Improvement of Analytic Solution to the Inverse Truly Nonlinear Oscillator by Extended Iterative Method

by

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Dedicated to My Family

Father Munshi Iqbal Hossain

Mother Bilkis Hossain

Brother Saifuzzaman

Sister Jannatul Ferdaus Mim

Declaration

This is to certify that the thesis work entitled "Improvement of Analytic Solution to the Inverse Truly Nonlinear Oscillator by Extended Iterative Method" has been carried out by Md. Asifuzzaman in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

Signature of Supervisor　　　　　　　　Signature of Candidate

Approval

This is to certify that the thesis work submitted by Md. Asifuzzaman entitled “Improvement of Analytic Solution to the Inverse Truly Nonlinear Oscillator by Extended Iterative Method " has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of M.Sc in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh in February 2017.

BOARD OF EXAMINERS

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Abstract

A new approach of the Mickens extended iterative method has been presented to obtain approximate analytic solutions for nonlinear oscillatory differential equation. To get modified approximate solution of the inverse nonlinear oscillator "$\ddot{x}+x^{-1}=0$", we have used the Fourier series and utilized indispensable truncated terms in each iterative step. In this thesis the solution gives more accurate result than other existing methods and shows a good agreement with its exact solution. The percentage of error between exact frequency and our third approximate frequency is as low as 0.0029%. We have compared our results with exact results and other existing results and the solution is convergent as well as consistent.

Publication

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CHAPTER I

Introduction

This chapter serves as an introduction to the central elements of the analysis of nonlinear dynamics systems. It is hoped that these discussions will provide a context that will help the readers to understand the importance of other chapters in this thesis.

Most phenomena in our world are essentially nonlinear and are described by nonlinear equations. A vast scientific knowledge has developed over a long period of time, devoted to a description of natural phenomena. Practically, most of the differential equations involving physical phenomena are nonlinear. These equations have also demonstrated their usefulness in ecology, business cycle and biology. Therefore the solution of such problems lies essentially in investigating the corresponding differential equations. In many cases it is possible to replace such a nonlinear equation by a related linear equation, which approximates the actual problem closely enough to give useful results. The method of small oscillations is a well-known example of the linearization of problems which are essentially nonlinear. However, such a linearization is not always feasible or possible; and when it is not, the original nonlinear equation itself must be considered.

One of the significant realizations is that the mathematical concept developed by modeling simple physical systems can be fruitfully applied to more complex systems. Some of which have great interest in the biomedical community (electrical signal propagation in cardiac tissue, neural networks or gene regulation). Often this leads to model different region of parameter space, and one region is found to exhibit quite similar to the real system. In many cases, the model behavior is rather sensitive to parameter variations, so if the model parameters can be measured in the real system the model shows realistic behavior at those values, and one can have some confidence that the model has captured the essential features of the system. Moreover, in many cases numerical simulation has to be done first in order to give some direction to theoretical studies. Though the catalogue of well characterized, generic behaviors of deterministic nonlinear systems is large and continues to grow, there is no method for classifying the expected behavior of a particular nonlinear dynamical system unless it can be directly mapped to a previously studied example. Rather than attempting a review of the state of the art in time-series analysis, numerical methods, and theoretical characterization of nonlinear dynamical systems, this thisis presents some of the essential concepts using an example.

Differential equation is one of the most attractive branch of mathematics and essential tool for modeling many physical situations like mechanical vibration, nonlinear circuits, chemical oscillation and space dynamics and so on. Therefore the solution of such problems lies essentially in investigating the corresponding differential equations. The differential equations may be linear or nonlinear, autonomous or non-autonomous.

A lot of differential equations that represent physical phenomena are nonlinear. Systems of nonlinear equations arise in many domains of practical importance such as engineering, mechanics, medicine, chemistry, and robotics. We can say that nonlinear equations are great in the range of importance. They help to predict a lot of things in our daily lives.

The ways of investigating linear differential equations are comparatively easy and highly developed. On the contrary, it is very little known to the general character about nonlinear equations. Ordinarily, the nonlinear problems are investigated by converting into linear equations by attributing some terms and conditions; but such linearization is not always possible. The equation is generally confined to a variety of rather special cases, and one must resort to various methods of approximation. Many methods exist for constructing analytical approximations to the solution of the oscillatory system, such as Perturbation method, Harmonic Balance (HB) method, Iterative method etc. Perturbation method is used only for weak nonlinearities, HB method is used for strong nonlinear problems. On the other hand Iterative method is used for weak as well as strong nonlinear oscillations. In the Perturbation method, the expansion of a solution to a differential equation is represented in a series of a small parameter. It is used to construct uniformly valid periodic solution to second-order nonlinear differential equations.

Harmonic Balance method is a procedure of determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation. An important advantage of the method is that it can be applied to nonlinear oscillatory problems for which the nonlinear terms are not weak i.e., no Perturbation parameter need to exist. A disadvantage of the method is that it is a priory difficult to predict for a given nonlinear differential equation whether a first order Harmonic Balance calculation will provide a sufficiently accurate approximation to periodic solution or not. The Iterative method introduces a reliable and efficient process for wide variety of scientific and engineering application for the case of nonlinear systems. There are two important advantages of Iterative method, one is "only linear, inhomogeneous differential equations are required to be investigated at each level of the calculation" and another is " the coefficients of the higher harmonic, for a given value of the Iterative index decrease rapidly with increasing harmonic number". The last point implies that higher order solutions may not be required. The important development of the theory of nonlinear dynamical systems, during these centuries, has essentially its origin in the studies of the "natural effects" encountered in these systems, and the rejection of non-essential generalizations. That is the study of concrete nonlinear systems has been possible due to the foundation of results from the theory or nonlinear dynamical system field.

Nonlinear phenomena are of fundamental importance in various fields of science and engineering, specially in fluid mechanics, solid state physics, plasma physics, plasma wave and chemical physics. The wide applicability of these equations is the main reason why they have attracted so much attention from many mathematicians. However, they are usually very difficult to investigate, either numerically or theoretically. For facilitating the solution procedure we have utilized the complete Fourier series (sometimes approximately) to expand the nonlinear terms in 'Cosine series'. In certain cases the coefficients of Fourier series have been reduced to a standard form.

The main intention of this thesis is to investigate the approximate analytic solutions using the modified Extended Iterative method to decompose the secular term, so that the solution can be obtained by Iterative procedure. This means that we can use Extended Iterative method to investigate many nonlinear problems. The main thrust of this technique is that the obtained solution rapidly converges to exact solutions.

The chapter outline of this thesis is as follows**:** In **Chapter II**, the review of literature is presented. In **Chapter III,** the Extended Iterative method has been described for obtaining modified approximate analytic solutions of the inverse truly nonlinear oscillator. In **Chapter IV,** the result of the adopted method has been shown. Finally, In **Chapter V,** some concluding remarks are included.

CHAPTER II

Literature Review

The review of literature is presented in this chapter. Here we examine some general techniques that can be used to illustrate the existence of periodic solutions for a given truly nonlinear equation. These methods also apply to the case of standard equation. Moreover, this chapter shows some existing methods and their solution procedure, which help us in comparative analysis.

2.1 Introduction

The natures of nonlinear differential equations are distinct. But the study of nonlinear problems is one of most fascinating parts in mathematics, physics and other science and engineering. The mathematical analysis of many of the oscillating phenomena that occur in nature leads to the solution of nonlinear differential equations or modification differential equations. A nonlinear system of equations is a set of simultaneous equations in which the unknowns appear as variables. Specially, a differential equation is regarded as linear if it gets linear in terms of the unknown function as well as its derivatives, even if nonlinear in terms of the other variables appearing in it.

Nonlinear equations are difficult to investigate and nonlinear systems are commonly approximated by linear equations. This works well up to some accuracy and some range for the input values, but some interesting phenomena such as chaos and singularities are hidden by linearization. It follows that some aspects of the behavior of a nonlinear system appear commonly to be chaotic, unpredictable or counterintuitive. Although such chaotic behavior may resemble random behavior, it is absolutely not random.

2.2 Description of the Different Methods

Analytical solutions of nonlinear differential equations or linear differential equations with variables coefficients play an important role in the study of nonlinear dynamical systems, but sometimes it is difficult to find solutions of these equations, especially for nonlinear problems with strong nonlinearities. There are several analytical approaches to find approximate solutions to nonlinear oscillatory system, such as: Perturbation method [1-6], Homotophy Perturbation method [7-10], Harmonic Balance (HB) method [11-22], Modified Linstedt-Poincaré method [23], Krylov-Bogoliubov-Mitropolskii (KBM) method [24-25], Energy Balance method [26], Cubication method [27], Iterative method [28-42], etc. Perturbation method is a well-known method for investigating to the differential equations in which the nonlinear term is small. The method of Lindstedt-Poincare [23], Krylov-Bogoliubov-Mtropolskii (KBM) method [24-25], Multiple Scales method [43] and Homotopy Perturbation method [7-10] are most momentous among all Perturbation methods.

The method of Lindstedt-Poincare [23] is an introductory method to investigate the following second order nonlinear differential equations

$$\ddot{x} + \omega_0^2 x + \varepsilon\, f(\ddot{x}, x) = 0, \tag{2.1}$$

where ω_0 is the unperturbed frequency and ε is a small parameter.

The fundamental idea in Lindstedt's technique is based on the observation that the nonlinearities alter the frequency of the system from the linear one ω_0 to $\omega(\varepsilon)$. To account for this change in frequency, He introduces a new variable $\tau = \omega t$ and expand ω and x in power of ε as

$$\begin{cases} x = x_0(\tau) + \varepsilon\, x_1(\tau) + \varepsilon^2 x_2(\tau) + \ldots \\ \omega = \omega_0 + \varepsilon\, \omega_1 + \varepsilon^2 \omega_2 + \ldots, \end{cases} \tag{2.2}$$

where ω_i, $i = 0, 1, 2, \ldots$, are unknown constants to be determined.

Substituting equation (2.2) into equation (2.1) and equating the coefficients of the various powers of ε, the following equations are obtained

$$\begin{cases} \ddot{x}_0 + x_0 = 0 \\ \ddot{x}_1 + x_1 = -2\omega_1 \ddot{x} - f(x_0, \dot{x}_0) \\ \ddot{x}_2 + x_2 = -2\omega_1 \ddot{x}_1 - f(x_0, \dot{x}_0) - (\omega_1^2 + 2\omega_2)\ddot{x}_0 \\ \qquad\qquad - f_x(x_0, \dot{x}_0)x_1 + f_{\dot{x}}(x_0, \dot{x}_0)(\omega_1 \dot{x}_0 + \dot{x}_1) \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ \ddot{x}_n + x_n = g_n(x_0, x_1, \ldots, x_{n-1}; \dot{x}_0, \dot{x}_1, \ldots, \dot{x}_{n-1}), \end{cases} \tag{2.3}$$

where over dot represents the differentiation with respect to τ .

Apparently equation (2.3) is a linear system and it is investigated by the elementary technique. This method is used only for finding the periodic solution, but the method cannot discuss the transient case.

Among various Perturbation methods, Krylov and Bogoliubov [24] introduced a technique to discuss transients of the same equation. This method starts with the solution of the linear equation, assuming that, in the nonlinear case, the amplitude and phase in the solution of the linear equation are time dependent function rather than constants [1]. The solution of corresponding unperturbed equation (i.e., for $\varepsilon = 0$) of equation (2.1) can be written as

$$x = a\cos\left(\omega_0 t + \theta\right), \tag{2.4}$$

where a and θ are two arbitrary constants respectively called amplitude and phase which are determined from the initial conditions $x(0) = x_0$ and $\dot{x}(0) = y_0$.

Now to determine an approximate solution of equation (2.1) for ε small but different from zero, Krylov and Bogoliubov [24] assumed that the solution is still given by equation (2.4) with varying a and θ subject to the conditions

$$\begin{cases} \dfrac{dx}{dt} = -a\omega_0 \sin\phi \\ \phi = \omega_0 t + \theta. \end{cases} \tag{2.5}$$

Differentiating equation (2.4) with respect to time, t and using equation (2.5), we obtain

$$\frac{da}{dt}\cos\phi - \frac{d\theta}{dt} a \sin\phi = 0. \tag{2.6}$$

Again differentiating equation (2.5) with respect to time, t , we obtain

$$\frac{d^2x}{dt^2} = -a\,\omega_0^2 \cos\phi - \omega_0 \frac{da}{dt}\sin\phi - a\omega_0 \frac{d\theta}{dt}\cos\phi. \tag{2.7}$$

Substituting equation (2.7) into equation (2.1) and using equation (2.4) and equation (2.5), we obtain

$$\frac{da}{dt}\omega_0 \sin\phi + \frac{d\theta}{dt} a\omega_0 \cos\phi = -\varepsilon\, f\left(a\cos\phi,\, -a\omega_0 \sin\phi\right). \tag{2.8}$$

Solving equation (2.6) and equation (2.8), $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ yields

$$\begin{cases} \dfrac{da}{dt} = -\dfrac{\varepsilon}{\omega_0}\sin\phi\, f\left(a\cos\phi,\, -a\omega_0 \sin\phi\right) \\ \dfrac{d\theta}{dt} = -\dfrac{\varepsilon}{a\omega_0}\cos\phi\, f\left(a\cos\phi,\, -a\omega_0 \sin\phi\right). \end{cases} \tag{2.9}$$

Here equation (2.4) together with equation (2.9) represents the first approximate solution of equation (2.1).

Further, the technique was modified and justified by Bogoliubov and Mitropolskii [25] in 1961. They assumed a solution of the nonlinear differential equation (2.1) of the form

$$x(t,\varepsilon) = a\cos\psi + \varepsilon\, x_1(a,\psi) + \cdots + \varepsilon^n\, x_n(a,\psi) + O\left(\varepsilon^{n+1}\right), \tag{2.10}$$

where $x_k,\ k = 1,2,\cdots,n$ is a periodic function of ψ with period $2\pi, a$ and ψ very with time, t according to

$$\begin{cases} \dfrac{da}{dt} = \varepsilon\, A_1(a) + \cdots + \varepsilon^n\, A_n(a) + O\left(\varepsilon^{n+1}\right) \\ \dfrac{d\psi}{dt} = \omega_0 + \varepsilon\, B_1(a) + \cdots + \varepsilon^n\, B_n(a) + O\left(\varepsilon^{n+1}\right), \end{cases} \tag{2.11}$$

where the function x_k , A_k and B_k are chosen such that equation (2.10) and equation (2.11) satisfy the differential equation (2.1).

Later this solution was used by Mitropolskii [44] to investigate similar system (i.e., equation (2.1)) in which the coefficient very slowly with time. Popov [45] extended this method to nonlinear strongly damped oscillatory systems. By Popov's [45] technique, Murty *et al.* [46] extended the method to over damped nonlinear system. Murty [47] further presented a unified KBM method to obtain under and over damped solution of a second order nonlinear differential equation. Shamsul and Sattar [48] extended Murty's [47] unified KBM method to investigate a third-order nonlinear differential equation.

Harmonic Balance method is the most useful technique for finding the periodic solutions of nonlinear system, which is patented by Mickens [11] and further work has been done by Hu [16], Beléndez *et al.* [22], Lim *et al.* [49], Wu *et al.* [50] and so on for investigating the strong nonlinear problems. If a periodic solution does not exist of an oscillator, it may be sought in the form of Fourier series, whose coefficients are determined by requiring the series to satisfy the equation of motion. However, in order to avoid investigating an infinite system of algebraic equations, it is better to approximate the solution by a suitable finite sum of trigonometric function. This is the main task of Harmonic Balance method. Thus approximate solutions of an oscillator are obtained by Harmonic Balance method using a suitable truncated Fourier series. The method is capable to determining analytic approximate solution to the nonlinear oscillator valid even for the case where the nonlinear terms are not small i.e., no particular parameter need exist. The formulation of the method of Harmonic Balance focuses primarily by Mickens [12]. However, it should be indicated that various generalizations of the method of Harmonic Balance has been made by an intrinsic method of harmonic analysis. Lately, combining the method of averaging and Harmonic Balance, Lim and Lai [51] presented analytic technique to obtain first approximate Perturbation solution; their solutions gives desired results for some non-conservative systems when the damping force is very small.

Mickens [35] has given the general procedure for calculating solutions by means of the method of direct Harmonic Balance as follows:

He considered the equation for all Truly Nonlinear (TNL) oscillators as

$$F(x,\dot{x},\ddot{x})=0, \tag{2.12}$$

where $F(x,\dot{x},\ddot{x})$ is of odd-parity, i.e.,

$$F(-x,-\dot{x},-\ddot{x})=-F(x,\dot{x},\ddot{x}). \tag{2.13}$$

A major consequence of this property is that the corresponding Fourier expansions of the periodic solutions only contain odd harmonics, i. e.,

$$x(t) = \sum_{k=1}^{\infty} \left\{ A_k \cos\left[(2k-1)\Omega t \right] + B_k \sin\left[(2k-1)\Omega t \right] \right\}. \tag{2.14}$$

The $N-th$ order Harmonic Balance approximation to $x(t)$ is the expression

$$x_N(t) = \sum_{k=1}^{N} \left\{ \bar{A}_k^N \cos\left[(2k-1)\bar{\Omega}_N t \right] + \bar{B}_k^N \sin\left[(2k-1)\bar{\Omega}_N t \right] \right\}, \tag{2.15}$$

where $\bar{A}_k^N$, $\bar{B}_k^N$, $\bar{\Omega}_N$ are approximations to A_k, B_k, Ω for $k = 1, 2, 3, ..., N.$

For the case of a conservative oscillator, equation (2.12) generally takes the form

$$\ddot{x} + f(x, \lambda) = 0, \tag{2.16}$$

where λ denotes the various parameters appearing in $f(x, \lambda)$ and $f(-x, \lambda) = -f(x, \lambda)$.

The following initial conditions are selected

$$x(0) = A, \quad \dot{x}(0) = 0, \tag{2.17}$$

and this has the consequence that only the cosine terms are needed in the Fourier expansions, and therefore we have

$$x_N(t) = \sum_{k=1}^{N} \bar{A}_k^N \cos\left[(2k-1)\bar{\Omega}_N t \right]. \tag{2.18}$$

Observe that $x_N(t)$ has $(N+1)$ unknowns, the N coefficients, and Ω_N, the angular frequency. These quantities may be calculated by carrying out the following ways:

Substituting equation (2.18) into equation (2.16), and expand the resulting form into an expression that has the following structure

$$\sum_{k=1}^{N} H_k \cos\left[(2k-1)\Omega_N t \right] + HOH \cong 0, \text{ HOH= Higher Order Harmonic} \tag{2.19}$$

where they H_k are functions of the coefficients, the angular frequency, and the parameters, i.e.,

$$H_k = H_k(\bar{A}_1^N, \bar{A}_2^N, ..., \bar{A}_N^N, \Omega_N, \lambda). \tag{2.20}$$

Herein equation (2.19), we only retain as many harmonics in our expansion as initially occur in the assumed approximation to the periodic solution. Set the functions H_k to zero, i.e.,

$$H_k = 0, \quad k = 1, 2, ..., N. \tag{2.21}$$

The action is justified since the cosine functions are linearly independent, as a result any linear sum of them that is equal to zero must have the property that the coefficient are all zero.

Solving the N equations in equation (2.21), for $(\bar{A}_2^N, \bar{A}_3^N, ...\bar{A}_N^N)$ and Ω_N, in terms of $\bar{A}_1^N$ and using the initial conditions, equation (2.17), we have for $\bar{A}_1^N$ the relation

$$x_N(0) = A = \bar{A}_1^N + \sum_{k=2}^{N} \bar{A}_k^N (\bar{A}_1^N, \lambda). \tag{2.22}$$

An important point is that equation (2.21) will have many distinct solutions and the one selected for a particular oscillator equation is that one for which we have known a priori restrictions on the behavior of the approximations to the coefficients. However, as the worked examples in the next section demonstrate, in general, no essential difficulties arise.

For the case of non-conservative oscillators, where $\dot{x}$ appears to an odd power the calculation of approximations to periodic solutions follows a procedure modified for the case of conservative oscillators presented above. Many of these equations take the form

$$\ddot{x} + f(x, \lambda_1) = g(x, \dot{x}, \lambda_2)\dot{x}, \tag{2.23}$$

where

$$\begin{cases} f(-x, \lambda_1) = -f(x, \lambda_1) \\ g(-x, -\dot{x}, \lambda_2) = -g(x, \dot{x}, \lambda_2), \end{cases} \tag{2.24}$$

and (λ_1, λ_2) denote the parameters appearing in $f(x, \lambda_1)$ and $g(x, \dot{x}, \lambda_2)$.

For this type of differential equation, a limit-cycle may exist and the initial conditions cannot, in general, be a priori specified.

Harmonic balancing, for systems where limit-cycles may exist, uses the following procedures:

The $N-th$ order approximation to the periodic solution to be

$$x_N(t) = \bar{A}_1^N \cos\left(\bar{\Omega}_N t\right) + \sum_{k=2}^{N} \left\{ \bar{A}_k^N \cos\left[\left(2k-1\right)\bar{\Omega}_N t\right] + \bar{B}_k^N \sin\left[\left(2k-1\right)\bar{\Omega}_N t\right] \right\}, \tag{2.25}$$

where the $2N$ unknowns $\bar{A}_1^N, \bar{A}_2^N, ..., \bar{A}_N^N$; $\bar{\Omega}_N, \bar{B}_2^N, ..., \bar{B}_N^N$ and $\bar{\Omega}_N$ are to be determined.

Substituting equation (2.25) into equation (2.23) and write the result as

$$\sum_{k=1}^{N}\left\{ H_k \cos\left[\left(2k-1\right)\Omega_N\, t\right] + L_k \sin\left[\left(2k-1\right)\Omega_N\, t\right]\right\} + HOH \cong 0, \tag{2.26}$$

where the $\{H_k\}$ and $\{L_k\}$, $k=1$ to N, are functions of the $2N$ unknowns which are mentioned above.

Next equating the $2N$ functions $\{H_k\}$ and $\{L_k\}$ to zero and solving them for the $(2N-1)$ amplitudes and the angular frequency. If a valid solution exists, then it corresponds to a limit-cycle. In general, the amplitudes and angular frequency will be expressed in terms of the parameters λ_1 and λ_2.

Here we present an example performed by Mickens with the Harmonic Balance method [35].

Mickens [35] has given the general procedure for calculating solutions by means of the method of direct Harmonic Balance as follows:

$$\ddot{x} + x^{-1} = 0. \tag{2.27}$$

For the first-order Harmonic Balance, the solution is $x_1(t) = A\cos\theta,\ \theta = \Omega_1 t.$ This calculation is best achieved if the TNL oscillator is written to the form

$$x\ddot{x} + 1 = 0. \tag{2.28}$$

Substituting $x_1(t)$ into this equation gives

$$\begin{cases} (A\cos\theta)(-\Omega_1^2\, A\cos\theta) + 1 + HOH \cong 0 \\ \left[-\left(\dfrac{\Omega_1^2\, A^2}{2}\right) + 1\right] + HOH \cong 0, \end{cases} \tag{2.29}$$

Therefore, in lowest order, the angular frequency is

$$\Omega_1(A) = \frac{\sqrt{2}}{A} = \frac{1.4142}{A}. \tag{2.30}$$

The second Harmonic Balance approximation is

$$x_2(t) = A_1 \cos\theta + A_2 \cos 3\theta,\ \theta = \Omega_2\, t. \tag{2.31}$$

Substituting this expression into equation (2.28) gives

$$(A_1\cos\theta + A_2\cos 3\theta)[-\Omega_2^2(A_1\cos\theta + 9A_2\cos 3\theta) + 1 \cong 0, \tag{2.32}$$

and on executing the required expansions, we obtain

$$\left[-\Omega_2^2\left(\frac{A_1^2 + 9A_2^2}{2}\right) + 1\right] - \Omega_2^2\left(\frac{A_1^2 + 10A_1A_2}{2}\right)\cos 2\theta + HOH \cong 0, \tag{2.33}$$

Setting the constant term and the coefficient of $\cos 2\theta$ to zero gives

$$-\Omega_2^2\left(\frac{A_1^2 + 9A_2^2}{2}\right) + 1 = 0,\ A_1^2 + 10A_1A_2 = 0, \tag{2.34}$$

with the solutions

$$A_2 = -\left(\frac{A_1}{10}\right),\ \Omega_2^2 = \frac{200}{109A_1^2}. \tag{2.35}$$

Therefore,

$$x_2(t) = A_1\left[\cos(\Omega_2 t) - \left(\frac{1}{10}\right)\cos(3\Omega_2 t)\right], \tag{2.36}$$

and requiring

$$x_2(0) = A = \left(\frac{9}{10}\right)A_1,\ A_1 = \left(\frac{10}{9}\right)A, \tag{2.37}$$

gives

$$x_2(t) = \left(\frac{10}{9}\right)A\left[\cos(\Omega_2 t) - \left(\frac{1}{10}\right)\cos(3\Omega_2 t)\right], \tag{2.38}$$

with

$$\Omega_2^2 = \frac{200}{109A_1^2} = \left(\frac{162}{109}\right)\frac{1}{A^2},\ \Omega_2(A) = \frac{1.273}{A}. \tag{2.39}$$

The percentage error is

$$\left|\frac{\Omega_{exact} - \Omega_2}{\Omega_{exact}}\right| \times 100 = 1.6\%.$$

Note that the first approximation gives

$$\left|\frac{\Omega_{exact}-\Omega_1}{\Omega_{exact}}\right|\times 100 = 12.84\%.$$

In this process the third approximation gives

$$\left|\frac{\Omega_{exact}-\Omega_3}{\Omega_{exact}}\right|\times 100 = 1.58\%.$$

Recently some authors used Iterative technique [28-42] for calculating approximations to the periodic solutions and corresponding frequencies of TNL oscillator differential equations for small and as well as large amplitude of oscillation. The method was originated by Mickens in 1987. He provided a general basis for Iterative methods as they are currently used in the calculation of approximations to the periodic solutions of various nonlinear oscillatory differential equations successfully. The general methodology of Iterative procedure by Mickens [35] and example will present below. The existence of a large percentage error suggests that we should try an alternative Iterative scheme and determine if a better result can be found. Further a generalization of this work was then given by Lim and Wu [29]. Their procedure is as follows:

They assumed the equation in the form

$$\ddot{x} + f(x) = 0, x(0) = A, \dot{x}(0) = 0, \tag{2.40}$$

where A is given positive constant and $f(x)$ satisfies the condition

$$f(-x) = -f(x). \tag{2.41}$$

Adding $\omega^2 x$ on both sides of equation (2.40), we obtain

$$\ddot{x} + \omega^2 \mathrm{x} = \omega^2 \mathrm{x} - f(x) \equiv g(x), \tag{2.42}$$

where ω is priory unknown frequency of the periodic solution $x(t)$ being sought.

They proposed the Iterative scheme of equation (2.42)

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = g(x_{k-1}) + g(x_{k-1})(x_k - x_{k-1});\ k = 0, 1, 2, \ldots, \tag{2.43}$$

where $g_x = \dfrac{\partial g}{\partial x}$ and the inputs of initial guess are

$$x_{-1}(t) = x_0(t) = A\cos(\omega t), \tag{2.44}$$

with the initial conditions

$$x_k(0) = A, \quad \dot{x}_k(0) = 0, \;\; k = 1,\, 2,\, 3, \ldots . \tag{2.45}$$

Then substituting equation (2.44) into equation (2.43) and expanding the right hand side of equation (2.43) into the Fourier series yields

$$g[x_{k-1}(t)] + g_x[x_{k-1}(t)][x_t(t) - x_{k-1}(t)] = a_1(A,\omega)\cos\omega t + \sum_{n=2}^{N} a_{2n-1}(A,\omega)\cos[(2n-1)\,\omega t], \tag{2.46}$$

where the coefficients $a_{2n-1}(A,\omega)$ are known functions of A and ω, and the integer N depends upon the function $g(x)$ of the right hand side of equation (2.42) , On view of equation (2.46), the solution of equation is taken to be

$$x_{k+1}(t) = B\cos\omega t - \sum_{n=2}^{N} \frac{a_{2n-1}(A,\omega)}{[(2n-1)^2 - 1]\omega^2}\cos[(2n-1)\omega t], \tag{2.47}$$

where B is, tentatively, an arbitrary constant.

In equation (2.47), the particular solution is chosen such that it contains no secular terms [35], which requires that the coefficient $a_1(A,w)$ of right-side term $\cos\omega t$ in equation (2.46) satisfy

$$a_1(A,w) = 0\,. \tag{2.48}$$

The equation (2.48) allows the determination of the frequency as a function A. Next, the unknown constant B will be computed by imposing the initial conditions in equation (2.45). Finally, putting these steps together gives the solution $x_{k+1}(t)$.

Mickens [35] has given the general procedure for calculating solutions by means of the method of direct Iterative method as follows:

Step-1. Assume that the differential equation of interest is

$$F(\ddot{x}, x) = 0, \;\; x(0) = A, \; \dot{x}(0) = 0, \tag{2.49}$$

and further assume that it can be rewritten to the form

$$\ddot{x} + f(\ddot{x}, x) = 0\,, \tag{2.50}$$

Step-2. Next, adding $\Omega^2 x$ to both sides to obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(x, \ddot{x}) \equiv G(x, \ddot{x}), \quad (2.51)$$

where the constant Ω^2 is currently unknown.

Step-3. Now, formulate the Iterative scheme in the following way

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \ddot{x}_k); \quad k = 0, 1, 2, \ldots, \quad (2.52)$$

with

$$x_0(t) = A\cos(\Omega_0 t), \quad (2.53)$$

such that the x_{k+1} satisfy the initial conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \quad (2.54)$$

Step-4. At each stage of the Iterative, k is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$.

Step-5. This procedure gives a sequence of solutions: $x_0(t),\ x_1(t),\ \ldots$. Since all solutions are obtained from investigating linear equations, they are, in principle, easy to calculate.

The only difficulty might be the algebraic intensity required to complete the calculations. At this point, the following observations should be noted:

i. The solution for $x_{k+1}(t)$ depends on having the solutions for k less than $(k+1)$.

ii. The linear differential equation for $x_{k+1}(t)$ allows the determination of Ω_k by the requirement that secular terms be absent. Therefore, the angular frequency, Ω appearing on the right-hand side of equation (2.52) in the function $x_k(t)$, is k.

iii. In general, if equation (2.50) is of odd parity, i.e.,

$$f(-\ddot{x}, -x) = -f(\ddot{x}, x), \quad (2.55)$$

then the $x_k(t)$ will only contain odd multiples of the angular frequency.

Here we present an Example performed by Mickens with the Direct Iterative [43] method.

The TNL oscillator differential equation of equation (2.27) can be written as

$$\begin{cases} x\ddot{x}+1=0 \\ \ddot{x}=-(\dot{x})^2 x \\ \ddot{x}+\Omega^2 x=\Omega^2 x-(\dot{x})^2 x. \end{cases} \tag{2.56}$$

This last expression suggests the following Iterative scheme

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=\Omega_k^2 x_k-(\dot{x}_k)^2 x_k\,. \tag{2.57}$$

For $k=0$ and $x_0(t)=A\cos\theta, \theta=\Omega_0\,t,$ we have

$$\begin{aligned} \ddot{x}_1+\Omega_0^2 x_1 &= (\Omega_0^2\,A\cos\theta)-(-\Omega_0^2\,A\cos\theta)^2(A\cos\theta) \\ &= \Omega_0^2\left[1-\frac{3A^2\Omega_0^2}{4}\right]A\cos\theta-\left(\frac{A^3\Omega_0^4}{4}\right)\cos 3\theta. \end{aligned} \tag{2.58}$$

The elimination of secular terms gives

$$1-\frac{3A^2\Omega_0^2}{4}=0,\ \ \Omega_0^2(A)=\left(\frac{4}{3}\right)\frac{1}{A^2}. \tag{2.59}$$

Therefore, $x_1(t)$ satisfies the equation

$$\ddot{x}_1+\Omega_0^2 x_1=-\left(\frac{A^3\Omega_0^4}{4}\right)\cos 3\theta. \tag{2.60}$$

The particular solution, $x_1^{(p)}(t)$, is

$$x_1^{(p)}(t)=\left(\frac{A^3\Omega_0^2}{32}\right)\cos 3\theta=\left(\frac{A}{24}\right)\cos 3\theta. \tag{2.61}$$

Therefore, the general solution is

$$x_1(t)=C\cos\theta+\left(\frac{A}{24}\right)\cos 3\theta. \tag{2.62}$$

Using $x_1(0)=A,$ then $C=23/24$ and

$$x_1(t)=A\left[\left(\frac{23}{24}\right)\cos\theta+\left(\frac{1}{24}\right)\cos 3\theta\right]. \tag{2.63}$$

If the calculation is stopped at this point, then

$$\begin{cases} x_1(t) = A\left[\left(\dfrac{23}{24}\right)\cos(\Omega_0\, t) + \left(\dfrac{1}{24}\right)\cos(3\Omega_0\, t)\right] \\ \Omega_0(A) = \dfrac{2}{\sqrt{3}A} = \dfrac{1.1547}{A}. \end{cases} \tag{2.64}$$

Note that

$$\Omega_{exact}(A) = \frac{\sqrt{2\pi}}{2A} = \frac{1.2533141}{A}, \tag{2.65}$$

and

$$\left|\frac{\Omega_{exact} - \Omega_0}{\Omega_{exact}}\right| \times 100 = 7.9\%\ error.$$

Proceeding to the second level of Iterative, $x_2(t)$ must satisfy the equation

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - (\ddot{x}_1)^2 x_1, \tag{2.66}$$

where

$$x_1(t) = A\left[\left(\frac{23}{24}\right)\cos(\Omega_1 t) + \left(\frac{1}{24}\right)\cos(3\Omega_1 t)\right]. \tag{2.67}$$

Let $\theta = \Omega_1\, t$ and substituting this $x_1(t)$ into the right-hand side of equation (2.66); doing so gives

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2\left[\alpha - \left(\frac{3}{4}\right)A^2\Omega_1^2 g(\alpha,\beta)\right]A\cos\theta + HOH, \tag{2.68}$$

where

$$\begin{cases} g(\alpha,\beta) = \alpha^3 + \left(\dfrac{19}{3}\right)\alpha^2\beta + 66\alpha\beta^2 + 27\beta^3, \\ \alpha = \dfrac{23}{24}, \quad \beta = \dfrac{1}{24}. \end{cases} \tag{2.69}$$

The absence of secular terms gives

$$\begin{cases} \Omega_1^2 = \left[\left(\dfrac{4}{3}\right)\dfrac{1}{A^2}\right]\left[\dfrac{\alpha}{g(\alpha,\beta)}\right] \\ \Omega_1(A) = \dfrac{1.0175}{A}, \end{cases} \tag{2.70}$$

and

$$\left|\frac{\Omega_{exact} - \Omega_1}{\Omega_{exact}}\right| \times 100 = 18.1\%\ error.$$

The existence of such a large percentage-error suggests that we should try an alternative Iterative scheme and determine if a better result can be found. This second scheme is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \ddot{x}_k - \Omega_k^2 (x_k)^2 \ddot{x}_k. \tag{2.71}$$

For $k = 0$, we have

$$\ddot{x}_1 + \Omega_0^2 x_1 = \ddot{x}_0 - \Omega_0^2 (x_0)^2 \ddot{x}_0, \tag{2.72}$$

with $x_0(t) = A\cos(\Omega_0 t),$ we find that

$$\Omega_0(A) = \sqrt{\frac{4}{3}}\left(\frac{1}{A}\right), \tag{2.73}$$

which is exactly the same result as previously given in equation (2.64). Similarly, we also determine that $x_1(t)$ is

$$x_1(t) = A\left[\left(\frac{25}{24}\right)\cos\theta - \left(\frac{1}{24}\right)\cos 3\theta\right], \tag{2.74}$$

a result which differs from the previous calculation, i.e., compare the coefficients in equation (2.63) and equation (2.74). Further, the value of $\Omega_1(A)$, for the Iterative scheme of equation (2.71), is

$$\Omega_1^2(A) = \left[\left(\frac{4}{3}\right)\frac{1}{A^2}\right]\left[\frac{\alpha}{h(\alpha,\beta)}\right], \tag{2.75}$$

where, for this case,

$$\begin{cases} h(\alpha,\beta) = \alpha^3 - \left(\frac{11}{3}\right)\alpha^2\beta + \left(\frac{38}{3}\right)\alpha\beta^2, \\ \alpha = \frac{25}{24}, \quad \beta = \frac{1}{24}, \end{cases} \tag{2.76}$$

with

$$\Omega_1(A) = \frac{1.0262}{A}, \tag{2.77}$$

and

$$\left|\frac{\Omega_{exact} - \Omega_1(A)}{\Omega_{exact}}\right| \times 100 = 18\%.$$

The general conclusion reached is that if the percentage error in the angular frequency is to be taken as a measure of the accuracy of this calculation, then the Iterative method does not appear to work well for this particular TNL oscillator. In fact, since the error for $\Omega_0(A)$ is less than that of $\Omega_1(A)$, the two schemes may give (increasing in value) erroneous results for the angular frequency as k becomes larger.

In 2005, this process was extended by Mickens [28] which is used in the calculation of approximations to the periodic solutions of nonlinear oscillatory differential equations. A generalization of this work was then given by Lim and Wu [29] and this was followed by an additional extension in Mickens. Actually Iterative method is a technique for calculating approximations to the periodic solutions of TNL oscillator which is presented by R.E. Mickens in [28].

Mickens [35] has given the general procedure for calculating solutions by means of the method of Extended Iterative method as follows:

He consider the equation as

$$\ddot{x} + f(\ddot{x}, \dot{x}, x) = 0, \; x(0) = A, \; \dot{x}(0) = 0, \tag{2.78}$$

where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ on both sides of equation (2.78), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x}, \dot{x}, x) \equiv G(x, \dot{x}, \ddot{x}) . \tag{2.79}$$

Now, formulate the Iterative scheme as

$$\begin{aligned} \ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = & G(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}) + G_x(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})(x_k - x_{k-1}) \\ & + G_{\dot{x}}(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})(\dot{x}_k - \dot{x}_{k-1}) + G_{\ddot{x}}(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})(\ddot{x}_k - \ddot{x}_{k-1}) , \end{aligned} \tag{2.80}$$

where

$$G_x = \frac{\partial G}{\partial x} , \; G_{\dot{x}} = \frac{\partial G}{\partial \dot{x}} , \; G_{\ddot{x}} = \frac{\partial G}{\partial \ddot{x}} . \tag{2.81}$$

And x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{2.82}$$

The initial guess are taken to be [30]

$$x_{-1}(t) = x_0(t) = A \cos(\Omega_0 t) . \tag{2.83}$$

The right hand side of equation (2.80) is essentially the first term in a Taylor series expansion of the function $G(x_k, \dot{x}_k, \ddot{x}_k)$ at the point $(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})$ [53]. To illustrate this point, note that

$$x_k = x_{k-1} + (x_k - x_{k-1}) , \tag{2.84}$$

and for some function $G(x)$, we have

$$G(x_k) = G[x_{k-1} + (x_k - x_{k-1})] = G(x_{k-1}) + G_x(x_k - x_{k-1}) + \dots . \tag{2.85}$$

An alternative, but very insightful, modification of above scheme was proposed by Hu [32]. He used the following equation in place of equation (2.84)

$$x_k = x_0 + (x_k - x_o). \tag{2.86}$$

Then, equation (2.85) is changed to

$$G(x_k) = G[x_0 + (x_k - x_0)] = G(x_0) + G_x(x_k - x_0) + \dots, \tag{2.87}$$

and the corresponding modification to equation (2.80) is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_0, \dot{x}_0, \ddot{x}_0); + G_x(x_0, \dot{x}_0, \ddot{x}_0)(x_k - x_0) \\ + G_{\dot{x}}(x_0, \dot{x}_0, \ddot{x}_0)(\dot{x}_k - \dot{x}_0) + G_{\ddot{x}}(x_0, \dot{x}_0, \ddot{x}_0)(\ddot{x}_k - \ddot{x}_0). \tag{2.88}$$

This scheme is computationally easier to work with, for $k \geq 2$, than the one given in equation (2.80). The essential idea is that if $x_0(t)$ is a good approximation, then the expansion should take place at $x = x_0$. Also, as pointed out by Hu [32], the $x_0(t)$ in $(x_k - x_o)$ is not the same for all k. In particular, $x_0(t)$ in $(x_1 - x_o)$ is the function $A\cos(\Omega_1 t)$, while the $x_0(t)$ in $(x_2 - x_o)$ is the function $A\cos(\Omega_2 t)$.

Here we present an example executed by Mickens with the Extended Iterative method [35]:

The TNL oscillator equation (2.27) has several possible Iterative schemes. We use the one derived from the relation

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x(\ddot{x})^2 = G(x, \ddot{x}, \Omega^2), \tag{2.89}$$

that is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \left[\Omega_k^2 x_0 - x_0(\ddot{x}_0)^2\right] + \left[\Omega_k^2 - (\ddot{x}_0)^2\right](x_k - x_0) - 2x_0\ddot{x}_1(\ddot{x}_k - \ddot{x}_0). \tag{2.90}$$

To obtain this relation the following formula was used for the Extended Iterative scheme

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_0, \ddot{x}_0, \Omega_k^2) + G_x(x_0, \ddot{x}_0, \Omega_k^2)(x_k - x_0) + G_{\ddot{x}}(x_0, \ddot{x}_0, \Omega_k^2)(\ddot{x}_k - \ddot{x}_0). \tag{2.91}$$

For k = 1, we have

$$\ddot{x}_2 + \Omega_1^2 x_2 = 2x_0(\ddot{x}_0)^2 + \left[\Omega_1^2 - (\ddot{x}_0)^2\right]x_1 - 2x_0\ddot{x}_0\ddot{x}_1, \tag{2.92}$$

with

$$\begin{cases} x_0(t) = A\cos\theta, \\ x_1(t) = A\left[\alpha\cos\theta + \beta\cos 3\theta\right], \\ \theta = \Omega_1 t, \ \alpha = \dfrac{23}{24}, \ \beta = \dfrac{1}{24}. \end{cases} \tag{2.93}$$

(Seeing equation (2.63) for $x_1(t)$). Substituting the items in equation (2.93) into the right-hand side of $\Omega_1(A) = \frac{1.189699}{A}$. Equation (2.92) gives, after some algebraic and trigonometric simplification, the result

$$\begin{aligned} \ddot{x}_2 + \Omega_1^2 x_2 &= (\Omega_1^2 A)\left[\alpha - (3-7\beta)\left(\frac{\Omega_1^2 A^4}{4}\right)\right]\cos\theta \\ &-\left(\frac{A\Omega_1^2}{4}\right)[(1+35\beta)\Omega_1^2 A^2 - 4\beta]\cos 3\theta - \left(\frac{19\beta}{4}\right)(\Omega_1^4 A^3)\cos 5\theta. \end{aligned} \tag{2.94}$$

Setting the coefficient of $\cos\theta$ to zero and solving for Ω_1^2 gives

$$\begin{cases} \Omega_1^2(A) = \left[\left(\frac{4}{3}\right)\frac{1}{A^2}\right]\left(\frac{69}{65}\right) = \Omega_0^2(A)\left[\frac{69}{65}\right] \\ \Omega_1(A) = \frac{1.189699}{A}. \end{cases} \tag{2.95}$$

Comparing $\Omega_1(A)$ with the exact value, $\Omega_{exact}(A)$, we find the following percentage error

$$\left|\frac{\Omega_{exact} - \Omega_1}{\Omega_{exact}}\right| \times 100 = 5.1\%.$$

Note that using the direct Iterative scheme, we found

$$\begin{cases} \Omega_0(A) = \frac{1.1547}{A} \ \ (7.9\% \ error), \\ \Omega_1(A) = \frac{1.0175}{A} \ \ (18.1\% \ error). \end{cases}$$

Therefore, the Extended Iterative procedure provides a better estimate of the angular frequency.

Replacing $\Omega_1^2 A^2$ in equation (2.94), by the expression of equation (2.95), we obtain

$$\ddot{x}_2 + \Omega_1^2 x_2 = -\left(\frac{A\Omega_1^2}{4}\right)\left(\frac{1292}{390}\right)\cos\theta - \left(\frac{A\Omega_1^2}{4}\right)\left(\frac{437}{390}\right)\cos 5\theta. \tag{2.96}$$

The corresponding particular solution takes the form

$$x_2^{(p)}(t) = D_1\cos 3\theta + D_2\cos 7\theta. \tag{2.97}$$

Substituting this into equation (2.96) and equating the coefficients, respectively, of the $\cos 3\theta$ and $\cos 7\theta$ terms, allows the calculation of D_1 and D_2; they are

$$\begin{cases} D_1 = \left(\dfrac{3876}{37440}\right)A, \\ D_2 = \left(\dfrac{437}{37440}\right)A. \end{cases} \tag{2.98}$$

Since the full solution for $x_2(t)$ is

$$x_2(t) = C\cos\theta + x_1^{(p)}(t), \tag{2.99}$$

with $x_2(0) = A$, it follows that

$$C = A - D_1 - D_2 = \left(\frac{33127}{37440}\right)A, \tag{2.100}$$

and

$$\begin{cases} x_2(t) = A\left[\left(\dfrac{33127}{37440}\right)\cos\theta + \left(\dfrac{3876}{37440}\right)\cos 3\theta + \left(\dfrac{437}{37440}\right)\cos 5\theta\right], \\ \theta = \Omega_1(t)t = \left[\dfrac{92}{65}\right]^{1/2}\left(\dfrac{1}{A}\right). \end{cases} \tag{2.101}$$

Inspection of $x_2(t)$ indicates that the coefficients of the harmonics satisfy the ratios

$$\begin{cases} \dfrac{a_1}{a_0} = \dfrac{3876}{33127} \approx 0.117, \\ \dfrac{a_2}{a_1} = \dfrac{437}{3876} \approx 0.113. \end{cases} \tag{2.102}$$

Now-a-day's Iterative method is used widely by Lim and Wu [29], Hu and Tang [33], Chen and Liu [34], Haque *et al.* [36-42] etc. which is valid for small together with large amplitude of oscillation to attain the approximate frequency and the harmonious periodic solution of such nonlinear problems. Mickens [28] provided a general basis for Iterative methods as they are currently used in the calculation of approximations to the periodic solutions of nonlinear oscillatory differential equations.

Here we present an Example executed by Haque *et al.* [36] with the Iterative method:

The equation (2.27) can be written as

$$\ddot{x}+\Omega^2 x=\Omega^2 x-x^{-1}. \tag{2.103}$$

According to equation (2.52), the Iterative scheme of equation (2.103) will be

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=\Omega_k^2 x_k-x_k^{-1}. \tag{2.104}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained from the solution of (putting $k=0$ in equation (2.104) and utilizing equation (2.53))

$$\ddot{x}_1+\Omega_0^2 x_1=\Omega_0^2 A\cos\theta-(A\cos\theta)^{-1}. \tag{2.105}$$

Now expanding $(\cos\theta)^{-1}$ in a Fourier Cosine series in interval $[0,\pi]$, the equation (2.103) reduces to

$$\ddot{x}_1+\Omega_0^2 x_1=\Omega_0^2 A\cos\theta-\frac{2}{A}\sum_{n=1}^{\infty}(-1)^{n-1}\cos(2n-1)\theta. \tag{2.106}$$

To check secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (2.106), and we obtain

$$\Omega^2{}_0=\frac{2}{A^2},\ \Omega_0=\frac{1.414}{A}. \tag{2.107}$$

Then solving equation (2.106) and satisfying the initial condition (according to equation (2.54)), we obtain

$$x_1(t)=A((1+\frac{1}{4}(-1+2\ln 2))\cos\theta-\sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta). \tag{2.108}$$

This is the second approximation of equation (2.27) and the related Ω_1 is to be determined. The second approximation $x_2(t)$ and the value of Ω_1 are obtained from the solution of

$$\ddot{x}_2+\Omega_1^2 x_2=\Omega_1^2 x_1-x_1^{-1}. \tag{2.109}$$

Substituting $x_1(t)$ from equation (2.108) into the right-hand side of equation (2.109), we obtain

$$\ddot{x}_2+\Omega_1^2x_2 = A\Omega_1^2((1+(-1+2\ln 2)/4)\cos\theta-\sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta)$$
$$-\frac{1}{A}\sum_{n=1}^{\infty}(-1)^{n-1}a_{2n-1}\cos(2n-1)\theta, \tag{2.110}$$

where

$$a_1=1.599611, a_3=0.983636, a_5=1.102235, a_7=1.079400, a_9=1.083797,\ldots . \tag{2.111}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (2.110). Thus we have

$$\Omega^2{}_1=\frac{1.599611}{A^2(1+(-1+2\ln 2)/4)},\ \Omega_1=\frac{1.208}{A}. \tag{2.112}$$

Then equation (2.110) becomes,

$$\ddot{x}_2+\Omega_1^2\,x_2=-A\Omega_1{}^2\sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta+\frac{1}{A}\sum_{n=2}^{\infty}(-1)^n a_{2n-1}\cos(2n-1)\theta. \tag{2.113}$$

The equation (2.113) approximately can be written as,

$$\ddot{x}_2+\Omega_1^2\,x_2=-A\Omega_1{}^2\sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta+\frac{1.1}{A}\sum_{n=2}^{\infty}(-1)^n a_{2n-1}\cos(2n-1)\theta. \tag{2.114}$$

Then solving equation (2.114) and satisfying the initial condition, we obtain the second approximation,

$$x_2(t)=A((1-(3-4\ln 2)/16+1.1(-1+2\ln 2)/(4z))\cos\theta A$$
$$+\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{(4(n-1)n)^2}+\frac{1.1(-1)^{n-1}}{4(n-1)n\Omega^2{}_1}\right)\cos(2n-1)\theta, \tag{2.115}$$

where

$$z=\frac{8}{(1+(-1+2\ln 2)/4)\sqrt{(3+\ln 2)(4+\ln 16)}}. \tag{2.116}$$

The third approximation x_3 and the value of Ω_2 are obtained from the solution of

$$\ddot{x}_3+\Omega_2^2\,x_3=\Omega_2^2\,x_2-x_2^{-1}. \tag{2.117}$$

Substituting $x_2(t)$ from equation (2.115) into the right-hand side of equation (2.117) and utilizing the same method, we obtain

$$\ddot{x}_3+\Omega_2^2 x_3=\sum_{n=2}^{\infty}\left(A\Omega^2{}_2\left(\frac{(-1)^n}{(4(n-1)n)^2}+\frac{1.1(-1)^{n-1}}{4\Omega^2{}_1(n-1)n}\right)+(-1)^n\frac{1.26}{A}\right)\cos(2n-1)\theta\,, \quad (2.118)$$

where

$$\Omega_2^2=1.693744/A^2(1-(3-4\ln 2)/16+1.1(-1+2\ln 2)/(4z)))\,,\ \Omega_2=\frac{1.265}{A}. \quad (2.119)$$

Then solving equation (2.118) and satisfying the initial condition, we obtain

$$x_3(t)=A\left(1.0672\cos\theta-\sum_{n=2}^{\infty}\left(\frac{1}{((n-1)n)^3}-\frac{1.1}{z((n-1)n)^2}+\frac{1.26}{z_1(n-1)n}\right)\cos\theta(2\cos 2\theta-1)\right), \quad (2.120)$$

where

$$z_1=1.693744/(1-(3-4\ln 2)/16+1.1(-1+2\ln 2)/4z)\,. \quad (2.121)$$

Therefore, $\Omega_0,\Omega_1,\Omega_2,...$, respectively obtained by equation (2.107), (2.112), (2.119),…., represent the approximation of frequencies of oscillator (2.27).

CHAPTER III

New Extended Iterative Method

3.1 Introduction

Inverse truly nonlinear oscillator (acceleration is inversely proportional to displacement) occurs as a model of certain phenomena in plasma physics [35]. Mickens [35] used the Iterative technique to calculate a higher-order approximation to the periodic solutions of a conservative oscillator. Hu [52] applied the modified Iterative technique of Mickens [35] to find approximate of nonlinear oscillators with fractional powers and quadratic nonlinear oscillator respectively. Haque *et al.* [36-42] has applied Mickens Iterative and modified Iterative method to determine approximate periodic solutions of a class of nonlinear equations. The main purpose of this thesis is to develop a modification of the Extended Iterative technique for the determination of approximate solution and angular frequency of "the inverse truly nonlinear oscillator". We compare the result with existing results obtained by various researchers and it is mentioned that our solution measure similar and sometimes better results than other existing procedures.

3.2 The Method

An Extended Iterative method will be used to obtain analytical solution of the inverse truly nonlinear oscillator. The procedure may be briefly described as follows.

A nonlinear oscillator will be modeled by

$$\ddot{x} + f(\ddot{x}, \dot{x}, x) = 0, \; x(0) = A, \; \dot{x}(0) = 0, \tag{3.1}$$

where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ to both sides of equation (3.1), we obtain

$$\ddot{x}+\Omega^2 x=\Omega^2 x-f(\ddot{x},\dot{x},x)\equiv G(\ddot{x},\dot{x},x). \tag{3.2}$$

The Extended Iterative scheme is

$$\begin{aligned}\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}&=G(x_{k-1},\dot{x}_{k-1},\ddot{x}_{k-1})+G_x(x_{k-1},\dot{x}_{k-1},\ddot{x}_{k-1})(x_k-x_{k-1})\\&+G_{\dot{x}}(x_{k-1},\dot{x}_{k-1},\ddot{x}_{k-1})(\dot{x}_k-\dot{x}_{k-1})+G_{\ddot{x}}(x_{k-1},\dot{x}_{k-1},\ddot{x}_{k-1})(\ddot{x}_k-\ddot{x}_{k-1}),\end{aligned} \tag{3.3}$$

where $G_x=\dfrac{\partial G}{\partial x}$, $G_{\dot{x}}=\dfrac{\partial G}{\partial \dot{x}}$, $G_{\ddot{x}}=\dfrac{\partial G}{\partial \ddot{x}}$.

The right hand side of equation (3.3) is essentially the first term in a Taylor series expansion of the function $G(x_k,\dot{x}_k,\ddot{x}_k)$ at the point $(x_{k-1},\dot{x}_{k-1},\ddot{x}_{k-1})$ [53].

An alternative, but very insightful, modification of above scheme was proposed by Hu [52] and the corresponding modification to equation (3.3) is

$$\begin{aligned}\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}&=G(x_0,\dot{x}_0,\ddot{x}_0);+G_x(x_0,\dot{x}_0,\ddot{x}_0)(x_k-x_0)\\&+G_{\dot{x}}(x_0,\dot{x}_0,\ddot{x}_0)(\dot{x}_k-\dot{x}_0)+G_{\ddot{x}}(x_0,\dot{x}_0,\ddot{x}_0)(\ddot{x}_k-\ddot{x}_0)\end{aligned} \tag{3.4}$$

And x_{k+1} satisfies the conditions

$$x_{k+1}(0)=A,\ \dot{x}_{k+1}(0)=0. \tag{3.5}$$

The initial guess are taken to be [29]

$$x_{-1}(t)=x_0(t)=A\cos(\Omega_0 t). \tag{3.6}$$

The above procedure gives the sequence of solutions $x_1(t),\ x_2(t), x_3(t),\cdots$. The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [28].

3.3 Solution Procedure

Let us consider the nonlinear inverse oscillator

$$\ddot{x}+x^{-1}=0 \tag{3.7}$$

Adding $\Omega^2 x$ on both sides of equation (3.7), we get

$$\ddot{x}+\Omega^2 x=\Omega^2 x-x^{-1}=G(x,\Omega^2) \tag{3.8}$$

where $G(x,\Omega^2)=\Omega^2 x-x^{-1}$, $G_x(x,\Omega^2)=\Omega^2+x^{-2}$.

According to equation (3.4), the Extended Iterative scheme of equation (3.8) is

$$\ddot{x}_{k+1} + \Omega_k^2\, x_{k+1} = (\Omega_k^2\, x_0 - x_0^{-1}) + (\Omega_k^2 + x_0^{-2})\,(x_k - x_0). \tag{3.9}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained by putting $k = 0$ in equation (3.9) and using equation (3.6) we get

$$\ddot{x}_1 + \Omega_0^2\, x_1 = (\Omega_0^2\, x_0 - x_0^{-1}) + (\Omega_0^2 + x_0^{-2})\,(x_0 - x_0)$$

$$= \Omega_0^2\, x_0 - x_0^{-1}, \tag{3.10}$$

where $x_0(t) = A\cos(\Omega_0 t) = A\cos\theta$.

Now substituting $x_0(t)$ and expanding the right-hand-side in a Fourier cosine series, then equation (3.10) reduces to

$$\ddot{x}_1 + \Omega_0^2\, x_1 = \Omega_0^2\, A\cos\theta - \left(\frac{2}{A}\cos\theta - \frac{2}{A}\cos 3\theta + \frac{2}{A}\cos 5\theta\right)$$

$$= \left(\Omega_0^2\, A - \frac{2}{A}\right)\cos\theta + \frac{2}{A}\cos 3\theta - \frac{2}{A}\cos 5\theta. \tag{3.11}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (3.11). Thus we have

$$\Omega_0^2 A - \frac{2}{A} = 0,\ \Omega_0 = \frac{\sqrt{2}}{A} = \frac{1.41421}{A}. \tag{3.12}$$

This is the first approximate frequency of the oscillator. Note that $\Omega_{exact}(A) = \dfrac{1.253314}{A}$.

After simplification the equation (3.11) reduces to

$$\ddot{x}_1 + \Omega_0^2\, x_1 = \frac{2}{A}\cos 3\theta - \frac{2}{A}\cos 5\theta. \tag{3.13}$$

The particular solution, $x_1^{(p)}(t)$ is

$$x_1^{(p)}(t) = \frac{2/A}{-9\Omega_0^2 + \Omega_0^2}\cos 3\theta - \frac{2/A}{-9\Omega_0^2 + \Omega_0^2}\cos 5\theta$$

$$= -\,0.125\, A\cos 3\theta + 0.041667\, A\cos 5\theta. \tag{3.14}$$

Therefore, the complete solution is

$$x_1(t) = C\cos\theta - 0.125\,A\cos 3\theta + 0.0416667\,A\cos 5\theta. \tag{3.15}$$

Using $x_1(0) = A$, we have $C = 1.083333\,A$. Then we obtain

$$x_1(t) = A(1.083333\cos\theta - 0.125\cos 3\theta). \tag{3.16}$$

This is the first approximate solution of the oscillator.

Proceeding to the second level of Iterative, $x_2(t)$ satisfies the equation

$$\ddot{x}_2 + \Omega_1^2\,x_2 = (\Omega_1^2\,x_0 - x_0^{-1}) + (\Omega_1^2 + x_0^{-2})(x_1 - x_0)$$

$$= \Omega_1^2\,x_1 - 2x_0^{-1} + x_0^{-2}x_1, \tag{3.17}$$

where $x_0(t) = A\cos(\Omega_0 t) = A\cos\theta$ and $x_1(t) = A(1.083333\cos\theta - 0.125\cos 3\theta)$.

Now substituting $x_0(t)$ and $x_1(t)$ and expanding the right- hand side in a Fourier cosine series, then equation (3.17) reduces to

$$\ddot{x}_2 + \Omega_1^2\,x_2 = \Omega_1^2\,A\left(1.083333\cos\theta - 0.125\cos 3\theta\right) - \left(\frac{1.583333}{A}cos\theta - \frac{1.083333}{A}cos3\theta\right)$$

$$= \left(1.083333\Omega_1^2\,A - \frac{1.583333}{A}\right)cos\theta - \left(0.125\Omega_1^2\,A - \frac{1.083333}{A}\right)cos3\theta. \tag{3.18}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (3.18). Thus we have

$$1.083333\Omega_1^2 A - \frac{1.583333}{A} = 0,\ \ \Omega_1 = \frac{1.20894}{A}. \tag{3.19}$$

This is the second approximate frequency of the oscillator. After simplification the equation (3.18) reduces to

$$\ddot{x}_2 + \Omega_1^2 x_2 = \frac{0.900641}{A}\cos 3\theta. \tag{3.20}$$

The particular solution, $x_2^{(p)}(t)$ is

$$x_2^{(p)}(t) = \frac{0.900641/A}{-9\Omega_1^2 + \Omega_1^2}\cos 3\theta \; = -0.077029\,A\cos 3\theta. \tag{3.21}$$

Therefore, the complete solution is

$$x_2(t) = D\cos\theta - 0.077029\,A\cos 3\theta. \tag{3.22}$$

Using $x_2(0) = A$, we have $D = 1.07703\,A.$ Then we obtain

$$x_2(t) = A(1.077029\cos\theta - 0.077029\cos 3\theta). \tag{3.23}$$

This is the second approximate solution of the oscillator.

Proceeding to the third level of Iterative, $x_3(t)$ satisfies the equation

$$\ddot{x}_3 + \Omega_2^2\,x_3 = (\Omega_2^2\,x_2 - x_0^{-1}) + (\Omega_2^2 + x_0^{-2})\,(x_2 - x_0)$$

$$= \Omega_2^2\,x_2 - 2x_0^{-1} + x_0^{-2}\,x_2\,, \tag{3.24}$$

where $x_0(t) = A\cos(\Omega_0 t) = A\cos\theta$ and $x_2(t) = A(1.077029\cos\theta - 0.077029\cos 3\theta).$

Now substituting $x_0(t)$ and $x_2(t)$ and expanding the right hand side in a Fourier cosine series, then equation (3.24) reduces to

$$\ddot{x}_3 + \Omega_2^2\,x_3 = \Omega_2^2\,A\big(1.077029\cos\theta - 0.077029\cos 3\theta\big) - \left(\frac{1.691886}{A}cos\theta - \frac{1.383772}{A}cos3\theta\right)$$

$$= \left(1.077029\;\Omega_2^2\,A - \frac{1.691886}{A}\right)cos\theta - \left(\;0.077029\Omega_2^2\,A - \frac{1.383772}{A}\right)cos3\theta. \tag{3.25}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (3.25). So we have

$$1.077029\Omega_2^2\,A - \frac{1.691886}{A} = 0,\; \Omega_2 = \frac{1.25335}{A}. \tag{3.26}$$

This is the third approximate frequency of the oscillator. After simplification the equation (3.25) reduces to

$$\ddot{x}_3 + \Omega_2^2\,x_3 = \frac{1.26277}{A}\cos 3\theta. \tag{3.27}$$

Then solving equation (3.27) and satisfying the initial condition $x_3(0) = A$, we obtain

$$x_3(t) = A(1.100482\cos\theta - 0.100482\cos 3\theta). \qquad (3.28)$$

This is the third approximate solution of the oscillator.

CHAPTER IV

Results and Discussion

In this chapter, we compare the results obtained by all of the methods that were used to calculate approximations to their periodic or oscillatory solutions. Consequently, one measure of the accuracy or quality of a given method is the difference between the exact value of the angular frequency and that determined using the approximation procedure.

4.1 Results

An Iterative approach is presented to obtain approximate solution of the "inverse truly nonlinear oscillator". Iterative methods utilized to approximate the solution of the oscillator. The present technique is very simple for investigating algebraic equations analytically and the approach is different from the existing other approach for taking truncated Fourier series. This process significantly improves the results.

Here we have calculated the first, second, third approximate frequencies Ω_0, Ω_1, Ω_2. All the results are given in the following Table-1. To compare the approximate frequencies we have also given the existing results determined by Mickens Direct and Extended Iterative method [35] and Mickens HB method [17], Haque *et al.* Iterative method [36] in Table-2. Fortunately our method gives significantly better result than other formula.

To show the accuracy, we have calculated the percentage of errors by the following definitions:

$$Error = \left| \frac{\Omega_e - \Omega_k}{\Omega_e} \right| \times 100\%$$

where $\Omega_k \ (k = 0,1,2,...)$ represents the approximate frequencies obtained by the present method and Ω_e represents the corresponding exact frequency of the oscillator.

Table-1:

Adopted approximate frequencies of $\ddot{x} + x^{-1} = 0$.

Exact frequency Ω_e			$\frac{1.253314}{A}$
Amplitude A	First approximate frequencies, Ω_0 Error (%)	Second approximate frequencies, Ω_1 Error (%)	Third approximate frequencies, Ω_2 Error (%)
Presented method	$\frac{1.41421}{A}$ 12.84	$\frac{1.20894}{A}$ 3.54	$\frac{1.25335}{A}$ 0.0029

Table-2:

Comparison of the approximate frequencies with exact frequency Ω_e of $\ddot{x} + x^{-1} = 0$.

Exact frequency Ω_e			$\frac{1.253314}{A}$
Amplitude A	First approximate frequencies, Ω_0 Error (%)	Second approximate frequencies, Ω_1 Error (%)	Third approximate frequencies, Ω_2 Error (%)
Mickens Direct Iterative method [35]	$\frac{1.155}{A}$ 7.9	$\frac{1.018}{A}$ 18.1	-----
Mickens Extended Iterative method [35]	$\frac{1.155}{A}$ 7.9	$\frac{1.189699}{A}$ 5.1	-----
Mickens HB method [17]	$\frac{1.414}{A}$ 12.84	$\frac{1.273}{A}$ 1.6	$\frac{1.2731}{A}$ 1.58
Haque *et al.* Iterative method [36]	$\frac{1.414}{A}$ 12.84	$\frac{1.208}{A}$ 3.63	$\frac{1.265}{A}$ 0.92
Presented method	$\frac{1.41421}{A}$ 12.84	$\frac{1.20894}{A}$ 3.54	$\frac{1.25335}{A}$ 0.0029

4.2 Convergence and Consistency Analysis

We know the basic idea of Iterative methods is to construct a sequence of solutions x_k (as well as frequencies Ω_k) that have the property of convergence

$$x_e = \lim_{k \to \infty} x_k \qquad \text{or,} \quad \Omega_e = \lim_{k \to \infty} \Omega_k$$

Here x_e is the exact solution of the given nonlinear oscillator.

In the present method, it has been shown that the solution yield the less error in each Iterative step compared to the previous Iterative step and finally $|\Omega_2 - \Omega_e| = |0.253350 - 0.253314| < \varepsilon$, where ε is a small positive number and A is chosen to be unity. From this, it is clear that the adopted method is convergent.

An Iterative method of the form represented by equation (3.4) with initial guess given in equation (3.5) is said to be consistent if

$$\lim_{k \to \infty} |x_k - x_e| = 0 \quad \text{or,} \quad \lim_{k \to \infty} |\Omega_k - \Omega_e| = 0$$

In the present analysis we see that

$$\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0, \text{ as } |\Omega_2 - \Omega_e| = 0.$$

Thus the consistency of the method is achieved.

4.3 Discussion

It is noted that Mickens [35] found only second approximate frequency by Direct Iterative method and Extended Iterative method. Mickens [17] and Haque *et al.* [36] also presented third approximate frequencies by Harmonic Balance method and Iterative method where result is comparatively not well.

CHAPTER V

Conclusions and Comments

In this final chapter, some concluding remarks have been included. Some essential recommendations about modified Extended Iterative method have also been presented.

5.1 Conclusions

The basic foundation behind Iterative methods is to re-express the original nonlinear differential equation that involves with an infinite sequence of equations, each of which can be solved, and such that at a particular stage of the calculation, knowledge of the solutions of the previous members of the sequence is required to solve the differential equation at that stage. The major issue is how to reformulate the original nonlinear differential such that a viable Iterative scheme can be constructed. The rewriting of a TNL differential equation to a new form raises several mathematical issues. The most significant is the relationship between the solutions of the original equations and those of the reformulated equations. In this thesis we used a simple but effective modification of the Extended Iterative method to investigate nonlinear differential equations. The results have improved when we truncated three terms to calculate the first approximate solution, two terms to calculate the second approximate solution and two terms to calculate the third approximate solution. This technique can be used as paradigms for many others applications in searching for periodic solution of other nonlinear oscillators. The obtained results show that the modification of the Extended Iterative method is more accurate than other methods and is valid for large region.

5.2 Comments

In the final analysis, the validity and value of a particular method and the solutions that it produces depend heavily on what we intend to do with the results obtained from the calculations. However, the following issues are of prime importance:

i. A given truly nonlinear (TNL) oscillator equation may have more than one possible Iterative scheme. At present, there are no a priori meta-principles which place limitations on the construction of Iterative schemes.
ii. Iterative methods may not provide accurate values for the angular frequencies when the original TNL oscillator differential equation contains “singular terms”.
iii. For level $k \geq 2$ calculations, the work required to determine the angular frequency and associated periodic solution may become algebraically intensive.
iv. The Extended Iterative method generally is easier to apply, for better result, in comparison with similar direct Iterative techniques.
v. In principle, Iterative methods may be generalized to higher-order differential equations.

We can get desirable solution or angular frequency from a truly nonlinear oscillator by the proper use of the term of the Fourier series. In each of the Iterative scheme, right choice of truncation is most important.

REFERENCES

1. Nayfeh A. H., *Perturbation Method*, Wiley, New York, 1973.

2. Nayfeh A. H., *Introduction to Perturbation Techniques*, Wiley, New York, 1981.

3. Rahman M. H., Haque B. M. I. and Akbar M. A., *Asymptotic Solution of Fourth Order Near Critically Damped Nonlinear Systems*, Journal of information and Mathematical Science, 1, 61-73, 2009.

4. Alam A., Rahman H., Haque B. M. I. and Akbar M. A., *Perturbation Technique for Analytical Solutions of Fourth Order Near Critically Damped Nonlinear Systems*, International Journal of Basic & Applied Sciences, 11(1), 2011.

5. Haque B. M. I., Alam A. and Rahman H., *Perturbation Solutions of Fourth Order Damped Oscillatory Nonlinear Systems*, International Journal of Basic & Applied Sciences IJBAS/IJENS., 11(3), 2011.

6. Rahman H., Haque B. M. I. and Akbar M. A., *An Analytical Solutions for Fourth Order Damped-Oscillatory Nonlinear Systems*, Research Journal of Mathematics and Statistics, 3, 12-19, 2011.

7. Belendez A., Pascual C., Gallego S., Ortuno M. and Neipp C., *Application of a modified He's homotopy Perturbation method to obtain higher-order approximations of a* $x^{1/3}$ *force nonlinear oscillator*, Physics Latter A, 371, N^0 5-6, 421-426, 2007.

8. Beléndez A., Pascual C., Fernandez E., Neipp C. and Beléndez T., *Higher-order approximate solutions to the relativistic and Duffing-harmonic oscillator by modified He's Homotopy method*, Physica Scripta, 77, 2008.

9. Belendez A., Pascual C., Belendez T. and Hernandez A., *Solution of an anti-symmetric quadratic nonlinear oscillator by a modified He's homotopy Perturbationmethod*", Nonlinear Anal. Real World Appl., 10, N^0 1,416-427, 2009.

10. Belendez A., Pascual C., Ortuno M., Belendez T. and Gallego S., *Application of a modified He's homotopy Perturbation method to obtain higher-order*

approximations to a nonlinear oscillator with discontinuities, Real World Applications, 10(2), 601-610, 2009.

11. Mickens R. E., *Nonlinear Oscillations*, Cambridge University Press, New York, 1961.

12. Mickens R. E., *Comments on the method of Harmonic Balance*, J. Sound Vib.94, 456-460, 1984.

13. Mickens R. E., *Periodic solutions of the relativistic harmonic oscillator*, Journal of Sound and Vibration, 212, 905-908, 1998.

14. Mickens R. E., *Mathematical and numerical study of Duffing-harmonic oscillator,* Journal of Sound and Vibration, 244(3), 563-567, 2001.

15. Hu H. and Tang J. H., *Solutions of Duffing-harmonic oscillator by the method of Harmonic Balance*, Journal of Sound and Vibration, 294(3), 637-639, 2006.

16. Hu H., *Solutions of a quadratic nonlinear oscillator by the method of Harmonic Balance*, Journal of Sound and Vibration, 293, 462-468, 2006.

17. Mickens R. E., *Harmonic Balance and Iterative calculations of periodic solutions to* $\ddot{y} + y^{-1} = 0$, J. Sound and Vib., 306, 968-972, 2007.

18. Alam M. S., Haque B. M. I. and Hossain M. B., *A new analytical technique to find periodic solutions of nonlinear systems*, Int. J. Nonlinear Mech., 24, 1035-1045, 2007.

19. Beléndez A. and Pascual C., *Harmonic Balance approach to the periodic solutions of the harmonic relativistic oscillator*, Physics Letters A., 371, 291- 299, 2007.

20. Beléndez A., Hernandez A., Beléndez T., Alvarez M. L., Gallego S., Ortuno M. and Neipp C., *Application of the Harmonic Balance method to a nonlinear oscillator typified by a mass attached to a stretched wire*, Journal of Sound and Vibration, Vol., 302(4), 1018-1029, 2007.

21. Beléndez A., Pascual C., Mendez D. I. and Neipp C., *Solution of the relativistic (an) harmonic oscillator using the Harmonic Balance method*, Journal of Sound and Vibration, 311, 3-5, 2008.

22. Belendez A., Gimeno E., Alvarez M. L. and Mendez D. I., *Nonlinear oscillator with discontinuity by generalized Harmonic Balanced method*, J. Computers and Math. with App., 58, 2117-2123, 2009.

23. He J. H., *Modified Lindstedt-Poincare methods for some non-linear oscillations. Part III: Double series expansion*, Int. J. Nonlinear Sci., Numer, Simul., 2, 317-320, 2001.

24. Krylov N. N. and Bogoliubov N. N., *Introduction to Nonlinear Mechanics*, Princeton University Press, New Jersey, 1947.

25. Bogoliubov N. N., Mitropolskii and Yu, *Asymptotic Methods in the Theory of NonlinearOscillations*, Gordan and Breach, New York, 1961.

26. Ozis T. and Yildrim A., Determination of the frequency-amplitude for a *Duffing-harmonic oscillator by Energy Balanced method*, Computer and Mathematics with Applications, Vol., 54(7-8), 1184-1187, 2007.

27. Bélendez A., Mendez D. I., Fernandez E., Marini S. and Pascual I., *An explicit approximate solution to the Duffing-harmonic oscillator by a Cubication method*, Physics Letters A., 373, 2805-2809, 2009.

28. Mickens R. E., *Iterative Procedure for determining approximate solutions to nonlinear oscillator equation*, J. Sound and Vib., 116, 185-188, 1987.

29. Lim C. W. and Wu B. S., *A modified procedure for certain non-linear oscillators*, J. Sound and Vib., 257, 202-206, 2002.

30. Mickens R. E., *A general procedure for calculating approximation to periodic solutions of truly nonlinear oscillators*, J. Sound and Vib., 287, 1045-1051, 2005.

31. Hu H., *Solutions of a quadratic nonlinear oscillator: Iterative procedure*, Journal of Sound and Vibration, 298, 1159-1165, 2006.

32. Hu H., *Solutions of Duffing-harmonic oscillator by an Iterative procedure*, J. Sound Vib., 298, 446-452, 2006.

33. Hu H. and Tang J. H., *A classical Iterative procedure valid for certain strongly nonlinear oscillator*, J. Sound Vib., 299, 397-402, 2006.

34. Chen Y. M. and Liu J. K., *A modified Mickens Iterative procedure for nonlinear oscillators*, J. Sound and Vib., 314, 465-473, 2008.

35. Mickens R. E., *Truly Nonlinear Oscillations*, World Scientific, Singapore, 2010.

36. Haque B. M. I., Alam M. S. and Majedur R. M., *Modified solutions of some oscillators by Iterative procedure*, Journal of the Egyptian Mathematical Society, 21, 68-73, 2013.

37. Haque B. M. I., *A New Approach of Iterative Method for solving Some Nonlinear Jerk Equations*, Global Journals Inc. (USA), 13, 11, 87-98, 2013.

38. Haque B. M. I., *A new approach of modified Mickens Iterative method for solving some nonlinear jerk equations,* British journal of Mathematics & Computer Science, 4(22), 3146-3162, 2014.

39. Haque B. M. I., Alam M. S., Rahman M. and Yeasmin I. A., *Iterative technique of periodic solutions to a class of non-linear conservative systems*, Int. J. Conceptions on Computation and Information technology, 2(1), 92- 97, 2014.

40. Haque B. M. I. and Hossain M. R., *An Analytic Solution of the Quadratic Nonlinear Oscillator by an Iterative Method*, BJMCS, 13(1), 3146-3162, 2016.

41. Haque B. M. I., Bostami M. B, Hossain M. M. A., Hossain M. R. and Rahman M. M., *Mickens Iterative Like Method for Approximate Solutions of the Inverse Cubic Truly Nonlinear Oscillator*, BJMCS.22823, 13(4), 1-7, 2016.

42. Haque B. M. I., Hossain M. M. A., Bostami M. B. and Hossain M. R., *Analytical Approximate Solutions to the Nonlinear Singular Oscillator: An Iterative Procedure*, BJMCS.23263, 14(3), 1-9, 2016.

43. Lakrad F. and Belhaq M., *Periodic solutions of strongly nonlinear oscillators by the Multiple Scales method*, Journal of Sound and Vibration, 258, 677-700, 2002.

44. Mitropolskii Y., *Problems on asymptotic methods of non-stationary oscillations*, Izdat, Nauka, Moscow, 1964.

45. Popov I. P., *A generalization of the Bogoliubov Asymptotic Methods in the Theory of Nonlinear Oscillations*, Dokl, Akad. Nauk. USSR, 3, 308-310, 1956.

46. Murty I. S. N., Deekshatulu B. L. and Krishna G., *On asymptotic method of Krylov-Bogoliubov for over-damped nonlinear systems*, Journal of Franklin Institute, 288, 49-64, 1969.

47. Murty I. S. N., *A unified Krylov-Bogoliubov method for solving second-order nonlinear systems*, International Journal of Nonlinear Mechanics, 6, 45-53, 1971.

48. Shamsul M. A. and Sattar M. A., *A unified Krylov-Bogoliubov-Mitropolskii method for solving third-order nonlinear systems,* India Journal of Pure Applied Mathematics, 28, 151-167, 1997.

49. Lim C. W., Lai S. K. and Wu B. S., *Accurate higher-order analytical approximate solutions to large-amplitude oscillating systems with a general non-rational restoring force,* Nonlinear dynamics, 42, 267-281, 2005.

50. Wu B. S., Sun W. P. and Lim C. W., *An analytical approximate technique for a class of strongly nonlinear oscillator,* Int. J. Nonlinear Mech., 41, 766- 774, 2006.

51. Lim C. W. and Lai S. K., *Accurate higher-order analytical approximate solutions to non-conservative nonlinear oscillators and application to van der pol damped oscillators*, International Journal of Mechanical Science, 48, 483-492, 2006.

52. Hu H., *A modified method of equivalent linearization that works even when the non-linearity is not small,* J. Sound and Vib., 276, 1145-1149, 2004.

53. Taylor A. E. and Mann W. R., *Advance Calculus,* Wiley, New York, 1983.