

Sequentially Renew Weighted Opportunity Cost Based Algorithm in Transportation Problems

by

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Declaration

This is to certify that the thesis work entitled "**Sequentially Renew Weighted Opportunity Cost Based Algorithm in Transportation Problems**" has been carried out by Pushpa Akhtar in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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This is to certify that the thesis work submitted by **Pushpa Akhtar** entitled "**Sequentially Renew Weighted Opportunity Cost Based Algorithm in Transportation Problems**" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of **Master of Science** in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh in December 2017.

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Dedication

To

My Parents

Sayed Makbul Hossain & Parul Begum

Acknowledgment

At first, I want to express my gratitude to my honorable teacher and supervisor Dr. A. R. M. Jalal Uddin Jamali, Professor, Department of Mathematics, Khulna University of Engineering & Technology, Khulna. His numerous suggestions and effective guidance always encouraged me to do better on this topic. Without his inspiration and supervision I could not reach at this stage. I am lucky to get him as my supervisor.

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Abstract

Transportation models are of multidisciplinary fields of interest. In classical transportation approaches, the flow of allocation is controlled by the cost entries and/or manipulation of cost entries – so called Distribution Indicator (DI) or Total Opportunity Cost (TOC). But these DI or TOC tables are formulated by the manipulation of cost entries only. None of them considers demand and/or supply entry to formulate the DI/ TOC table. In this research we have developed Weighted Opportunity Cost (WOC) matrix, which is off course a new idea, for the control of the flow of allocations. It is noted that this WOC matrix is formulated by the manipulation of supply and demand entries along with cost entries as well. In this WOC matrix, the supply and demand entries act as weighted factors. Now it is known that, in Least Cost Matrix method, the flow of allocations are controlled by the least cost entries only and we do not need to change allocation direction in sub-sequence steps. On the other hand in Vogel's Approximation Method, the flow of allocation is controlled by the DI table and this table is updated after each allocation step. Motivated by this idea, we have reformed the WOC matrix as Sequentially Updated Weighted Opportunity Cost (SUWOC) matrix. The significance difference of these two matrices is that, WOC matrix is invariant through all over the allocation procedures whereas SUWOC matrix is updated in each step of allocation procedures. Note that here update (/invariant) means changed (/unchanged) the weighted opportunity cost of the cells. Finally by incorporating this SUWOC matrix in LCM, we have developed a new approach to find out Initial Feasible Basic Solution of Transportation Problems. Some experiments have been carried out to justify the validity and the effectiveness of the proposed SUWOC-LCM approach. Experimental results have shown that the SUWOC-LCM approach outperforms. Moreover sometime this approach is able to find out optimal solution too.

Publications

The following articles have been extracted from this thesis work:

1. Jamali, A. R. M. Jalal Uddin, Jannat, F. and **Akhtar, P.**, 2017, “Weighted cost opportunity based algorithm for initial basic feasible solution: A new approach in transportation problem”, Journal of Engineering Science, Vol. 8, no. 1, pp. 63-70.
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3. A. R. M. Jalal Uddin Jamali, **Pushpa Akhtar** and Fatima Jannat, “Weighted cost based distribution opportunity table in transportation problem”, Proceedings of the 4th International Conference on Mechanical Engineering and Renewable Energy, (ICMERE 2017) Chittagong, Bangladesh, 18 – 20, December 2017.
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CHAPTER I

Introduction

1.1 Background

Transportation problem is a particular class of linear programming which is associated with day to day activities in our life and mainly deals with logistics. It helps in solving problems on distribution and transportation of resources from one place to another. The goods are transported from a set of sources (e.g., factory) to a set of destinations (e.g., warehouse) to meet the specific requirements.

The transportation problem arises in many fields of applications e.g. industry, communication network, planning, scheduling transportation and allotment etc. Transportation problem deals with the problem how to plan production and transportation in such an industry given several plants at different location and large number of customers of their products. The transportation problem received this name because many of its applications involve in determining how to optimally transport goods [Asase (2011)].

Because of its major application in solving problems which involving several products sources and several destinations of products, this type of problem is frequently called “The Transportation problem”. The classical transportation problem is referred to as special case of Linear Problem (LP) and its model is applied to determine an optimal solution of delivery available amount of satisfied demand in which the total transportation cost is minimized [Gupta and Hira (2000)].

1.2 Literature Review

The first linear programming formulation of a problem that is equivalent to the general linear programming problem was given by Kantorovich (1939), who also proposed a

method for solving it. He developed it during World War II as a way to plan expenditures and returns so as to reduce costs to the army and increase losses incurred by the enemy. About the same time as Kantorovich, the Dutch-American economist Koopman (1949) formulated classical economic problems as linear programs. Kantorovich and Koopmans later shared the Nobel Prize in economics. Khachiyan (1979) devised the ellipsoid method. More recent, Karmarkar (1984a-1984b) developed a new method to solve LPP.

The basic transportation problem was originally developed by Hitchcock (1941) and later discussed in details by Koopman (1949). Efficient methods of solution are derived from the Simplex algorithm and were developed in 1947. The transportation problem can be converted as a standard linear programming problem, which can be solved by the Simplex method. However, because of its very special mathematical structure, it was recognized early that the Simplex method applied to the transportation problem can be made quite efficient in terms of how to evaluate the necessary Simplex-method information (variable to enter the basis, variable to leave the basis and optimality conditions).

The simplest procedure for finding an initial basic feasible solution was proposed by Dantzig (1951) and was termed the Northwest Corner rule by Charnes and Cooper (1954). Later Dantzig (1963) proposed the method of solving Linear LPP by Simplex method in 1963. Simplex algorithm was used to solve the LPP. But it was laborious. For this reason, researchers try, wherever possible, to simplify the way of calculations. Resultant of one such effort is Transportation Model.

Charnes and Cooper (1954) developed the Stepping Stone Method which provides an alternative way of determining the simplex-method information. Dantzig (1963) used the Simplex method in the transportation problem as the Primal Simplex transportation method. An initial basic feasible solution for the transportation problem can be obtained by using the North West Corner (NWC) rule.

Arsham et al. (1989) introduced a new algorithm for solving the transportation problem. The proposed method used only one operation, the Gauss Jordan pivoting method, which was used in Simplex method. The final table can be used for the post optimality analysis of

transportation problem. This algorithm is faster than Simplex, more general than Stepping Stone and simpler than both in solving general transportation problem.

Many researchers have developed a numbers of transportation algorithms and also research works are ongoing for better results. Moreover for finding Initial Basic Feasible Solution (IBFS) much of the research works are concerned with cost matrix and manipulation of cost matrix. It is noted that in TP, all the optimized algorithms initially need an IBFS to obtain the optimal solution [Korte and Vygen (2012), Sifaleras (2013) and Bazaraa (2009)].

There are various simple heuristic methods available to get an IBFS, such as, North-West Corner method, Row Minimum method, Column Minima method, Least Cost Matrix method etc. [Taha (2003)]. Among all the simple heuristic methods, the Least Cost Matrix (Matrix Minima) is relatively efficient and this method considers the lowest cost cell of the Transportation Table (TT) for making allocation in every stage.

Vogel's Approximation Method is a well-known algorithm for IBFS [Reinfeld and Vogel (1958)] which provides good IBFS too. In this method, author developed a Distribution Indicator (DI) table for the purpose of allocations. In DI table author introduced the penalties which are determined from the difference of smallest and next to the smallest cost entries. The allocation flow is done according to the DI Table. Vogel's Approximation Method (VAM) provides comparatively better Initial Basic Feasible Solution. It is noted that, the DI table namely penalty rows and columns are updated after each allocation.

After VAM method, researchers proposed several versions of the VAM method by modifying some tricks [Shimshak et al., (1981), Goyal (1984), Ramakrishnan (1988), Islam (2012), Korukoglu and Balli (2011), Amirul (2012), Hakim (2012), Kawser (2016) and Azad (2017)].

Recently, Sharma and Bhadane (2016) presented an alternative method to North West Corner method by using statistical tool called Coefficient of Range (CoR). On the other hand, Azad et al. [Azad (2017)] at first developed TOC then they formed DI tableau for allocation by considering the average of TOC of cells along each row identified as Row Average Total Opportunity Cost (RATOC) and the average of TOC of cells along each

column identified as Column Average Total Opportunity Cost (CATOC). Allocations of costs are started in the cell along the row or column which has the highest RATOCs or CATOCs.

As Transportation Problems (TP) is playing very important role to ensure in time availability of raw materials and finished goods from different sources to distinct destinations, many researchers are continuously research to develop better transportation algorithms. Most of the research works concern with the cost matrix, we mean manipulation of cost entries to form Distribution Indicator (DI) /or Total Opportunity Cost Matrix whatever be the structure of supply row and demand column.

1.3 Goal of the Study

It is noted that, all the approaches discussed above are concerned with the cost entries and /or the manipulation of cost entries to form DI or TOC table whatever be the structure of supply and demand. None of them considered supply/ demand to formulate DI or TOC in allocation procedures. But it might be assumed that, supply and demand play a vital role in the formulation of cost allocation table to obtain a better solution.

As we hope, there are significant effects of demand and supply entries in DI/TOC table, we have tried to develop an allocation flow indicator matrix by considering demand and supply entries as a weight factor correspond to each cost entry. It is no doubt that, it was very difficult to form a significant weight factor by considering demand and supply entries. After finding effective weight parameter we have paid attention to formulate weighted based distribution indicator.

Therefore, the first aim of research is to formulate a virtual weighted opportunity cost (WOC) table by considering supply and demand entries as a weight factor. Then incorporating the concept of VAM upon WOC, a virtual dynamic weighted cost opportunity matrix is formulated. Finally we have developed a new sequential updated weighted-cost based algorithm embedded on LCM method to find IBFS of TP.

For effectiveness and efficiency of the proposed algorithm intensive experiments have

been carried out. From the experimental results it may conclude that our proposed cost minimization based transportation algorithm is efficient and comparable to other existing approaches regarding primal solutions.

1.4 Arrangement of the Thesis

In **Chapter I** the introduction of the research works is presented. Moreover the background, literature review and goal of the study are discussed in this chapter. **Chapter II** presents about the preliminaries. In **Chapter III**, extensive investigations are carried out to formulate weighted opportunity cost matrix. In **Chapter IV** several experiments are performed by incorporating the WOC matrix upon LCM approach. We have developed an algorithm named SUWOC-LCM which is discussed in the **Chapter IV**. Moreover in this chapter we have presented some numerical instances to justify and the effectiveness of the proposed algorithm. The optimality of the IBFS of the approach is discussed in this chapter too. Finally concluding remarks about the research works are given in **Chapter VI**. The list of the references is at the end of the thesis as well.

CHAPTER II

Preliminaries

2.1 Introduction

In this chapter we will present some basic concepts related to the research works. Mainly we will focus on the transportation problem (TP). It is known that TP is a special class of linear programming problem, which deals with shipping commodities from source to destinations with certain constraints. The objective of the TP is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply and demand limits. In a transportation problem, we have certain origins, which may represent factories where items are produced. On the other hand the produced items are supplied to a certain number of destinations according to their demands. This must be done in such a way as to maximize the profit or minimize the transportation cost. Thus we have the places of production as origins and the places of supply as destinations. Sometimes the origins and destinations are also termed as sources and sinks, respectively.

2.2 Network of Transportation Model

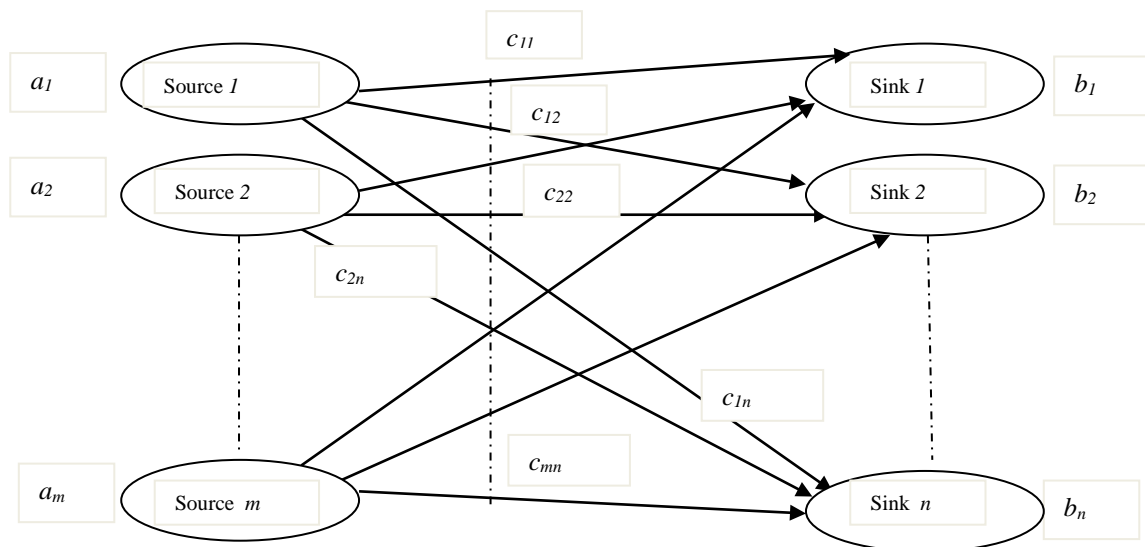


Figure.2.1 Schematic view of the Transportation Network Model

The general transportation problem is represented by the network in Figure 2.1. There are m sources and n destinations (sinks), each represented by a node. The arcs represent the routes linking the sources and the destinations. Arc (i,j) joining source i to sink j carries two types of information: the unit transportation cost, c_{ij} , and the amount shipped, x_{ij} . The amount of supply at source i is a_i and the amount of demand at sink j is b_j . The objective of the model is to determine the unknown x_{ij} that will minimize the total transportation cost while satisfying all supply and demand restrictions [Taha (2003)].

2.3 Mathematical Model of the TP

The above network model of the transportation problem can be presented as a mathematical model especially Linear Programming (LP) model as follows:

$$\text{Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n x_{ij} c_{ij} \quad (2.1)$$

Subject to

$$\sum_{j=1}^n x_{ij} = a_i \forall i = 1, 2, 3, \dots, m \quad (2.2)$$

$$\sum_{i=1}^m x_{ij} = b_j \forall j = 1, 2, 3, \dots, n \quad (2.3)$$

$$x_{ij} \geq 0; a_i \geq 0; b_j \geq 0 \quad \forall i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n \quad (2.4)$$

where Z : Total transportation cost to be minimized, which is the objective function.

c_{ij} : Unit transportation cost of the commodity from each source i to destination j .

x_{ij} : Number of units of commodity sent from source i to destination j , which is called decision variable.

a_i : Level of supply at each source i .

b_j : Level of demand at each destination j .

The Equations (2.2) indicate supply constraints and (2.3) indicate demand constraints. In brief Equations (2.2) and (2.3) are called capacity constraints whereas constraint (2.4) is called non-negative restrictions conditions. Note that the problem is called balanced if Total Supply = Total Demand. Mathematically,

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j \quad (2.5)$$

Otherwise the problem will be unbalanced.

Now Equation (2.1) indicates that the above TP has mn variable, Equation (2.2) have m constraints and Equation (2.3) have n constraints. That is the above TP has $m+n$ constraints excluding the non-negativity constraints (2.4). Now for a balanced transportation problem we have $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$. A consequence of this is that the problem is defined by $n + m - 1$ supply and demand variables. Since, if $a_i, i = 1, 2, \dots, m$ and $b_j, j = 1, 2, \dots, n$ are specified, then one of a_i can be found from (2.5). This means that one of the constraint equations is not required. Thus, a balanced transportation model has $n + m - 1$ independent constraint equations. Since the number of basic variables in a basic solution is the same as the number of constraints, solutions of this problem should have $n + m - 1$ basic variable, x_{ij} , (which is non-zero for non-degenerated solution) and all the remaining variables will be non-basic and thus have the value zero.

2.4 Assumptions of Transportation Problem

- a) Only a single type of commodity is being shipped from an origin to a destination.
- b) Total supply is equal to the total demand. $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, where both a_i (supply) and b_j (demand) are all positive integers.
- c) The unit transportation cost of the item from all the sources to destinations is certainly and preciously known.
- d) The objective is to minimize the total cost.

2.5 Transportation Tableau

Though the TP problem can be solved as a regular Linear Programming model but its special structure allows us to solve the problem more conveniently using the Transportation Tableau shown in the Table 2.1. In the Transportation Tableau O_i indicates i th source with amount of availability is a_i , which is shown in the far right column. On the other hand D_j denotes j th destination with demand b_j , which is shown in the bottom row of the tableau. The unit shipping cost from origin, O_i , to destination, D_j , be c_{ij} which is shown in the (i,j) cell of the cost matrix $[c_{ij}]$ of the table.

Table 2.1 Tabular view of a Transportation Problem (TP)

		Destinations					Supply
		D_1	D_2	D_n	
Origins	O_1	c_{11}	c_{12}	c_{1n}	a_1
	O_1	c_{21}	c_{22}	c_{2n}	a_2
	⋮	⋮	⋮		
	⋮	⋮	⋮		
	O_m	c_{m1}	c_{m2}	c_{mn}	a_m
	Demand	b_1	b_2	b_n	
	Requirement						

2.6 Some Important Definitions of Transportation Problem

(a) **Feasible Solution (FS):** A Set of non-negative allocations $x_{ij} > 0$ which satisfies the row and column restrictions of the TP is known as feasible solution.

(b) **Basic Feasible Solution (BFS):** A feasible solution to an m origin and n destination TP is said to be basic feasible solution if the number of positive allocations are $(m+n-1)$.

(c) **The Initial Basic Feasible Solution (IBFS):** If the sum of origin capacities equals the sum of destination requirements of a transportation problem involving m origins and n destinations, then a feasible solution of the TP always exists. Any feasible solution satisfying $m+n-1$ of the $m+n$ constraints is a redundant one and hence can be deleted. This means that a feasible solution to a TP can have at most only $m+n-1$ strictly positive component.

(d) **Degenerate Solution and Non- degenerate Basic Feasible Solution:**

A feasible solution to a m - origins and n - destinations problem is said to be Basic Feasible Solution if the number of positive allocations are $(m + n - 1)$. If the number of allocations in basic feasible solution are less than $(m + n - 1)$, it is called **Degenerate** Basic Feasible Solution (DBFS) otherwise **non-degenerate**. That is a basic feasible solution to Equations (2.1 – 2.4) is called degenerate if one or more the basic variables are zero. Note that to

resolve *degeneracy*, we make use of an artificial quantity, d , which is very small. The quantity d is assigned to that unoccupied cell, which has the minimum transportation cost. Again it is noted that a non-degenerate basic feasible solution is a basic feasible solution with exactly $m+n-1$ positive x_{ij} , that is, all basic variables are positive.

2.7 Solution Algorithm for the Transportation Problem

The solution algorithm to a transportation problem can be summarized into following steps:

a) Formulate the problem and set up in the matrix form:

The formulation of transportation problem is similar to LP problem formulation. Here the objective function is the total transportation cost and constrains are the supply and demand available at each source and destination, respectively.

b) Obtain an initial basic feasible solution:

There are several methods exist in literature to find out Initial Basic Feasible Solution (IBFS) like Least Cost Method (LCM), Vogel's Approximation Method (VAM) etc.

c) Find out optimal solution:

There are also several methods exist in literature to find out optimal solution from the IBFS such as Modified Distribution (MODI) method and Stepping Stone Method.

We have briefly presented the algorithm of LCM method, since our proposed algorithm is developed based on LCM method. Moreover since, in our proposed algorithm, the allocation procedures is updated after each allocation like VAM method, so we have also briefly shown the algorithm of VAM method below.

2.7.1 Least Cost Matrix (LCM) Method

Least Cost Matrix (LCM) or Matrix minimum method is a method for computing IBFS of a transportation problem where the basic variables are chosen according to the minimum unit cost of transportation. The main steps of LCM algorithm are given below:

Step 1. Identify the smallest cost in the cost matrix of the transportation table, and allocate maximum possible allocation to the corresponding cell such that the allocation will be minimum between the corresponding supply (row) and demand (column) units.

Step 2. Cross out the satisfied supply (row) or demand (column) i.e. which is minimum. Also update the unallocated amounts to the row or column.

Step 3. Repeat step 1 and step 2 for the resulting reduced transportation table until all the requirements are satisfied.

It is noted that whenever the minimum cost is not unique, make an arbitrary choice among the minima.

2.7.2 Vogel's Approximation Method (VAM)

VAM is an improved version of the Least-Cost Matrix method that generally, but not always, produces better starting solutions. VAM is based upon the concept of minimizing opportunity (or penalty) costs. The Opportunity cost for a given supply row or demand column is defined as the difference between the lowest cost and next lowest cost alternative. The main steps involved in determining an initial solution using VAM are as follows:

Step 1. Compute the difference between the minimum each row and each column cost and the next minimum cost corresponding to each row and each column which is called penalty cost.

Step 2. Identify the row or column with the largest penalty and assign highest possible value to the variable having smallest shipping cost in that row or column. Suppose it is the cell C_{ij} . Then allocate $\min(a_i, b_j)$ to this cell C_{ij} .

Step 3. Now if the $\min(a_i, b_j) = a_i$ then the availability of the i th origin is exhausted and demand at the j th destination remains as $b_j - a_i$ and the i th row is cross out. Again if $\min(a_i, b_j) = b_j$, then demand at the j th destination is fulfilled and the availability at the i th origin remains to be $a_i - b_j$ and the j th column is cross out.

Step 4. Compute new penalties with same procedure until one row or column is left out.

Step 5. Repeat steps 2, 3, and 4 with the remaining table until all origins are exhausted and all demands are fulfilled.

2.8 Optimality Test and Procedure for Optimal Solution of TP

A feasible solution is said to be optimal if it minimizes the total transportation cost. There are basically two well known methods available to test the optimality as well as to find

Optimal solution (if the IBFS is not optimal) from the obtained IBFS of the TP. The well-know methods are given below:

- (a) Modified Distribution Method (MODI)
- (b) Stepping Stone Method.

Anyway here we have discussed the MODI only. The main steps of the method are given below:

Steps

1. Determine an initial basic feasible solution using any method such as Least Cost Method.
2. Determine the values of dual variables, u_i and v_j , using $u_i + v_j = c_{ij}$
3. Compute the opportunity cost using $\Delta_{ij} = (u_i + v_j) - c_{ij}$
4. Check the sign of each opportunity cost.
 - (a) If the opportunity costs of all the unoccupied cells are either positive or zero, the given solution is the optimal solution. On the other hand,
 - (b) If one or more unoccupied cell has negative opportunity cost, the given solution is not an optimal solution and further savings in transportation cost are possible.
5. Select the unoccupied cell with the smallest negative opportunity cost as the cell to be included in the next solution.
6. Draw a closed path or loop for the unoccupied cell selected in the previous step. Note that, the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.
7. Assign alternate plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.
8. Determine the maximum number of units that should be shipped to this unoccupied cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Now, add this quantity to all the cells on the corner points of the closed path marked with plus signs, and subtract it from those cells marked with minus signs. In this way, an occupied cell becomes an occupied cell.
9. Repeat the whole procedure until an optimal solution is obtained.

CHAPTER III

Formulation of Weighted Opportunity Cost Matrix

3.1 Introduction

Transportation network flow models are multidisciplinary field of interest. Here we will consider a simple real five example in business arena. Then the physical problem is formulated as a mathematical model as a Linear Programming (LP) Model. Then this LP will be represented as TP in transportation tableau in which we have a cost matrix along with supply and demand entries. Finally, we will develop a Weighted Opportunity Cost (WOC) matrix which is actually be a distribution indicator matrix.

3.2 Transportation Model of a Physical Problem

It is mentioned earlier that we will consider a real life physical problem. We have considered the following simple Example 3.2.1.

Example 3.2.1: A company has 5 production centers i.e. factories O_1, O_2, O_3, O_4 and O_5 in given locations with production capacities of 10, 25, 15, 20 and 30 ton (per day), respectively, of a certain product with which it must supply 5 warehouses, D_1, D_2, D_3, D_4 and D_5 , where the demand of the warehouses are 20, 10, 5, 30 and 35 ton (per day), respectively, The unit costs of transportation, from factory O_i to destination (warehouse) D_j is shown in the (i,j) th, cell C_{ij} , of the cost matrix $[c_{ij}]$ which is shown in the Table 3.1.

Table 3.1. The unit cost of transportation from the factories to the warehouses

		Warehouse				
		D_1	D_2	D_3	D_4	D_5
Factories	O_1	1	2	3	4	5
	O_2	2	3	4	5	6
	O_3	3	4	5	6	7
	O_4	4	5	6	7	8
	O_5	5	6	7	8	9

3.2.1 Formulation of mathematical model

Step 1: Key decision to be made is to find how much quantity of production from which factories to which warehouses is shipped so as to satisfy the constraints and minimize the cost. Since there are 5 factories (Origins) and 5 warehouse (Destinations), so there are 5×5 i.e. 25 possible variables: $x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{31}, x_{32}, x_{33}, x_{34}, x_{35}, x_{41}, x_{42}, x_{43}, x_{44}, x_{45}, x_{51}, x_{52}, x_{53}, x_{54}$ and x_{55} . These variables represent the quantities of product to be shipped from different factories to different warehouses and can be represented in the form of a matrix shown in the Table 3.2 below:

Table 3.2. Amount of transportation commodity in matrix form

		Warehouse				
		D_1	D_2	D_3	D_4	D_5
Factories	O_1	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
	O_2	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}
	O_3	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}
	O_4	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}
	O_5	x_{51}	x_{52}	x_{53}	x_{54}	x_{55}

In general, we can say that the key decision to be made is to find the quantity of units to be transported from each origin to each destination. Thus, if there are m origins and n destinations, then x_{ij} are the decision variables (quantities to be found), where $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

Step 2 (*set non negativity constraint*): Feasible alternatives are sets of values of x_{ij} , where $x_{ij} > 0$.

Step 3 (set objective function): Objective is to minimize the cost of transportation i.e.,

$$\begin{aligned} \text{Minimize } Z = & \{1x_{11} + 2x_{12} + 3x_{13} + 4x_{14} + 5x_{15} \\ & + 2x_{21} + 3x_{22} + 4x_{23} + 5x_{24} + 6x_{25} \\ & + 3x_{31} + 4x_{32} + 5x_{33} + 6x_{34} + 7x_{35} \\ & + 4x_{41} + 5x_{42} + 6x_{43} + 7x_{44} + 8x_{45} \\ & + 5x_{51} + 6x_{52} + 7x_{53} + 8x_{54} + 9x_{55} \} \end{aligned}$$

In general, we can say that if c_{ij} is the unit cost of shipping from i th source to j th destination, the objective is

$$\text{Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij}$$

Step 4 (set capacity constraints): Constraints are

(i) According to the availability or supply:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 10 \quad (\text{for factory 1})$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 25 \quad (\text{for factory 2})$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 15 \quad (\text{for factory 3})$$

$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} = 20 \quad (\text{for factory 4})$$

$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} = 30 \quad (\text{for factory 5})$$

Thus in all, there are 5 constraints (equal to the number of factories).

In general, there will be m constraints, if number of origins is m , with n number of destinations, which can be expressed as

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, 2, 3, \dots, m.$$

(ii) According to the requirement or demand:

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 20 \quad (\text{for warehouse 1})$$

$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 10 \quad (\text{for warehouse 2})$$

$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 5 \quad (\text{for warehouse 3})$$

$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} = 30 \quad (\text{for warehouse 4})$$

$$x_{15} + x_{25} + x_{35} + x_{45} + x_{55} = 35 \quad (\text{for warehouse 5})$$

In general, there will be n constraints, if number of destinations is n , with m number of origins, which can be expressed as

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, 3, \dots, n.$$

Thus we have found that the given situation involves ($5 \times 5 = 25$) variables and ($5 + 5 = 10$) constraints. In general, such a situation will involve ($m \times n$) variables and ($m+n$)

constraints.

Step 5 (Balanced TP): It is observed that total availability of factories (supply) is

$$\sum_{i=1}^5 a_i = 100, (10+25+15+20+30),$$

and total capacity of warehouse (demand) is

$$\sum_{j=1}^5 b_j = 100, (20+10+5+30+35).$$

i.e. $\sum_{i=1}^5 a_i = \sum_{j=1}^5 b_j$, the problem is balanced TP.

In general, if number of origins is m and number of destinations is n , then for balanced problem

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j.$$

Since, in general, the transportation model is balanced, so one of these constraints must be redundant. Thus, the model has $m + n - 1$ independent constraint equations, which means that the starting basic feasible solution consists of $m + n - 1$ basic variables. It is observed that in this model, the objective function and the constraints are linear functions. The following points may be noted in a transportation model:

- a) All supply as well as demand constraints are of equality type.
- b) They are expressed in terms of only one kind of unit.
- c) Each variable occurs only once in the supply constraints and only once in the demand constraints.
- d) Each variable in the constraints has unit coefficient only.

Therefore, the transportation model is a special case of general L.P model where in the above four conditions hold good and can be solved by a special technique called the transportation technique (namely Transportation tableau) which is easier and shorter than the other technique.

3.2.2 Transportation model (Transportation Tableau)

The above mathematical LP model of the given physical problem can be reformed as a Transportation Tableau so that we can solve the problem by any transportation algorithm. The transportation tableau of the given problem is displayed in the Table 3.3.

Table 3.3. A transportation Table of the given problem

		Destinations					Supply
		D_1	D_2	D_3	D_4	D_5	
Origins	O_1	1	2	3	4	5	10
	O_2	2	3	4	5	6	25
	O_3	3	4	5	6	7	15
	O_4	4	5	6	7	8	20
	O_5	5	6	7	8	9	30
Demand		20	10	5	30	35	Total 100

After formulation of transportation tableau we have to check whether $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ is true or not. If yes, the problem is said to be a balanced or self contained or standard problem. If not, a dummy origin or destination (as the case may be) is added to balance the supply and demand.

3.3 Formulation of Weighted Opportunity Cost Matrix

In classical transportation approaches, like North-West Corner Rule, Least Cost Matrix (LCM) method etc, the flow of allocation is controlled by the cost entries only; and/or manipulation of cost entries – so called Distribution Indicator (DI) such as VAM method or Total Opportunity Cost (TOC) like modified VAM method. But these DI tables are formed by the manipulation of cost entries only. None of them considers demand and/or supply entry to formulate the DI/ TOC table. We have a new idea (i.e. incorporating demand /supply) for the control of the flow of allocations which is called Weighted Opportunity Cost (WOC) matrix. It is noted that, this weighted opportunity cost matrix is formulated by the manipulation of supply and demand entries along with cost entries. In this WOC matrix, the supply and demand entries act as weighted factors on the WOC matrix.

3.3.1 Finding cell weight

At first we will find out the maximum possible allocation of any cell C_{ij} where transportation cost is c_{ij} (unit cost from origin i to destination j). Since the availability of the origin i is a_i (units) and the demand at destination j is b_j (units). So the maximum

possible allocation at cell C_{ij} is obviously $\min(a_i, b_j)$. Now since the maximum possible ability of allocation of each cell C_{ij} is $\min(a_i, b_j)$, so the total possible maximum allocation of all cells be $\sum_{i=1}^m \sum_{j=1}^n \min(a_i, b_j)$. Therefore for each cell C_{ij} , its weight factor is:

$$w_{ij} = \min(a_i, b_j) / \sum_{i=1}^m \sum_{j=1}^n \min(a_i, b_j) \quad (3.1)$$

So that $\sum_{i=1}^m \sum_{j=1}^n w_{ij} = 1$. But since the factor “ $1 / \sum_{i=1}^m \sum_{j=1}^n \min(a_i, b_j)$ ” is common to $w_{ij} \forall i, j$, so we might ignore this factor. Therefore the weight factor at cell C_{ij} be

$$w_{ij} = \min(a_i, b_j) \quad (3.2)$$

It is noted that this reduces a significant amount of computational cost.

3.3.2 Finding appropriate weighted opportunity cost entries

After successful formulation of cell weight, out next task is to formulation of weighted opportunity cost matrix. But we have a problem – since cell with lower cost has preference for allocation, on the other hand the cell with larger weight has preference for allocation. So we cannot simply multiply weight to cell cost to find meaning full elements of weighted cost matrix. To overcome this difficulty and for the formulation of meaningful weighted cost matrix, we should transform one of the two so that the multiplication of the two will be meaningful. This can be done by inversing the cost elements. Therefore the virtual weighted opportunity cost corresponding cell cost c_{ij} be

$${}^w c_{ij} = \frac{1}{c_{ij}} \times \min(a_i, b_j) \quad (3.3)$$

where ${}^w c_{ij}$ and c_{ij} denote weighted virtual cell cost and actual value of cost at the cell c_{ij} respectively.

But one problem again takes place. What happen if cell cost is zero? So to prevail over this difficulty we need some more special attentions. We can overcome this shortcoming by considering zero costs and costs which is greater than zero but less than one. So if there exist any cell whose cost entry is zero, then we can formulate the weighted cost to the cell C_{pq} as follows:

- i. If $\{c_{ij}: 0 < c_{ij} < 1 \forall i, j\} \neq \Phi$ (i.e. null set), then set ${}^w c_{pq} = M \times \min(a_i, b_j)$
where $M = \max \{a_i, b_j \forall i, j\} / [\min \{c_{ij}: 0 < c_{ij} < 1 \forall i, j\}]$.
- ii. Else if $\{c_{ij}: 0 < c_{ij} < 1 \forall i, j\} = \Phi$, set ${}^w c_{pq} = N \times \min(a_i, b_j)$
where $N = \max \{a_i, b_j \forall i, j\}$

3.3.3 Algorithm of weighted opportunity cost matrix

Now we are able to overcome all the shortcomings to formulate the weighted cost.

Therefore the algorithm of Weighted Cost Matrix (WOC) [${}^w c_{pq}$] is as follows:

(a) If $c_{pq} = 0$ and $\{c_{ij}: 0 < c_{ij} < 1 \forall i, j\} \neq \Phi$ (i.e. null set), then set ${}^w c_{pq} = M \times \min(a_i, b_j)$

where $M = \max \{a_i, b_j \forall i, j\} / [\min \{c_{ij}: 0 < c_{ij} < 1 \forall i, j\}]$.

(b) Else if $c_{pq} = 0$ and $\{c_{ij}: 0 < c_{ij} < 1 \forall i, j\} = \Phi$, set ${}^w c_{pq} = N \times \min(a_i, b_j)$

where $N = \max \{a_i, b_j \forall i, j\}$

(c) Else if $c_{pq} > 0 \forall i, j$ then set ${}^w c_{pq} = \frac{1}{c_{pq}} \times \min(a_i, b_j)$

3.4 Numerical Illustration

For numerical illustration we consider the above Example 3.1 whose cost matrix along with its supply and demand entries is given in the Table 3.3. But for the better visualization we again copy it here – which is shown at (a) of the Table 3.4. It is observed that in this Example 3.1, all the cost entries are greater than zero i.e. $c_{ij} > 0$, so the algorithm executes only the third case of the WOC algorithm (section 3.3.3). Formally

(c) Else if $c_{ij} > 0 \forall i, j$ then set ${}^w c_{ij} = \frac{1}{c_{ij}} \times \min(a_i, b_j)$

So, for the cell $c_{11} > 0$, the weighted opportunity cost

$${}^w c_{11} = \frac{1}{c_{11}} \times \min(a_1, b_1) = \left(\frac{1}{1}\right) \times \min(10, 20) = 10/1$$

Table 3.4. Transportation Tableau and corresponding WOC Tableau

(a) Transportation Tableau							(b) WOC Tableau						
	D_1	D_2	D_3	D_4	D_5	S		D_1	D_2	D_3	D_4	D_5	S
O_1	1	2	3	4	5	10	O_1	10/1	10/2	5/3	10/4	10/5	10
O_2	2	3	4	5	6	25	O_2	20/2	10/3	5/4	25/5	25/6	25
O_3	3	4	5	6	7	15	O_3	15/3	10/4	5/5	15/6	15/7	15
O_4	4	5	6	7	8	20	O_4	20/4	10/5	5/6	20/7	20/8	20
O_5	5	6	7	8	9	30	O_5	20/5	10/6	5/7	30/8	30/9	30
D	20	10	5	30	35		D	20	10	5	30	35	

Similarly we can find out all the weighted opportunity cost according to the algorithm of WOC. The final WOC Tableau corresponding to the Transportation Tableau 3.4(a) is shown in 3.4(b).

In the Example 3.1, all the cost entries are greater than zero, so we have considered another Example 3.2 whose transportation tableau is shown in the Table 3.5.

Example 3.2: Cost of transportation from the factories to the warehouses, demand and supply are shown in the Table 3.5.

Table 3.5 Cost of transportation from the factories to the warehouses

	D_1	D_2	D_3	D_4	Supply
O_1	0	0	4	5	20
O_2	1	4	2	15	25
O_3	3	2	1	4	10
O_4	4	5	6	3	10
Demand	5	10	30	20	

It is observed in the Table that there are two entries whose values are zero i.e. $c_{11} = c_{12} = 0$. Moreover, we observed in the transportation tableau 3.5 that there is no any cell cost which lies between zero and one. So for finding out weighted opportunity cost for cells C_{11} and C_{12} we need case (b) of the WOC algorithm. Formally

(b) Else if $\{c_{ij}: 0 < c_{ij} < 1 \forall i, j\} = \Phi$, set ${}^w c_{pq} = N \times \min(a_i, b_j)$

$$\text{where } N = \max \{a_i, b_j \forall i, j\}$$

On the other hand except cells C_{11} and C_{12} , all other cell costs are greater than zero. Therefore in these situations we need case (c) of the WOC algorithm. Formally

(c) Else if $c_{ij} > 0 \forall i, j$ then set ${}^w c_{pq} = \frac{1}{c_{ij}} \times \min(a_i, b_j)$

Now for find out weighted cost of the cells whose cost entries are zero, we need to find out the value $N = \max \{a_i, b_j \forall i, j\}$. It is observed that $N = 30$. Again for the cell C_{11} , the min {supply, demand} is 5. Therefore, the weighted cost for the cell C_{11} is ${}^w c_{11} = 30 \times 5 = 150$. Similarly we can able to find out well weighted cost entries. The complete WOC is

displayed in the table 3.6.

Table 3.6. The WOC table for the Transportation tableau (Table 3.5) of Example 3.2

	D_1	D_2	D_3	D_4	Supply
O_1	$30 \times 5 = 150$	$30 \times 10 = 300$	$20/4$	$20/5$	20
O_2	$5/1$	$10/4$	$25/2$	$20/15$	25
O_3	$5/3$	$10/2$	$10/1$	$10/4$	10
O_4	$5/4$	$10/5$	$10/6$	$10/3$	10
Demand	5	10	30	20	

It is observed in the table 3.6 that though both the cell C_{11} and C_{12} have zero transportation cost but according to WOC for allocation procedure the cell C_{12} prefers first as its weight factor is larger than that of C_{11} i.e. ${}^w c_{12}(= 300) > {}^w c_{11}(= 150)$.

CHAPTER IV

Proposed Algorithm

4.1 Introduction

In the previous chapter we have successfully developed weighted opportunity cost (WOC) matrix by incorporating supply and demand as weight factor. Now we need to solve the transportation problem (TP) by applying this concept. In order to develop an algorithm by incorporating WOC matrix, we need a base algorithm which is able to solve the transportation problem (TP).

4.2 Proposed Algorithm of SUWOC-LCM

It is known that Least Cost Matrix (LCM) is very simple and effective to find out IBFS of TP. In LCM, the allocation flows are directly controlled by cost matrix – least cost prefers first over larger costs for allocation. In LCM, there is no any special Distribution Indicator (DI), rather than cost matrix, which forces the direction of flow of allocation. On the other hand VAM's method, which is more efficient to find out IBFS, has DI for control the flow of allocations. As we mentioned earlier that it's DI is formulated by the manipulation of only cost entries. Moreover, it has one more significant feature – the DI table is updated after each allocation steps. It is noted that we have developed the WOC matrix by incorporating supply/demand entries as weighted factor. But we know after each allocation the amount of demand /supply is changed. So the weighted factor should be changed. By exploiting these emerging ideas, we have developed here a Sequentially Updated Weighted Opportunity Cost (SUWOC) matrix. Finally embedded this SUWOC upon LCM algorithm, here we have developed a modified algorithm named SUWOC-LCM approach for finding IBFS of TP.

In the proposed SUWOC-LCM, the flow of allocation is controlled by the SUWOC matrix

rather than cost matrix such that the cell with larger weight factor is preferred first for allocation. Moreover after each step of allocation procedures, SUWOC matrix may be changed and updated according to the present status of demand /supply entries. The algorithm of SUWOC-LCM is given below:

Alg. SUWOC-LCM()

Step 1 (Input): read cost matrix $[c_{ij}]$, supply $[a_i]$ and demand $[b_j]$

Step 2 (Find Allocation Units): find possible maximum allocation units of each cell $[c_{ij}]$:

$$\min(a_i, b_j).$$

Step 3 (Find Weighted Opportunity Cost of each Cell ${}^w c_{ij}$):

- (i) If $c_{ij} > 0$, set ${}^w c_{ij} = \frac{1}{c_{ij}} \times \min(a_i, b_j)$
- (ii) else if $c_{ij} = 0$ and $\{c_{pq}: 0 < c_{pq} < 1 \forall p, q\} \neq \Phi$ [null set] then set
 ${}^w c_{ij} = M \times \min(a_i, b_j)$
 where $M = \max\{a_p, b_q \forall p, q\} / [\min\{c_{pq}: 0 < c_{pq} < 1 \forall p, q\}]$.
- (iii) else if $c_{ij} = 0$ and $\{c_{pq}: 0 < c_{pq} < 1 \forall p, q\} = \Phi$, then set
 ${}^w c_{ij} = N \times \min(a_i, b_j)$ where $N = \max\{a_i, b_j \forall i, j\}$

Step 4 (Formulation of WOC matrix i.e. $[{}^w c_{ij}]$): Do Step 2 and Step 3 for each i and j until the WOC matrix i.e. $[{}^w c_{ij}]$ is formed.

Step 5 (Allocation procedure):

Allocate amount of $\min(a_i, b_j)$ at cell C_{ij} s.t. ${}^w c_{ij} = \max\{{}^w c_{pq}; \forall p, q\}$

Step 6 (Updating transportation tableau and WOC):

- (i) If $a_i = \min(a_i, b_j)$ then set $a_i = 0$ and cross out c_{iq} and ${}^w c_{iq} \forall q$ and then
 - (a) update demand $b_j' = |b_j - a_i|$;
 - (b) update j^{th} column of WOC matrix $[{}^w c_{ij}]$ s.t.
 ${}^w c_{pj}' = {}^w c_{pj} \times \min\{a_p, b_j'\} / \min(a_p, b_j) \forall p$
- (ii) Else set $b_j = 0$ and cross out c_{pj} and ${}^w c_{pj} \forall p$ and then
 - (a) update supply $a_i' = |b_j - a_i|$;
 - (b) update i^{th} row of WOC matrix $[{}^w c_{ij}]$ s.t.
 ${}^w c_{iq}' = {}^w c_{iq} \times \min\{a_i', b_q\} / \min(a_i, b_q) \forall q$

Step 7 (Termination condition): Repeated the **Step 5** and **Step 6** unless **termination**

condition meets i.e. $a_i = 0 \forall i$ or $b_j = 0 \forall j$

[i.e. Continuing the allocation procedures until possible all allocations will be completed].

4.3 Experiments and Discussions

To justify the effectiveness and to examine the validity of the proposed algorithm we will consider the previous typical Example. 3.1. For better visualization we would like to display it again, mainly the Transportation tableau which is shown in the Table 4.1.

Table 4.1 Transportation tableau of Example 3.1

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	1	2	3	4	5	10
O_2	2	3	4	5	6	25
O_3	3	4	5	6	7	15
O_4	4	5	6	7	8	20
O_5	5	6	7	8	9	30
Demand	20	10	5	30	35	

Solution: Since the WOC matrix of the Example 3.1 is already formed in the previous Chapter III, here we just copy it and shown in the Table 4.2. Note that in this problem all the cost entries are greater than zero i.e. $c_{ij} > 0$, so for initial formulation of WOC matrix we need only the formula: ${}^w c_{ij} = \frac{1}{c_{ij}} \times \min(a_i, b_j)$

Table 4.2 Initial Weighted Opportunity Cost (WOC) Matrix

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	10/1	10/2	5/3	10/4	10/5	10
O_2	20/2	10/3	5/4	25/5	25/6	25
O_3	15/3	10/4	5/5	15/6	15/7	15
O_4	20/4	10/5	5/6	20/7	20/8	20
O_5	20/5	10/6	5/7	30/8	30/9	30
Demand	20	10	5	30	35	

For hand calculation, we have now incorporated this WOC matrix into the corresponding Transportation Tableau (TT). The schematic view of the incorporated WOC matrix into TT

is shown in the Table 4.3. In this table each weighted opportunity cost is given to the upper left corner of each corresponding cell whereas each actual cost is given to the upper right corner of each corresponding cell.

Table 4.3. The schematic view of WOC Matrix and TT of a TP

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	10/1 1	10/2 2	5/3 3	10/4 4	10/5 5	10
O_2	20/2 2	10/3 3	5/4 4	25/5 5	25/6 6	25
O_3	15/3 3	10/4 4	5/5 5	15/6 6	15/7 7	15
O_4	20/4 4	10/5 5	5/6 6	20/7 7	20/8 8	20
O_5	20/5 5	10/6 6	5/7 7	30/8 8	30/9 9	30
Demand	20	10	5	30	35	

Now we will allocate to each cell according to the SUWOC-LCM approach. For better understanding we want to discuss the allocation procedures step by step as below:

Table 4.4 After 1st Step : the SUWOC matrix and TT of the TP

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	10/1 1 ×	10/2 2	5/3 3	10/4 4	10/5 5	10
O_2	20/2 2 20	10/3 , 5/3 3	5/4 4	25/5 , 5/5 5	25/6 , 5/6 6	25 , 5
O_3	15/3 3 ×	10/4 4	5/5 5	15/6 6	15/7 7	15
O_4	20/4 4 ×	10/5 5	5/6 6	20/7 7	20/8 8	20
O_5	20/5 5 ×	10/6 6	5/7 7	30/8 8	30/9 9	30
Demand	20	10	5	30	35	

1st Step: It is observed in the Table 4.3 that there are two equal largest weighted opportunity cost cells namely ${}^w c_{11} = {}^w c_{21} = 10$, though their corresponding cell cost are not identical namely $c_{11} = 1$ and $c_{21} = 2$. Now we have randomly selected one of the two cells, say cell C_{21} . Then according to the algorithm we have allocated amount of $\min(a_2, b_1) = \min(25, 20) = 20$ to this cell, C_{21} . Therefore we have $x_{21} = 20$. Now we have to update relevant issues. Here $b_1 (= 20) < a_2 (= 25)$, therefore the demand of

destinations D_1 is satisfied completely and hence cross out off the first column ($j = 1$) namely and diminish a_2 by b_1 as $a'_2 = |25 - 20| = 5$. Then we have to update the remaining weight costs of that row ($i = 2 \forall j$). After first Step the pictorial view of the WOC matrix as well as TT is shown in the Table 4.4.

2nd Step: It is observed in the Table 4.4 that after first step the remained largest weight opportunity cost is ${}^w c_{12} = 10/2$. Therefore according to the algorithm we have allocated amount of $\min(a_1, b_2) = \min(10, 10) = 10$ to the cell C_{12} , and so $x_{21} = 10$. Since here $a_1 = 10 = b_2$, therefore the supply of source O_1 and demand of destination D_2 are both satisfied completely and hence cross out off the first row and second column simultaneously. After second Step the pictorial view of the WOC matrix incorporated in TT is shown in the Table 4.5.

Table 4.5 After 2nd Step : the SUWOC matrix and TT of the TP

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	10/1 1 ×	10/2 2 10	5/3 3 ×	10/4 4 ×	10/5 5 ×	40
O_2	20/2 2 20	10/3, 5/3 3 ×	5/4 4	25/5, 5/5 5	25/6, 5/6 6	25 , 5
O_3	15/3 3 ×	10/4 4 ×	5/5 5	15/6 6	15/7 7	15
O_4	20/4 4 ×	10/5 5 ×	5/6 6	20/7 7	20/8 8	20
O_5	20/5 5 ×	10/6 6 ×	5/7 7	30/8 8	30/9 9	30
Demand	20	40	5	30	35	

3rd Step: After second step, it is noticed in the reduced Table 4.5 that the largest weighted opportunity cost is now ${}^w c_{54} = 30/8$. Therefore, allocate the amount of $x_{54} = \min(a_5, b_4) = \min(30, 30) = 30$ to the cell C_{54} . Now since $a_5 = 30 = b_4$, therefore the supply of source O_5 and demand of destination D_4 are both satisfied completely and hence cross out off the fifth row and fourth column again simultaneously. So after third step, the WOC matrix incorporated in TT is given in the Table 4.6.

Table 4.6 After 3rd Step : the SUWOC matrix and TT of the TP

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	10/1 1 ×	10/2 2 10	5/3 3 ×	10/4 4 ×	10/5 5 ×	10
O_2	20/2 2 20	10/3,5/3 3 ×	5/4 4	25/5,5/5 5 ×	25/6,5/6 6	25, 5
O_3	15/3 3 ×	10/4 4 ×	5/5 5	15/6 6 ×	15/7 7	15
O_4	20/4 4 ×	10/5 5 ×	5/6 6	20/7 7 ×	20/8 8	20
O_5	20/5 5 ×	10/6 6 ×	5/7 7 ×	30/8 8 30	30/9 9 ×	30
Demand	20	10	5	30	35	

4th Step: After third step, we observed in the table 4.6 that the remained largest weighted opportunity cost is now ${}^w c_{45} = 20/8$. Therefore allocate the amount of $x_{45} = \min(a_4, b_5) = \min(20, 35) = 20$ to the cell C_{45} . Now since $a_4 = 20 < 30 = b_5$, therefore the supply of the origin O_4 is satisfied completely and hence crosses out off the fourth row and diminishes b_5 by a_4 as $|35 - 20| = 5$. Then we have updated the remaining weighted opportunity costs of that column $j=5$. So after 4th step, Table 4.7 represents the updated WOC matrix as well as TT of the TP.

Table 4.7 After 4th Step: the SUWOC matrix and TT of the TP

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	10/1 1 ×	10/2 2 10	5/3 3 ×	10/4 4 ×	10/5 5 ×	10
O_2	20/2 2 20	10/3,5/3 3 ×	5/4 4	25/5,5/5 5 ×	25/6,5/6 6	25, 5
O_3	15/3 3 ×	10/4 4 ×	5/5 5	15/6 6 ×	15/7 7	15
O_4	20/4 4 ×	10/5 5 ×	5/6 6 ×	20/7 7 ×	20/8 8 20	20
O_5	20/5 5 ×	10/6 6 ×	5/7 7 ×	30/8 8 30	30/9 9 ×	30
Demand	20	10	5	30	35, 15	

Table 4.8 After 5th Step: the SUWOC matrix and TT of the TP

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	10/1 1 ×	10/2 2 10	5/3 3 ×	10/4 4 ×	10/5 5 ×	40
O_2	20/2 2 20	10/3,5/3 3 ×	5/4 4	25/5,5/5 5	25/6,5/6 6 ×	25, 5
O_3	15/3 3 ×	10/4 4 ×	5/5 5 ×	15/6 6 ×	15/7 7 15	15
O_4	20/4 4 ×	10/5 5 ×	5/6 6 ×	20/7 7 ×	20/8 8 20	20
O_5	20/5 5 ×	10/6 6 ×	5/7 7 ×	30/8 8 30	30/9 9 ×	30
Demand	20	10	5	30	35, 15	

5th Step: After fourth step, it is noticed in the table 4.7 that largest weighted opportunity cost remained is $^w c_{35} = 15/7$. Therefore allocate the amount of $x_{35} = \min(a_3, b_5) = \min(15, 15) = 15$ to the cell C_{35} . Now since $a_3 = 15 = 15 = b_5$, therefore the supply of source O_3 and demand of destination D_5 are both satisfied completely and hence cross out off the third row and fifth column. So after 5th step, the WOC matrix incorporated in TT is given in the Table 4.8.

Table 4.9 After 6th Step: the SUWOC matrix and TT of the TP

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	10/1 1 ×	10/2 2 10	5/3 3 ×	10/4 4 ×	10/5 5 ×	40
O_2	20/2 2 20	10/3,5/3 3 ×	5/4 4 5	25/5,5/5 5 ×	25/6,5/6 6 ×	25, 5
O_3	15/3 3 ×	10/4 4 ×	5/5 5 ×	15/6 6 ×	15/7 7 15	15
O_4	20/4 4 ×	10/5 5 ×	5/6 6 ×	20/7 7 ×	20/8 8 20	20
O_5	20/5 5 ×	10/6 6 ×	5/7 7 ×	30/8 8 30	30/9 9 ×	30
Demand	20	10	5	30	35, 15	

6th Step: After fifth step, it is noticed in the Table 4.8 that only the remain unoccupied cell is C_{23} and undistributed supply is 5 which is obviously equal to demand to the corresponding column. Therefore all the rest amount (= 5) is allocated in this cell, C_{23} . So

the supply of source O_2 and the demand of destinations D_3 are both satisfied. Finally after 6th step the pictorial view of the WOC matrix as well as TT is shown in the Table 4.9. It is observed that all the allocation is completed. So the initial basic feasible solution obtained by the proposed method is as follow:

$$x_{12} = 10, x_{21} = 20, x_{23} = 5, x_{35} = 15, x_{45} = 20, \text{ and } x_{54} = 30.$$

Therefore the total transportation cost according to the proposed method is

$$\begin{aligned} &= 2 \times 10 + 2 \times 20 + 4 \times 5 + 7 \times 15 + 8 \times 20 + 8 \times 30 \\ &= 515. \end{aligned}$$

Now to investigate the efficiency and effectiveness of the proposed algorithm, we will compare this result with LCM and VAM methods. The results of comparison among the three approaches regarding the example 3.1 are shown in the Table 4.10. We notice that the result of our proposed algorithm SUWOC-LCM is best compared to all the other approaches considered here namely LCM and VAM method.

Table 4.10 Comparison among SUWOC-LCM, LCM and VAM approaches of the Ex. 3.1

Method	SUWOC-LCM	LCM	VAM
Result	515	585	585

4.4 Further Experiments and Discussions

Now to compare the performance of the proposed SUWOC-LCM method to existing approaches we have performed further experiments. For this experimental study we have considered some instances given in the first column of the Table 4.11. For the comparison study we consider two well-known approaches namely LCM and VAM methods. The experimental results are displayed in the Table 4.11.

It is observed in the Table 4.11 that out of 12 instances, the proposed SUWOC-LCM outperform in 2 instances namely Example No. 4 and 5 compared to both LCM and VAM methods. It is also noticed that in two cases (Example No. 7 and 11) the results of VAM and proposed method are identical but better that of LCM method. Again in one case namely Example No. 3, the solutions of LCM and SUWOC-LCM are identical but better than that of VAM approach. In Example No. 6 the proposed method performed better than

VAM method but worse than LCM method. In 4 cases namely Example No.1, 2, 10 and 12, the results of all the three methods are identical. But it is also noticed that in Example No. 8 and Example No. 9 the results of the proposed method are worst compared to other two methods. It is worthwhile to mention here that out of 12 instances in 5 instances the proposed method is able to find out optimal solutions.

Table 4.11: Comparison of SUWOC-LCM, LCM and VAM approaches in TP

Ex. No.	Problem	SUWOC-LCM	LCM	VAM	Optimal Value
1	$c_{ij}:\{(5,7,9,11); (7,9,11,13); (9,11,13,15); (11,13,15,17)\}$ $S: (10, 25, 30, 35)$ $D: (20, 30,15, 35)$	1210	1210	1210	1210
2	$c_{ij}:\{(6,4,1); (3,8,7); (4,4,2)\}$ $S: (50,40,60)$ $D: (20,95,35)$	555	555	555	555
3	$c_{ij}:\{(2,4,1,3); (4,3,5,2); (5,2,3,6)\}$ $S: (10,20,10)$ $D: (9,11,6,14)$	85	85	90	85
4	$c_{ij}:\{(3,3,5); (6,5,4); (6,10,7)\}$ $S: (9,8,10)$ $D: (7,12,8)$	131	159	143	125
5	$c_{ij}:\{(9,8,5,7); (4,6,8,7); (5,8,9,5)\}$ $S: (12,14,16)$ $D: (8,18,13,3)$	240	248	248	240
6	$c_{ij}:\{(4,19,22,11); (1,9,14,14); (6,6,16,14)\}$ $S: (100,30,70)$	2160	2090	2170	2040

	$D: (40,20,60,80)$				
7	$c_{ij}: \{(9,12,9,6,9,10); (7,3,7,7,5,5); (6,5,9,11,3,11); (6,8,11,2,2,10)\}$ $S: (5,6,2,9)$ $D: (4,4,6,2,4,2)$	112	114	112	112
8	$c_{ij}: \{(4,5,8,4); (6,2,8,1); (8,7,9,10)\}$ $S: (52,57,54)$ $D: (60,45,8,50)$	795	674	674	674
9	$c_{ij}: \{(5,2,4,1); (5,2,1,4); (6,4,8,2); (4,6,5,4) (2,8,4,5)\}$ $S: (30,20,12,30,46)$ $D: (31,50,30,27)$	429	423	391	381
10	$c_{ij}: \{(2,5,4); (6,1,2); (4,5,2)\}$ $S: (4,6,6)$ $D: (3,7,6)$	29	29	29	29
11	$c_{ij}: \{(19,30,50,10); (70,30,40,60); (40,8,70,20)\}$ $S: (7,9,18)$ $D: (5,8,7,14)$	779	814	779	743
12	$c_{ij}: \{(10,0,20,11); (12,7,9,20); (0,14,16,18)\}$ $S: (20,25,15)$ $D: (10,15,15,20)$	480	480	480	460

From these primary experimental investigations, we may conclude that the performance of the proposed algorithm comparatively better than both LCM and VAM method. But it is noted that solution obtained by the proposed method is IBFS and there is no any guaranty that the solution to be optimal.

4.5 Optimal Solution

In this section we will test the optimality of the IBFS. If the solution is not optimal, then we need to apply other approach which optimizes the IBFS. An optimal solution is one where there is no other set of transportation routes that will further reduce the total transportation cost. To test the optimality and/or to obtain optimal solution here we have considered the well-known method - Modified Distribution Indicator (MODI) method.

4.5.1 Modified Distribution Indicator (MODI) method

To obtain an optimal solution by making successive improvements to initial basic feasible solution until no further decrease in the transportation cost is possible. Thus, we have to evaluate each unoccupied cell in the transportation table in terms of an opportunity of reducing total transportation cost. An unoccupied cell with the largest negative opportunity cost is selected to include in the new set of transportation routes (allocations). This value indicates the per unit cost reduction that can be achieved by raising the shipment allocation in the unoccupied cell from its present level of zero. This is also known as an incoming cell (or variable). The outgoing cell (or variable) in the current solution is the occupied cell (basic variable) in the unique closed path (loop) whose allocation will become zero first as more units are allocated to the unoccupied cell with largest negative opportunity cost. That is, the current solution cannot be improved further. This is the optimal solution.

4.5.2 An Optimal solution by using (MODI) method

To find an optimal solution of the initial basic feasible solution we have considered another example 4.2 of a transportation problem.

Example 4.2: The transportation tableau of a transportation problem is shown in the Table 4.12.

Table 4.12 A typical example of TP

	D_1	D_2	D_3	D_4	Supply
O_1	9	8	5	7	12
O_2	4	6	8	7	14
O_3	5	8	9	5	16
Demand	8	18	13	3	

Solution: It first by the proposed SUWOC-LCM method, we need to find IBFS of the problem. So we have carried out experiment on this problem and the pictorial view of the final solution (IBFS) is shown in the Table 4.13.

Table 4.13 Solution of TP by using proposed algorithm SUWOC-LCM

	D_1	D_2	D_3	D_4	Supply
O_1	8/9 9 ×	12/8 8 ×	12/5 5 12	3/7 7 ×	12
O_2	8/4 4 ×	14/6 6 14	13/8, 1/8 8 ×	3/7 7 ×	14
O_3	8/5 5 8	16/8, 4/8 8 4	13/9, 1/9 9 1	3/5 5 3	16, 8, 5, 1
Demand	8	18, 4	13, 1	3	

So the initial basic feasible solution obtained by the proposed method is

$$x_{13} = 12, x_{22} = 14, x_{31} = 8, x_{32} = 4, x_{33} = 1, \text{ and } x_{34} = 3.$$

Therefore the total transportation cost = $5 \times 12 + 6 \times 14 + 5 \times 8 + 8 \times 4 + 9 \times 1 + 5 \times 3 = 240$.

Now by using MODI algorithm we will find out the optimal solution of the problem in which we will use this IBFS obtained by the proposed method. Therefore we have to calculate the cell evaluations corresponding to all non-basic or unoccupied cells.

Here we find the values u_i and v_j using the relation $u_i + v_j = c_{ij}$ for all basic (occupied) cells.

For basic cell:

$$(1, 3): u_1 + v_3 = 5 \dots (1)$$

$$(2, 2): u_2 + v_2 = 6 \dots (2)$$

$$(3, 1): u_3 + v_1 = 5 \dots (3)$$

$$(3, 2): u_3 + v_2 = 8 \dots (4)$$

$$(3, 3): u_3 + v_3 = 9 \dots (5)$$

$$(3, 4): u_3 + v_4 = 5 \dots (6)$$

Let arbitrarily $u_3=0$ as third row contains the maximum number of occupied cells.

$$\therefore (3) \Rightarrow v_1 = 5$$

$$(4) \Rightarrow v_2 = 8$$

$$(5) \Rightarrow v_3 = 9$$

$$(6) \Rightarrow v_4 = 5$$

$$(1) \Rightarrow u_1 = -4$$

$$(2) \Rightarrow u_2 = -2$$

$\therefore u_1 = -4, u_2 = -2, u_3 = 0, v_1 = 5, v_2 = 8, v_3 = 9$ and $v_4 = 5$

Which are displayed right side and below the Table 4.13.

Table 4.14 Optimal Solution of TP

	D_1	D_2	D_3	D_4	u_i
O_1	-8 9	-4 8	12 5	-6 7	-4
O_2	-1 4	14 6	-1 8	-4 7	-2
O_3	8 5	4 8	1 9	3 5	0
v_j	5	8	9	5	

Cell evaluation for unoccupied cells: $\Delta_{ij} = (u_i + v_j) - c_{ij}$

For cell:

$$(1, 1): \Delta_{11} = (u_1 + v_1) - c_{11} = -4 + 5 - 9 = -8$$

$$(1, 2): \Delta_{12} = (u_1 + v_2) - c_{12} = -4 + 8 - 8 = -4$$

$$(1, 4): \Delta_{14} = (u_1 + v_4) - c_{14} = -4 + 5 - 7 = -6$$

$$(2, 1): \Delta_{21} = (u_2 + v_1) - c_{21} = -2 + 5 - 4 = -1$$

$$(2, 3): \Delta_{23} = (u_2 + v_3) - c_{23} = -2 + 9 - 8 = -1$$

$$(2, 4): \Delta_{24} = (u_2 + v_4) - c_{24} = -2 + 5 - 7 = -4$$

Displayed these values of cell evaluation for unoccupied cells on the proposed Sequentially Updated Weighted Opportunity Cost matrix based Least Cost Method (SUWOC-LCM) of the corresponding cell. Here all $\Delta_{ij} \leq 0$ i.e. all the cell evaluation is negative the solution is optimum and optimum solution is

$x_{13} = 12, x_{22} = 14, x_{31} = 8, x_{32} = 4, x_{33} = 1,$ and $x_{34} = 3$ and minimum cost is $z = 240$ units.

It is worthwhile to remarks that, the result obtained by the proposed algorithm is optimum which 240. On the other the IBFS of the LCM method and VAM method are 248 and 248 respectively and obviously these are not optimum.

CHAPTER V

Conclusion

Transportation Problem, which is a special type of linear programming problem, has been playing very important role in business arena. The main task in TP is to minimize transportation cost as well as other relevant issues so that the profit is maximized. Researchers are continuously hunting for finding better transportation algorithms. Regarding the methods of finding initial basic feasible solution, much of the research works concern with the transportation cost entries, and/or the manipulation of cost entries to form Distribution Indicator (DI) /or Total Opportunity Cost Matrix whatever be the structure of supply entries and demand entries.

But in the real market it is observed that supply and demand plays a vital role in business field. Exploiting this idea, we have formulated a weighted opportunity cost (WOC) matrix in which supply and demand entries act as a weight factor upon transportation cost entries. After successful formulation of WOC matrix we intend to develop an effective algorithm for solving TP. It is known that in LCM method, the strategy of allocation flow is initially defined i.e. least cost prefers first. On the other hand in the VAM method, the strategy of allocation flow is depended upon Distribution Indicator (DI). The motivated feature of VAM is that after each allocation the DI is updated. Therefore by incorporating the concept of VAM upon WOC, a virtual dynamic weighted cost opportunity matrix is formulated. Finally we have developed a new sequential updated weighted-cost based algorithm embedded on LCM method (SUWOC-LCM) to find IBFS of TP.

Several experiments have been performed to investigate the performance of the proposed algorithm. Experimental results reveal that the proposed method is effective as well as

efficient to find out the IBFS of TP. Moreover some time the proposed method able to find out optimal solution too. On the average exception of few instances the proposed method has performed better than VAM or LCM methods to find out IBFS of TP.

Anyway our main contribution of this research is that we have incorporated a new and unique idea in transportation arena. As it is a new way to think about solving TP, we hope by further intensive research some excellent outputs might come out.

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