1 is a constant of integration.

Further, we have from (v) of equation (2.38),

$$a_2 = \gamma(UL)_\xi \quad (3.3)$$

We get from equations (3.1) and (3.3),

$$\gamma(UL)_\xi = (\gamma UL)_\xi$$

$$\text{or, } \frac{\gamma UL(UL)_\xi}{UL} = (\gamma UL)_\xi$$

$$\text{or, } \frac{(UL)_\xi}{UL} = \frac{(\gamma UL)_\xi}{\gamma UL}$$

Integrating, we obtain

$$\ln(\gamma UL) = \ln(UL) + \ln(K_1)$$

$$\text{or, } \ln(\gamma UL) = \ln(K_1 UL)$$

$$\text{or, } \gamma UL = K_1 UL$$

$$\therefore \gamma = K_1 \quad (3.4)$$

Without loss of generality we may write $UL = u_e$ for combined convection flows. By virtue of equations (3.3) – (3.4) the general conditions for similarity requirements (2.38) yield relations between the constants (i.e. a 's). These relations are

$$a_4 = -a_0, \quad a_1 = 0, \quad a_0, a_2 \text{ arbitrary,}$$

$$a_3 = \frac{r_w^2}{2\gamma}$$

$$\text{or, } a_3 = \frac{r_w^2}{2K_1} \quad \left[\because \gamma = K_1 \right]$$

$$a_5 = -\frac{\gamma}{UL} g_x \beta_T \Delta T$$

$$= -\frac{K_1}{K_1^{-1} \{a_2 \xi + A\}} g_x \beta_T \Delta T \quad \left[\because UL = \frac{(a_2 \xi + A)}{K_1} \right]$$

$$\text{or, } a_5 = -g_x \beta_T \Delta T K_1^2 (a_2 \xi + A)^{-1}$$

$$a_6 = \frac{\gamma}{UL} \left\{ (u_e)_\tau + u_e (u_e)_\xi \right\}$$

$$\text{or, } a_6 = \frac{\gamma}{UL} \left\{ (UL)_\tau + UL(UL)_\xi \right\} \quad \left[\because UL = u_e \right]$$

$$\text{or, } a_6 = \frac{\gamma}{UL} \left\{ UL(UL)_\xi \right\} \quad \left[\because (UL)_x = 0 \right]$$

$$\text{or, } a_6 = \gamma(UL)_\xi$$

$$\text{or, } a_6 = a_2 \quad \left[\because a_2 = \gamma(UL)_\xi \right]$$

$$a_7 = \gamma(\log \Delta T)_\tau$$

$$= \gamma \frac{\partial}{\partial \tau} \left\{ \log \left(\frac{-a_5 UL}{g_x \beta_T \gamma} \right) \right\} \quad \left[\because \Delta T = \frac{-a_5 UL}{g_x \beta_T \gamma} \right]$$

$$= 0 \quad \left(\text{Since } UL \text{ and } \gamma \text{ are function of } \xi \text{ or constant only} \right)$$

$$\text{i.e. } \therefore a_7 = 0$$

$$a_8 = \gamma UL \{ \log \Delta T \}_\xi$$

$$\text{or, } a_8 = \gamma UL \frac{\partial}{\partial \xi} \{ \log \Delta T \}$$

$$\text{or, } a_8 = \gamma UL \frac{\partial}{\partial \xi} \left\{ \log \left(\frac{-a_5 UL}{g_x \beta_T \gamma} \right) \right\} \quad \left[\because \Delta T = \frac{-a_5 UL}{g_x \beta_T \gamma} \right]$$

$$\text{or, } a_8 = \gamma UL \frac{\partial}{\partial \xi} \{ \log(-a_5) + \log(UL) - \log g_x - \log \beta_T - \log(\gamma) \}$$

$$\text{or, } a_8 = \gamma UL \left\{ \frac{-1}{a_5} (a_5)_\xi + \frac{1}{UL} (UL)_\xi \right\}$$

$$\text{or, } a_8 = -\gamma UL \frac{(a_5)_\xi}{a_5} + \frac{\gamma UL}{UL} (UL)_\xi$$

$$\text{or, } a_8 = -\gamma UL \frac{(a_5)_\xi}{a_5} + \gamma (UL)_\xi$$

$$\text{or, } a_8 = -\gamma UL \frac{(a_5)_\xi}{a_5} + a_2 \quad \left[\because a_2 = \gamma (UL)_\xi \right]$$

$$\text{or, } a_8 = -\frac{\gamma UL}{a_5} g_x \beta_T \Delta TK_1^2 (a_2 \xi + A)^{-2} a_2 + a_2 \quad \left[\because a_5 = -g_x \beta_T \Delta TK_1^2 (a_2 \xi + A)^{-1} \right]$$

$$\text{or, } a_8 = \frac{-\gamma UL g_x \beta_T \Delta TK_1^2 (a_2 \xi + A)^{-2} a_2}{-g_x \beta_T \Delta TK_1^2 (a_2 \xi + A)^{-1}} + a_2$$

$$\text{or, } a_8 = a_2 \gamma UL (a_2 \xi + A)^{-1} + a_2$$

$$\text{or, } a_8 = a_2 \left\{ 1 + K_1 UL (a_2 \xi + A)^{-1} \right\} \quad \left[\because \gamma = K_1 \right]$$

$$\text{or, } a_8 = a_2 \left\{ 1 + K_1 K_1^{-1} (a_2 \xi + A)(a_2 \xi + A)^{-1} \right\} \quad \left[\because UL = \frac{a_2 \xi + A}{K_1} \right]$$

$$\text{or, } a_8 = a_2(1+1)$$

$$\text{or, } a_8 = 2a_2 \quad \text{and } a_9 \text{ is arbitrary.}$$

In view of the above relations, the general equations (2.36) and (2.37) take the following forms in this case:

$$2\nu \left\{ (\bar{\phi} + a_3) \tilde{f}_{\bar{\phi}\bar{\phi}} \right\}_{\bar{\phi}} + \left\{ a_0 (\bar{\phi} + a_3) + a_9 \right\} \tilde{f}_{\bar{\phi}\bar{\phi}} + (a_1 + a_2) \tilde{f} \tilde{f}_{\bar{\phi}\bar{\phi}} - a_2 \tilde{f}_{\bar{\phi}}^2 - a_4 \tilde{f}_{\bar{\phi}} + a_5 \vartheta + a_6 = 0$$

$$\text{or, } 2\nu \left\{ (\bar{\phi} + a_3) \tilde{f}_{\bar{\phi}\bar{\phi}} \right\}_{\bar{\phi}} + \left\{ a_0 (\bar{\phi} + a_3) + a_9 \right\} \tilde{f}_{\bar{\phi}\bar{\phi}} + a_2 \tilde{f} \tilde{f}_{\bar{\phi}\bar{\phi}} - a_2 \tilde{f}_{\bar{\phi}}^2 + a_0 \tilde{f}_{\bar{\phi}} + a_5 \vartheta + a_2 = 0 \quad (3.5)$$

$$\left[\because a_1 = 0, \quad a_4 = -a_0, \quad a_6 = a_2 \right]$$

and

$$\frac{2\nu}{Pr} \left[\left\{ \bar{\phi} + a_3 \right\} \vartheta_{\bar{\phi}} \right]_{\bar{\phi}} + \left\{ a_0 (\bar{\phi} + a_3) + a_9 \right\} \vartheta_{\bar{\phi}} + (a_1 + a_2) \tilde{f} \vartheta_{\bar{\phi}} - (a_7 + a_8) \tilde{f}_{\bar{\phi}} \vartheta = 0$$

$$\text{or, } \frac{2\nu}{Pr} \left[\left\{ \bar{\phi} + a_3 \right\} \vartheta_{\bar{\phi}} \right]_{\bar{\phi}} + \left\{ a_0 (\bar{\phi} + a_3) + a_9 \right\} \vartheta_{\bar{\phi}} + a_2 \tilde{f} \vartheta_{\bar{\phi}} - 2a_2 \tilde{f}_{\bar{\phi}} \vartheta = 0 \quad (3.6)$$

$$\left[\because a_1 = 0, \quad a_7 = 0, \quad a_8 = 2a_2 \right]$$

Substituting $\tilde{f} = \alpha_1 f$

$$\eta = \frac{1}{\alpha_1} \bar{\phi}$$

$$\begin{aligned} \text{We obtain } \frac{\partial}{\partial \bar{\phi}} &= \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial \bar{\phi}} \\ &= \frac{1}{\alpha_1} \frac{\partial}{\partial \eta} \end{aligned}$$

Then equation (3.5) becomes

$$\begin{aligned} 2\nu \left[(\alpha_1 \eta + a_3) \frac{\alpha_1}{\alpha_1^2} f_{\eta\eta} \right]_{\eta} \frac{1}{\alpha_1} + \left\{ a_0 (\alpha_1 \eta + a_3) + a_9 \right\} \frac{\alpha_1}{\alpha_1^2} f_{\eta\eta} + a_2 \frac{\alpha_1 f}{\alpha_1^2} \alpha_1 f_{\eta\eta} \\ - a_2 \left(\frac{\alpha_1 f_{\eta}}{\alpha_1} \right)^2 + a_0 \frac{\alpha_1}{\alpha_1} f_{\eta} + a_5 \vartheta + a_2 = 0 \end{aligned}$$

$$\text{or, } \frac{2\nu}{\alpha_1} \left[\eta f_{\eta\eta} + \frac{a_3}{\alpha_1} f_{\eta\eta} \right]_{\eta} + a_0 \eta f_{\eta\eta} + \frac{a_0 a_3}{\alpha_1} f_{\eta\eta} + \frac{a_9}{\alpha_1} f_{\eta\eta} + a_2 \tilde{f} f_{\eta\eta} - a_2 f_{\eta}^2 + a_0 f_{\eta} + a_5 \vartheta + a_2 = 0$$

$$\text{or, } \frac{2\nu}{\alpha_1} \left[\eta f_{\eta\eta} + \frac{r_w^2}{2K_1\alpha_1} f_{\eta\eta} \right] + a_0 \eta f_{\eta\eta} + \frac{a_0}{\alpha_1} \frac{r_w^2}{2K_1} f_{\eta\eta} + \frac{a_9}{\alpha_1} f_{\eta\eta} + a_2 f f_{\eta\eta} - a_2 f_\eta^2 + a_0 f_\eta + a_5 \vartheta + a_2 = 0$$

Dividing by $\frac{\nu}{\alpha_1}$,

$$2 \left[\eta f_{\eta\eta} + \frac{r_w^2}{2K_1\alpha_1} f_{\eta\eta} \right] + \frac{\alpha_0\alpha_1}{\nu} \eta f_{\eta\eta} + \frac{a_0\alpha_1}{\nu} \frac{r_w^2}{2K_1\alpha_1} f_{\eta\eta} + \frac{r_w\nu_w}{\nu} f_{\eta\eta} + \frac{a_2\alpha_1}{\nu} f f_{\eta\eta} - \frac{a_2\alpha_1}{\nu} f_\eta^2 + \frac{a_0\alpha_1}{\nu} f_\eta + \frac{a_5\alpha_1}{\nu} \vartheta + \frac{a_2\alpha_1}{\nu} = 0 \quad \left[\because a_9 = -r_w\nu_w \right]$$

Choosing $\frac{a_2}{\nu} \alpha_1 = \beta$ and $\frac{a_0\alpha_1}{\nu} = 1$, finally writing $\frac{r_w^2}{2K_1\alpha_1} = R_0$ the transformed equations of (3.5) and (3.6) are

$$2(\eta f_{\eta\eta} + R_0 f_{\eta\eta})_\eta + \eta f_{\eta\eta} + R_0 f_{\eta\eta} + f_w f_{\eta\eta} + \beta f f_{\eta\eta} - \beta f_\eta^2 + f_\eta + \frac{U_F^2}{u_e^2} \vartheta + \beta = 0$$

$$\left[\because R_0 = \frac{r_w^2}{2K_1\alpha_1}, \frac{a_0\alpha_1}{\nu} = 1, \beta = \frac{a_2\alpha_1}{\nu}, F_w = -\frac{r_w\nu_w}{\nu}, \frac{a_5\alpha_1}{\nu} = \frac{U_F^2}{u_e^2} \right]$$

$$\text{or, } 2[(\eta + R_0) f_{\eta\eta} + f_{\eta\eta}] + [(\eta + R_0 + F_w) + \beta f] f_{\eta\eta} + \beta(1 - f_\eta^2) + f_\eta + \frac{U_F^2}{u_e^2} \vartheta = 0$$

$$\text{or, } 2(\eta + R_0) f_{\eta\eta} + (2 + \eta + R_0 + F_w + \beta f) f_{\eta\eta} + \beta(1 - f_\eta^2) + f_\eta + \frac{U_F^2}{u_e^2} \vartheta = 0 \quad (3.7)$$

and

$$\frac{2\nu}{\text{Pr}} \left[(\alpha_1 \eta + a_3) \frac{1}{\alpha_1} \vartheta_\eta \right]_\eta \frac{1}{\alpha_1} + [a_0(\alpha_1 \eta + a_3) + a_9] \frac{1}{\alpha_1} \vartheta_\eta + a_2 \alpha_1 f \frac{1}{\alpha_1} \vartheta_\eta - 2a_2 \frac{\alpha_1}{\alpha_1} f_\eta \vartheta = 0$$

$$\text{or, } \frac{2\nu}{\alpha_1 \text{Pr}} \left[\eta \vartheta_\eta + \frac{a_3}{\alpha_1} \vartheta_\eta \right]_\eta + a_0 \eta \vartheta_\eta + \frac{a_0 a_3}{\alpha_1} \vartheta_\eta + \frac{a_9}{\alpha_1} \vartheta_\eta + a_2 f \vartheta_\eta - 2a_2 f_\eta \vartheta = 0$$

$$\text{or, } \frac{2\nu}{\text{Pr} \alpha_1} [(\eta + R_0) \vartheta_\eta]_\eta + a_0 \eta \vartheta_\eta + \frac{a_0}{\alpha_1} \frac{r_w^2}{2K_1} \vartheta_\eta - \frac{r_w\nu_w}{\alpha_1} \vartheta_\eta + a_2 f \vartheta_\eta - 2a_2 f_\eta \vartheta = 0$$

$$\left[\because R_0 = \frac{a_3}{\alpha_1}, a_3 = \frac{r_w^2}{2K_1}, a_9 = -r_w\nu_w \right]$$

Dividing by $\frac{\nu}{\alpha_1 \text{Pr}}$,

$$2[(\eta + R_0) \vartheta_\eta]_\eta + \frac{a_0\alpha_1}{\nu} \text{Pr} \eta \vartheta_\eta + \frac{a_0\alpha_1}{\nu} R_0 \vartheta_\eta \text{Pr} + F_w \vartheta_\eta \text{Pr} + \frac{a_2\alpha_1}{\nu} f \text{Pr} \vartheta_\eta - 2 \frac{a_2\alpha_1}{\nu} f_\eta \text{Pr} \vartheta = 0$$

$$\left[\because R_0 = \frac{r_w^2}{2K_1\alpha_1}, F_w = -\frac{r_w\nu_w}{\nu} \right]$$

$$\text{or, } 2(\eta + R_0)\mathcal{G}_{\eta\eta} + 2\mathcal{G}_\eta + \text{Pr}\eta\mathcal{G}_\eta + \text{Pr}R_0\mathcal{G}_\eta + \text{Pr}F_w\mathcal{G}_\eta + \text{Pr}\beta f\mathcal{G}_\eta - 2\text{Pr}\beta f_\eta\mathcal{G} = 0$$

$$\left[\because \frac{a_0\alpha_1}{\nu} = 1, \quad \beta = \frac{a_2\alpha_1}{\nu} \right]$$

$$\text{or, } 2(\eta + R_0)\mathcal{G}_{\eta\eta} + P_r\left(\frac{2}{\text{Pr}} + \eta + R_0 + F_w + \beta f\right)\mathcal{G}_\eta - 2\text{Pr}\beta f_\eta\mathcal{G} = 0 \quad (3.8)$$

$$\text{The boundary conditions are } f(0) = f_\eta(0) = \mathcal{G}(\infty) = 0, \quad f_\eta(\infty) = \mathcal{G}(0) = 1 \quad (3.9)$$

Here the free convection velocity U_F , caused by the temperature difference $(T_w - T_0)$, is connected with by the following equation:

$$U_F^2 = -g_x\beta_T\Delta T(\xi + \xi_0) \quad (3.10)$$

where $(\xi + \xi_0)$, i.e. $(x + x_0)$, is the local characteristic length. The controlling parameters established here are Pr , the Prandtl number of the fluid, R_0 , the body radius

parameter, $\frac{U_F^2}{u_e^2}$, the square of the ratio between the free convection velocity U_F

(Ostrach [65]) and the forcing velocity u_e and β , the exponent of the external velocity variation $(u_e \propto (x + x_0)^\beta)$. The latter exponent is consequently related to the exponent of the surface temperature variation $(\Delta T \propto (x + x_0)^{2\beta-1})$. For the above class of similarity solution, the following restrictions must be satisfied:

$$(i) \quad u_e \propto (x + x_0)^\beta$$

$$(ii) \quad (T_w - T_0) \propto (x + x_0)^{2\beta-1}$$

$$(iii) \quad r_W \propto (x + x_0)^{\frac{1-\beta}{2}} \quad \left[\because R_0 = \frac{r_W^2 u_e}{2\nu(x + x_0)} \right]$$

The case $\beta = 0$ results in the outer flow moving with constant velocity around a vertical body which is a paraboloid of revolution with latusrectum of length $\frac{2\nu R_0}{u_e}$ and the surface

temperature varies inversely as the distance along the axis measured from the stagnation point. The restrictions (i) and (iii) are basically for forced flow, originally established by

Probstein and Elliott [66]. The similarity function, similarity variable, the velocity components and the body-radius parameter (R_0) related to the equations (3.8) and (3.9)

are

$$\begin{aligned}
\psi &= \gamma U F + \psi(\xi, \phi_0, \tau) \\
&= \gamma U L \tilde{f}(\phi) + \psi(\xi, \phi_0, \tau) \\
&= \gamma U L \alpha_1 f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= \gamma U L \frac{\nu}{a_1 + a_2} f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= \gamma U L \frac{\nu}{0 + a_2} f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= \gamma U L \frac{\nu \beta}{a_2} f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= \gamma U L \frac{\nu \beta}{\gamma (UL)_\xi} f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= \nu \frac{U L \beta}{(UL)_\xi} f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= \nu \frac{\beta}{\log(UL)} f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= \nu \log(UL)^{-\beta} f(\eta) + \psi(\xi, \phi_0, \tau) \\
\text{i.e. } \psi &= \nu(x + x_0) f(\eta) + \psi(\xi, \phi_0, \tau)
\end{aligned}$$

$$\text{Hence } f(\eta) = \{\nu(x + x_0)\}^{-1} \{\psi(\xi, \phi, \tau) - \psi(\xi, \phi_0, \tau)\} \quad (3.11)$$

$$\alpha_1 \eta = \bar{\phi}$$

$$\text{or, } \alpha_1 \eta = \frac{r^2 - r_w^2}{2\gamma}$$

$$\text{or, } \eta = \frac{r^2 - r_w^2}{2\alpha_1 \gamma}$$

$$= \frac{r^2 - r_w^2}{2 \frac{\nu}{a_1 + a_2} \gamma}$$

$$= \frac{r^2 - r_w^2}{2 \frac{\nu}{0 + a_2} \gamma}$$

$$= \frac{(r^2 - r_w^2) a_2}{2\nu\gamma\beta}$$

$$\begin{aligned}
&= \frac{(r^2 - r_w^2)\gamma(UL)_\xi}{2\nu\gamma\beta} \\
&= \frac{(r^2 - r_w^2)(UL)_\xi UL}{2\nu\beta UL} \\
&= \frac{(r^2 - r_w^2) \frac{\partial}{\partial \xi} \{\log(UL)\} u_e}{2\nu\beta} \\
&= \frac{(r^2 - r_w^2)u_e}{2\nu(x+x_0)^\beta}
\end{aligned}$$

$$\text{Hence } \eta = \frac{(r^2 - r_w^2)u_e}{2\nu(x+x_0)} \quad (3.12)$$

$$\begin{aligned}
u &= \frac{\partial}{\partial \phi} \left(\frac{\psi}{\gamma} \right) \\
&= \frac{\partial}{\partial \phi} \left\{ \frac{\gamma U F + \psi(\xi, \phi_0, \tau)}{\gamma} \right\} \\
&= \frac{\partial}{\partial \phi} \left(\frac{\gamma U F}{\gamma} \right) + \frac{\partial}{\partial \phi} \left\{ \frac{\psi(\xi, \phi_0, \tau)}{\gamma} \right\} \\
&= \frac{\partial}{\partial \phi} (U F) \\
&= \frac{\partial}{\partial \phi} \{U L \tilde{f}(\phi)\} \\
&= U L \frac{\partial \tilde{f}(\phi)}{\partial \phi} \\
&= U L \tilde{f}_\phi
\end{aligned}$$

$$\text{Hence } u = u_e f_\eta(\eta) \quad (3.13)$$

$$-v = \frac{v}{r} [f(\eta) - (1-\beta)(\eta + R_0)f_\eta(\eta) - r_w v_w] \quad (3.14)$$

$$\begin{aligned}
R_0 &= \frac{r_w^2}{2\alpha_1 B} \\
&= \frac{r_w^2}{2 \frac{\nu}{a_1 + a_2}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{r_w^2 (a_1 + a_2)}{2\nu} \\
&= \frac{r_w^2 (0 + a_2)}{2\nu} \\
&= \frac{r_w^2 \gamma (UL)_\xi}{2\nu \beta}
\end{aligned}$$

$$\text{Hence } R_0 = \frac{r_w^2 u_e}{2\nu (x + x_0)} \quad (3.15)$$

It is interesting to mention here that in the absence of buoyancy effects (e.g. $U_F^2 = 0$) the similarity solution of the momentum equation given by Glauert and Lighthill [67] may be obtained for $\beta=0$ in present case with a minor change in similarity variable

$$\left(\eta \text{ to } 2\eta_a \right)$$

3.2 Flow Parameters

The viscous drag of a body moving through a fluid is obtained from the shearing stress distribution at the wall. This is one of the boundary-layer characteristics of interest. The shearing stress at the wall is

$$\tau_w = \mu \left. \frac{\partial u}{\partial r} \right|_{r=r_w} \quad (3.16)$$

and in terms of the similarity variable,

$$\begin{aligned}
\tau_w &= \mu \frac{\partial u}{\partial r} \quad \text{at} \quad r = r_w \\
&= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(\frac{\psi}{\gamma} \right) \right\} \\
&= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(\frac{\gamma U F + \psi(\xi, \phi_0, \tau)}{\gamma} \right) \right\} \\
&= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} (U F) \right\} \\
&= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} (U L \tilde{f}) \right\} \\
&= \mu \frac{r_w U L}{\gamma} \tilde{f}_{\phi\phi}
\end{aligned}$$

$$\begin{aligned}
&= \mu \frac{r_w u_e}{\gamma} \frac{1}{\alpha_1} f_{\eta\eta}(0) \\
&= \mu u_e \frac{r_w}{\gamma} \frac{a_1 + a_2}{\nu} f_{\eta\eta}(0) \\
&= \mu u_e \frac{r_w}{\gamma} \frac{0 + a_2}{\nu} f_{\eta\eta}(0) \\
&= \mu u_e \frac{r_w}{\gamma} \frac{a_2}{\nu \beta} f_{\eta\eta}(0) \\
&= \mu u_e \frac{r_w}{\gamma} \frac{\gamma(UL)_\xi}{\nu \beta} f_{\eta\eta}(0) \\
&= \mu u_e r_w \frac{(UL)_\xi}{\nu \beta} f_{\eta\eta}(0)
\end{aligned}$$

$$\text{Hence } \tau_w = \frac{\mu r_w}{\nu} \frac{u_e^2}{(\xi + \xi_0)} f_{\eta\eta}(0) \quad (3.17)$$

Usually, the skin friction and Reynolds number are combined to give the boundary-layer shear stress at the wall in the form,

$$C_f \sqrt{R_{e_x}} = 2\sqrt{2R_0} f_{\eta\eta}(0) \quad (3.18)$$

where

$$C_f = \frac{2\lambda_w}{\rho u_e^2} \text{ (local skin friction)}$$

$$R_{e_x} = \frac{u_e(x + x_0)}{\nu} \text{ (local Reynolds number)}$$

$$R_0 = \frac{r_w^2 u_e}{2\nu(x + x_0)} \text{ (local body radius parameter)}$$

and $(x + x_0)$ is the local characteristic length.

It is necessary to know the heat transfer rate between the boundary-layer and the wall. The heat transfer rate at the wall is given by Fourier's law and is related to the non-dimensional temperature function by

$$\begin{aligned}
q_w &= -K \frac{\partial T}{\partial r}, \quad \text{at } r = r_w \\
&= -K \frac{r}{\gamma} \frac{\partial T}{\partial \phi}
\end{aligned}$$

$$\begin{aligned}
&= -K \frac{r_w}{\gamma} \Delta T \frac{\partial \theta}{\partial \phi} \\
&= -K \frac{r_w}{\gamma} \Delta T \frac{1}{\alpha_1} \theta_\eta(0) \\
&= -K \frac{r_w}{\gamma} \Delta T \frac{a_1 + a_2}{\nu} \theta_\eta(0) \\
&= -K \frac{r_w}{\gamma} \Delta T \frac{0 + a_2}{\nu} \theta_\eta(0) \\
&= -K \frac{r_w}{\gamma} \Delta T \frac{a_2}{\nu \beta} \theta_\eta(0) \\
&= -K \frac{r_w}{\gamma} \Delta T \frac{\gamma (UL)_\xi}{\nu \beta} \theta_\eta(0) \\
&= -K \frac{r_w}{\nu} \Delta T \frac{(UL)_\xi}{\beta} \theta_\eta(0)
\end{aligned}$$

$$\text{Hence } q_w = -\frac{K \Delta T r_w u_e}{\nu (x + x_0)} \theta_\eta(0) \quad (3.19)$$

Hence the counterpart of the equation (3.17) is

$$\frac{N_{u_r}}{\sqrt{R_{c_s}}} = -\sqrt{2R_0} \theta_\eta(0) \quad (3.20)$$

Due to the non-linearity of the ordinary differential equation it is not possible to obtain analytical solutions of the equations (3.7)–(3.8) together with the boundary conditions (3.9). Therefore numerical solutions are obtained for specified values of the controlling

parameters F_w , Pr , R_0 , β and $\frac{U_F^2}{u_e^2}$.

3.3 Numerical Scheme and Procedure

The set of differential equations (3.7)–(3.8) with the boundary conditions (3.9) are solved numerically by using computer software MATLAB. The velocity f_η and temperature θ are determined as a function of coordinate η . The numerical results thus obtained in terms of the similarity variables are displayed in graphs for several selected values of the

established parameters F_w , β , $\frac{U_F^2}{u_e^2}$, R_0 and Pr below.

CHAPTER IV

4.1 Numerical Results and Discussion

To obtain the solution of differential equations (3.7)–(3.8) with the boundary conditions (3.9), we have adopted a numerical procedure based on computer software. The effects of suction parameter F_w , driving parameter β (the ratio between the changes of local boundary-layer thickness with regard to position and time), the buoyancy parameter $\frac{U_F^2}{u_e^2}$

(the square of the ratio between the fluid velocity caused by buoyancy effects and external velocity for the forced flow), the body radius parameter R_0 and the Prandtl number Pr are plotted in the Figure 4.1–4.10. To observe the effect of F_w , the other four

parameters β , $\frac{U_F^2}{u_e^2}$, R_0 and Pr are taken constants. Similarly, we observe the effect of the

parameters β , $\frac{U_F^2}{u_e^2}$, R_0 and Pr by taking the rest four parameters constant respectively.

The effect of F_w on the velocity and temperature fields are plotted in Figure 4.1 and Figure 4.2 respectively. From Figure 4.1, it is observed that in all cases (suction ($F_w > 0$) and blowing ($F_w < 0$)) the velocity is starting at zero, then velocity increases with the increase of η near the leading edge. The maximum velocity appears at $\eta \approx 2.7$ and finally moves towards 1.0 asymptotically. Here we see that for the case of suction ($F_w > 0$), the velocity increases with decreasing F_w but for blowing case ($F_w < 0$), velocity increases with the increase of the magnitude of blowing. The usual stabilizing effect of the suction parameter on the boundary layer growth is also evident from this figure. From Figure 4.2 it is observed that for the both cases suction and blowing temperature decreases quickly close to the leading edge and away from it temperature decreases asymptotically and finally leads to zero with the increasing of η . For the case of suction ($F_w > 0$) the temperature decreases slowly with the decreasing suction. But of

the case of blowing ($F_w < 0$) the temperature decreases, more with decreasing of the magnitude of blowing and finally asymptotically leads to zero for larger η .

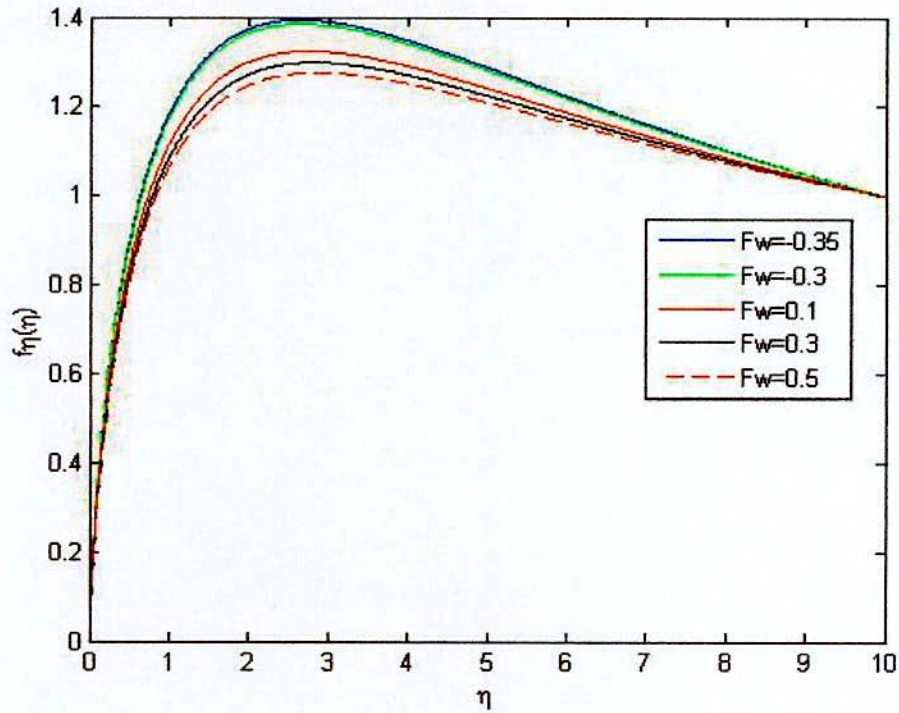


Figure 4.1: Velocity profiles for different values of F_w (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $R_0 = 0.1$, $Pr = 0.71$ and $\beta = 0.5$).

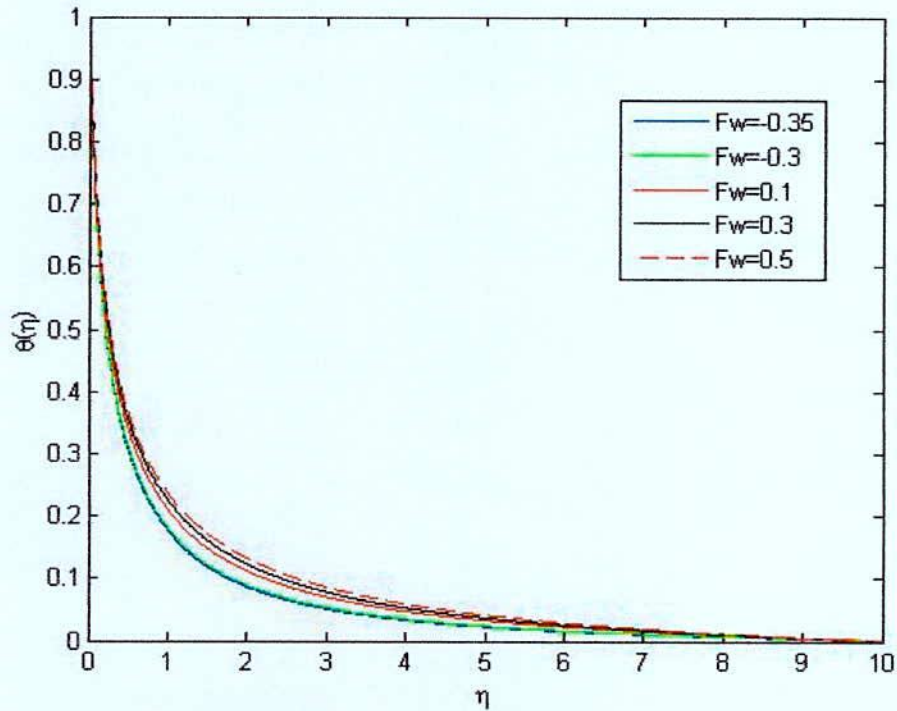


Figure 4.2: Temperature profiles for different values of F_w (with fixed values

$$\text{of } \frac{U_F^2}{u_e^2} = 0.5, R_0 = 0.1, Pr = 0.71 \text{ and } \beta = 0.5).$$

The other controlling parameter is the Prandtl number Pr , ($= \frac{\mu C_p}{k}$) which depends on the properties of the medium. Here velocity exhibits minor changes while temperature exhibits significant changes with the variation of Prandtl number Pr . As it is observed from Figure 4.3 that, the velocity is starting at zero, then velocity increases with the increase of η near the leading edge. The maximum velocity appears at $\eta \approx 2.4$ and finally moves towards 1.0 asymptotically. Velocity increases negligibly with the increase of Pr before being 1.0 asymptotically in all cases for large value of η . From Figure 4.4 we see that in all cases the temperature is starting from 1.0 and temperature decreases quickly with decreasing Pr close to the leading edge and finally leads to zero smoothly for larger η .

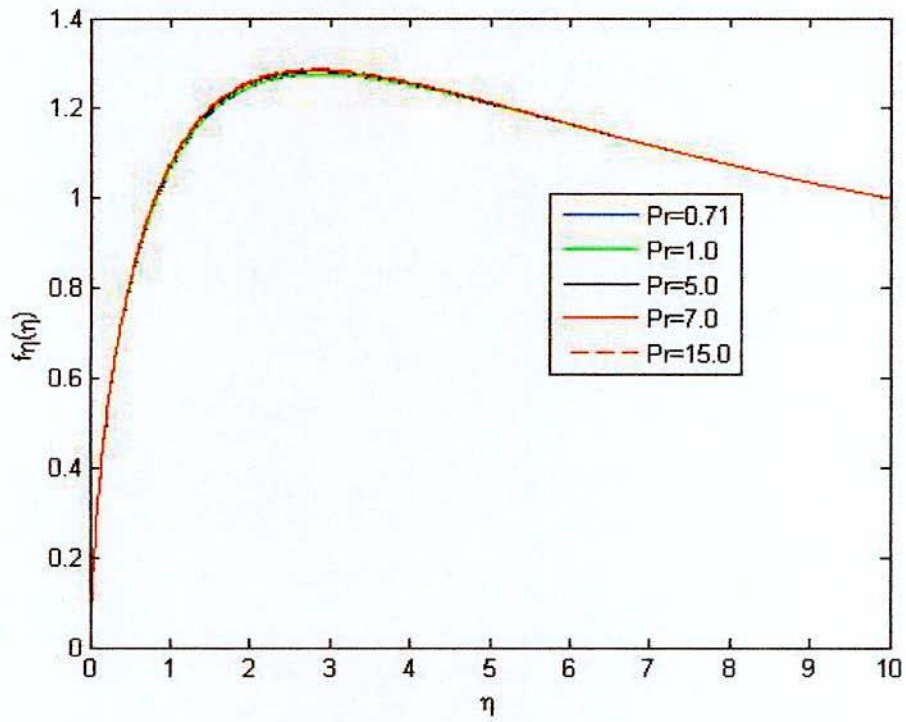


Figure 4.3: Velocity profiles for different values of Pr (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $\beta = 0.5$, $F_w = 0.5$ and $R_0 = 0.1$).

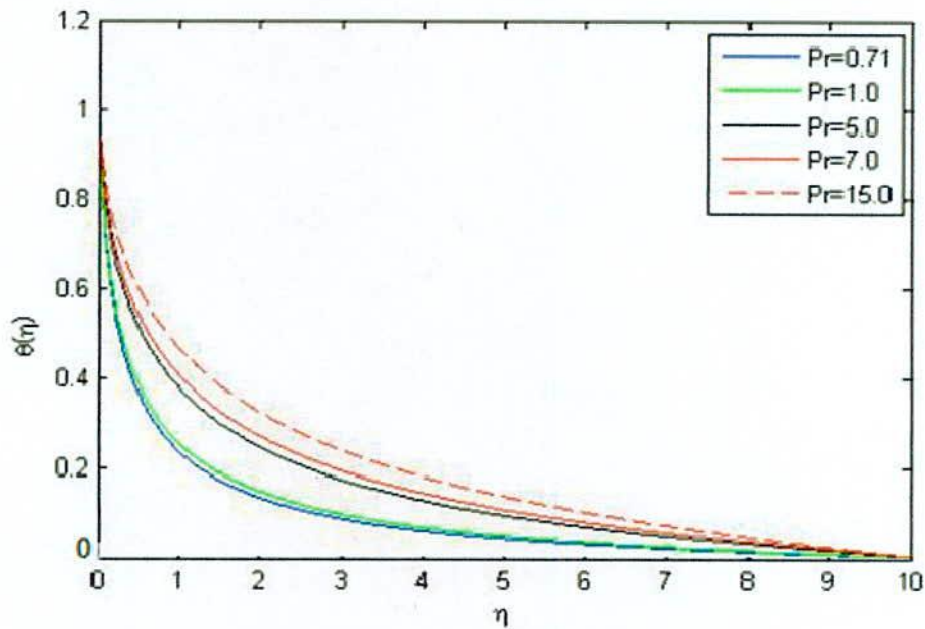


Figure 4.4: Temperature profiles for different values of Pr (with fixed values of

$$\frac{U_F^2}{u_e^2} = 0.5, F_w = 0.5, R_0 = 0.1 \text{ and } \beta = 0.5).$$

The body radius parameter R_0 depends on the shape of the slender body. There is a remarkable consequence of the variation of R_0 on the velocity and temperature fields as are observed in Figure 4.5 and Figure 4.6. Like before, for all values of R_0 , velocity starts from zero and then increases with the increase of η and the maximum velocity appears when $\eta \approx 2.4$ and then finally leads to 1.0, whereas the temperature starting from 1.0, then decreases with the increasing of η and finally asymptotically leads to zero for larger η . We see from Figure 4.5 that within the boundary layer, velocity highly increases with the increase of R_0 , before asymptotically being 1.0 for large value of η . From Figure 4.6 it is observed that temperature decreases sharply with the increases in R_0 before being zero asymptotically for larger value of η .

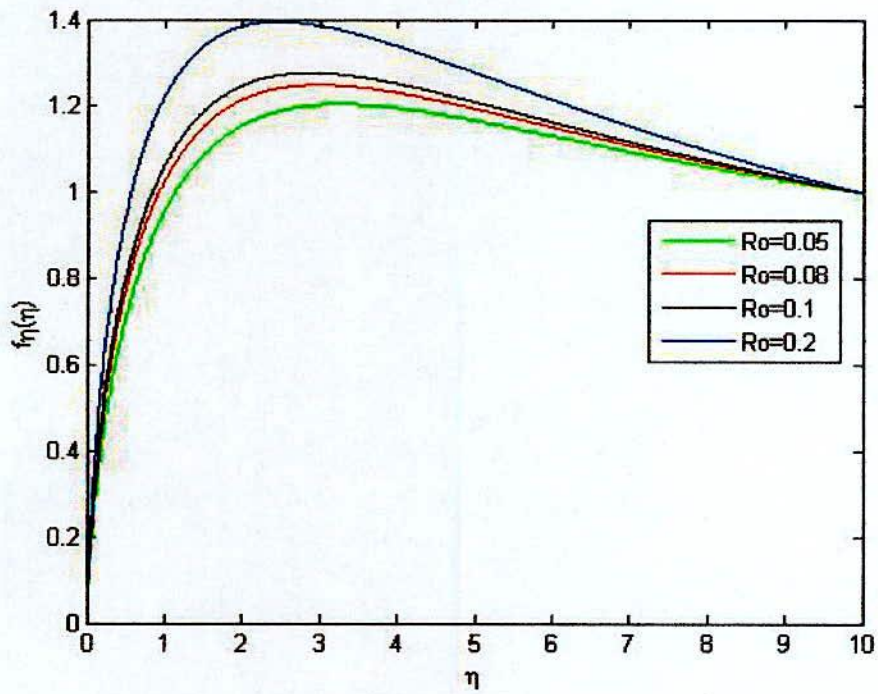


Figure 4.5: Velocity profiles for different values of R_0 (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $F_w = 0.5$, $Pr = 0.71$ and $\beta = 0.5$).

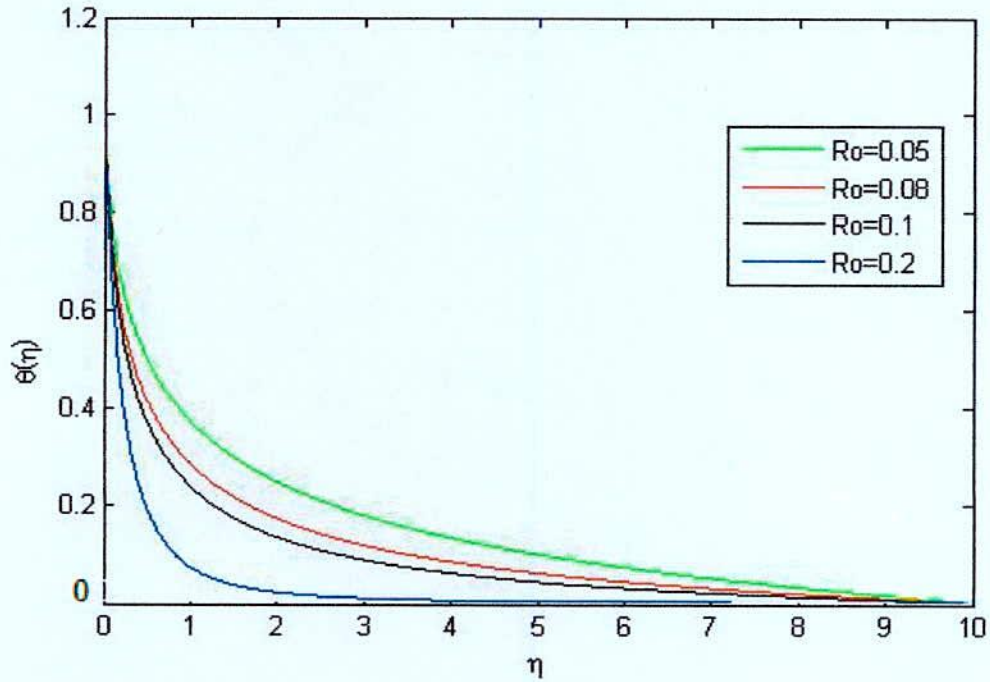


Figure 4.6: Temperature profiles for different values of R_0 (with fixed values of

$$\frac{U_F^2}{u_e^2} = 0.5, Pr=0.71, F_w = 0.5 \text{ and } \beta = 0.5).$$

Figure 4.7 and Figure 4.8 show the effect of buoyancy parameter $\frac{U_F^2}{u_e^2}$ on the velocity and temperature fields respectively. We see from the Figure 4.7 that all cases the velocity is starting at zero and increasing asymptotically to 1.0. But with increasing $\frac{U_F^2}{u_e^2}$ the rate of change of velocity increases slightly. The maximum velocity appears when $\eta \approx 2.6$. Thus before being asymptotically goes to 1.0 far away; velocity is higher for higher values of $\frac{U_F^2}{u_e^2}$ within the boundary layer. Since the energy equation is independent of the buoyancy parameter $\frac{U_F^2}{u_e^2}$, no effect of this parameter on the temperature field is observed as shown in Figure 4.8.

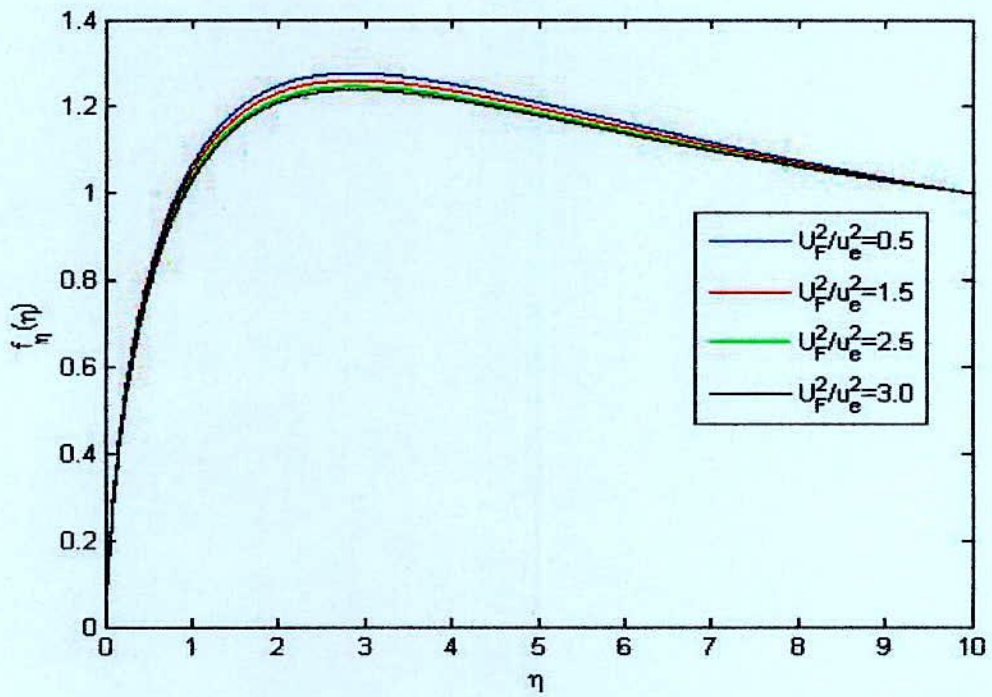


Figure 4.7: Velocity profiles for different values of $\frac{U_F^2}{u_e^2}$ (with fixed values of $\beta = 0.5$, $Pr = 0.71$, $F_w = 0.5$ and $R_0 = 0.1$).

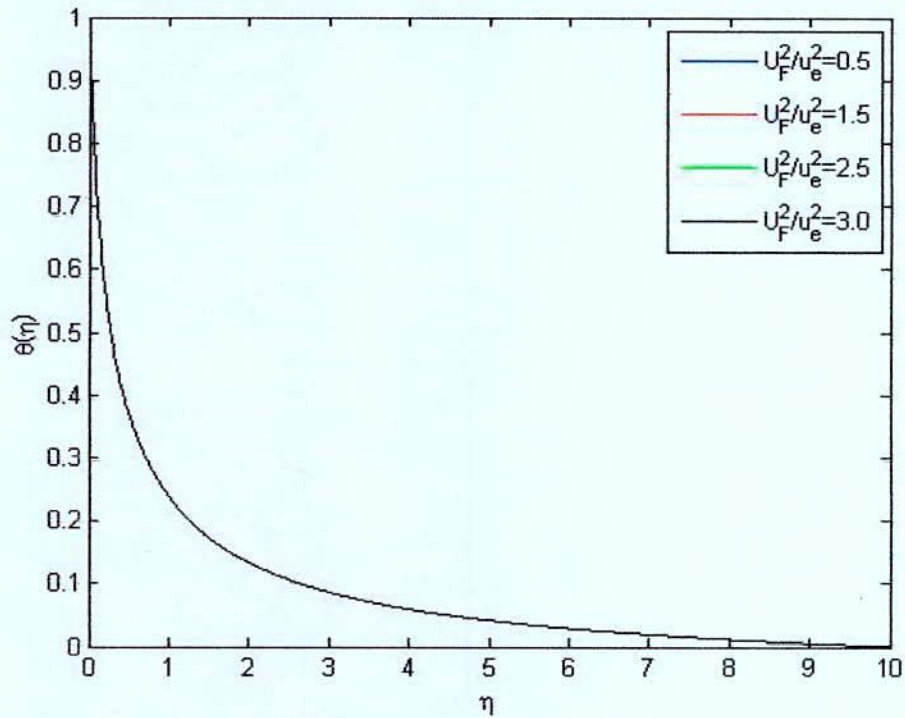


Figure 4.8: Temperature profiles for different values of $\frac{U_F^2}{u_e^2}$ (with fixed values of $\beta = 0.5$, $Pr = 0.71$, $F_w = 0.5$ and $R_0 = 0.1$).

Figure 4.9 and Figure 4.10 exhibit the effects of the driving parameter β on the velocity and temperature fields, respectively. The velocity and temperature fields exhibit remarkable changes with the variation of β as observed from Figure 4.9 and Figure 4.10 respectively. From Figure 4.9 it is observed that in all cases the velocity is starting at zero and increasing asymptotically to 1.0. But the velocity increases with increasing β and maximum velocity appears when $\eta \approx 2.3$ for $\beta = 1.0$ and then finally leads to 1.0 asymptotically for larger value of η . Like before, temperature decreases faster with the decreasing β and finally leads to zero asymptotically for larger value of η as shown in Figure 4.10.

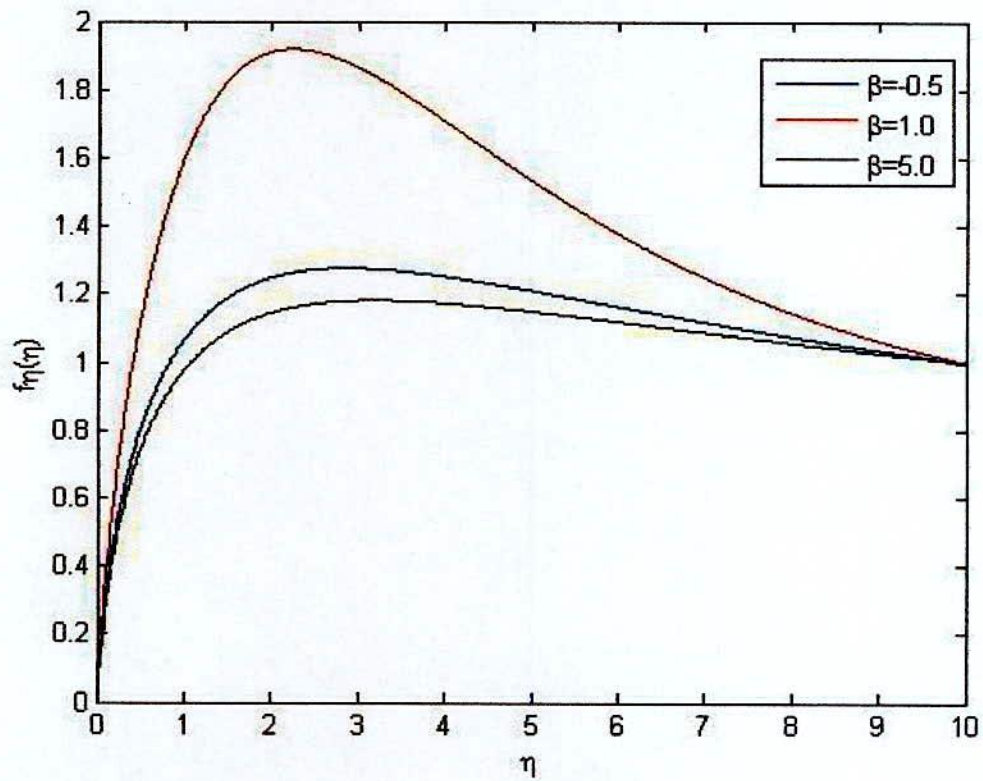


Figure 4.9: Velocity profiles for different values of β (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $Pr=0.71$, $F_w = 0.5$ and $R_0 = 0.1$).

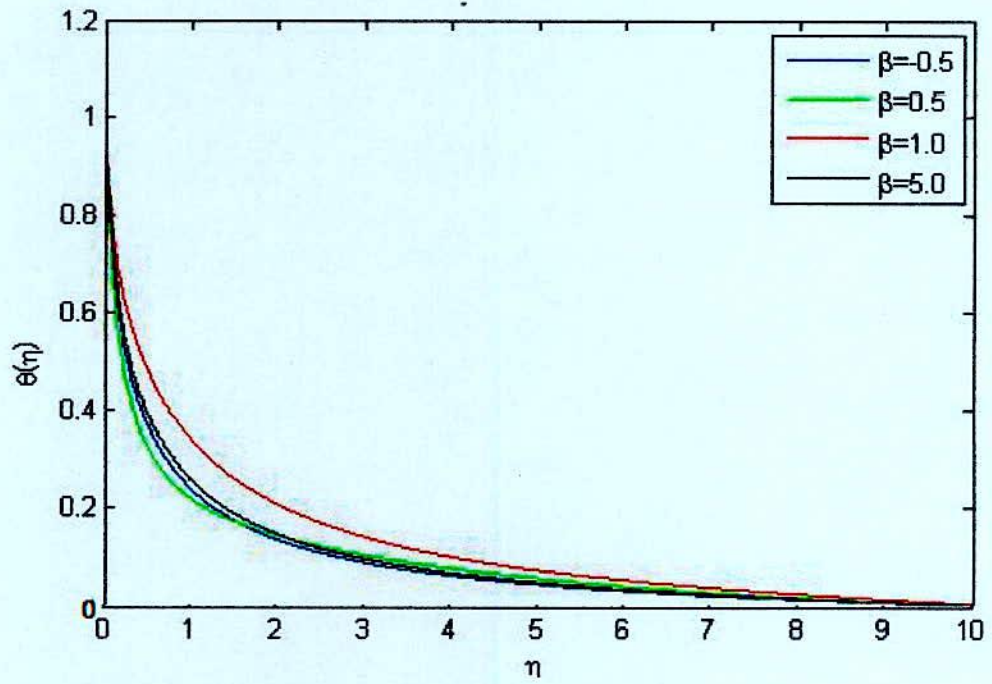


Figure 4.10: Temperature profiles for different values of β (with fixed values of

$$\frac{U_F^2}{u_e^2} = 0.5, Pr = 0.71, F_w = 0.5 \text{ and } R_0 = 0.1).$$

CHAPTER V

Conclusions and Recommendations

By using the technique of similarity solutions, the governing boundary layer equations for the two dimensional mixed convective laminar boundary layer flow around a vertical slender body have been solved in this present thesis. It is considered that the surface of the slender body is porous so that the effects of suction and blowing are taken into account. Different similarity cases arise with the choice of $\frac{dA}{d\tau}$ and $\frac{dB}{d\xi}$ either zero or not equal to zero or constants. Similarity solutions for one of the cases, namely, $\frac{\partial A}{\partial \tau} = 0$ and $\frac{\partial B}{\partial \xi} = 0$, have been studied in this thesis. Through out the studies very small suction or blowing value have been considered. On the basis of findings, it is observed that:

- (a) Velocity increases with decreases of suction but for the case of blowing the velocity increases with the increase of magnitude of blowing.
- (b) Temperature decreases with decreasing suction but temperature decreases more with decrease of the magnitude of blowing.
- (c) Velocity increases negligibly with the increase of Pr and finally the least to the value 1.0 asymptotically for larger value of η .
- (d) Temperature decreases quickly with decreasing Pr close to the leading edge and finally leads to zero smoothly for larger η .
- (e) Velocity increases with the increase of R_0 and finally leads to 1.0 asymptotically for large η .

- (f) Temperature decreases sharply with the increase of R_0 and finally approaches zero asymptotically for large η .
- (g) With the increase in the buoyancy parameter $\frac{U_F^2}{u_e^2}$ the rate of change of velocity increases slightly and finally leads to 1.0 asymptotically for larger η .
- (h) No effect of buoyancy parameter $\frac{U_F^2}{u_e^2}$ on the temperature field is observed.
- (i) The velocity increases with increasing driving parameter β and finally leads to 1.0 asymptotically for larger value of η .
- (j) Temperature decreases faster with the decreasing β and finally leads to zero asymptotically for larger value of η .

Rest of the similarity cases should be analyzed for better understanding of the problem.

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