A STUDY ON SOME INTEGRAL TRANSFORMS USED IN ENGINEERING SCIENCES

by

Chhanda Rani Mondal

Roll No. 1451560

A thesis submitted in partial fulfillment of the requirements for the degree

of

Master of Science in Mathematics

Department of Mathematics

Khulna University of Engineering & Technology

Khulna-9203, Bangladesh

March 2017

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Chhanda Rani Mondal

Roll No. 1451560

DEPARTMENT OF MATHEMATICS

KHULNA UNIVERSITY OF ENGINEERING & TECHNOLOGY

Khulna, Bangladesh

Declaration

This is to certify that the thesis work entitled **"A Study on Some Integral Transforms Used in Engineering Sciences"** has been carried out by Chhanda Rani Mondal, Roll No.-1451560 in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

Signature of Supervisor

Signature of Student

APPROVAL

Dedication

To

My beloved parents

Bishnu Pada Mondal & Kanika Rani Mondal

ACKNOLEDGEMENT

All praise of Almighty God who has created us and given me a greatest status among all creations. Thanks God to give me strength, patience and ability to complete this study.

I would like to express my deep gratitude to my reverend supervisor Dr. Mohammad Arif Hossain, Professor, Department of mathematics, Khulna University of Engineering & Technology, for his invariable guidance, generous advice, constructive criticism, inspiration and encouragement conveyed during my research work and for the valuable efforts made for the preparation of this thesis. I also remember the patience of my supervisor during my working hour, gratefully.

On this occasion of a major work in my life, I like to pay due respect the memory of my beloved father and mother to whom I am indebted for everything of my life.

I am obliged to express my heartiest thanks to my two sisters for their cooperation, nice and understanding behavior during my research work.

Finally, I like to share my satisfaction of completing this task with my well-wishers, friends but the responsibility of errors and deficiencies that still remain devolves on me alone.

Abstract

Transform methods, have their great importance in the field of applied sciences especially in engineering sciences, are divided in two categories continuous and discrete, of which our matter of interest is the continuous one. To most of us, Laplace transform a member of the continuous transform category is well known and we are acquainted to solve differential equations with this important tool. The procedure of adopting a transform consists of three steps- use of the transform, algebraic manipulation and finally inversion. In case of Laplace transform the inversion with the definition is not usually an adopted one, but is a powerful tool. In this study we have rigorously handled the Laplace transform and complex inversion formula is utilized to obtain the inverse. Hankel transform, another member of the same category which has specialty in handling cylindrical coordinates with circular symmetry is also studied. It has been observed that when periodic, impulsive or similar forces are applied to a system, is is not that much difficult to obtain the solution of the system when complex inversion formula is used while Laplace transform method is used as a solving tool. Thus as a concluding remark, it has been suggested to pay proper attention to the complex inversion formula.

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Chapter I

Introduction

Transform means shift from one from to another. The methods which transform something from one form to some other form are termed as transform methods. Generally it is required or used to shift variables from one type to other type (e.g. $t \rightarrow s$). As variables or parameters have two different forms (e.g. continuous and discrete) so the transforms method will have also two types; one will handle continuous and the other will handle discrete variable /parameters. When the whole of the space is to be considered then for continuous variable one require integration and for discrete variable summation is used. We generally used the terminology “Integral Transform” for the purpose. Similarly for discrete variables “Discrete Transform” is used. For both the transforms some kernel is to be used. For integral transform integration is to be performed over the domain after multiplication by the kernel. In a similar fashion summation is taken over the domain after multiplication by the kernel. On the basis of the kernels the transformed are labeled. Sometimes the domain may be finite, in these cases they are labeled as finite transforms (e.g. Finite Fourier Transforms). Transform methods have their own merits in the field of applied sciences, especially in the field of engineering sciences. When a physical system is modeled sometimes differential equations (Ordinary or Partial) arises. For example when a simple circuit is modeled, a differential equation is raised in which inductance, capacitance, resistance and e. m. f. will be present. The differential equation can be solved by general mathematical tools for solving differential equations, but also can be easily solved by Laplace Transform method. Because, after introducing the Laplace transform to the differential equation, one will require some algebraic manipulation and finally the inverse transform will provide the required result. If the initial or boundary conditions were given, the arbitrariness present in the solution can be removed to get particular solutions. The Laplace transform is very much useful in solving ordinary differential equations with less effort. If partial differential equations are there (of two independent variables), Laplace transform reduces the form to ordinary differential equations which are less tedious than partial differential equations. From these discussion it is clear that Laplace transform is a useful tool especially to applied scientists and engineers. In a similar fashion it is observed that when z transform is applied to difference equation, one get a form which after algebraic manipulation and inverse transform provide the solution of the difference equation. Difference equation arises in case of discrete functions as differential equations arises in case of continuous functions. Thus it is observed that transform methods, both integral and discrete are essential tools to be familiarized to applied scientist and engineers. In the field of signal processing time and frequency are the matter of interest. So in the field of signal processing both integral transforms and discrete transforms are used. Many common integral transforms used in the field of signal processing have their discrete counter-part (e. g. Fourier and Wavelet Transforms have their discrete counter parts as Discrete Fourier Transform (DFT), Discrete Sine Transform (DST), Discrete Cosine Transform (DCT), Discrete Wavelet Transform (DWT), etc.).

With the advantage of fast and cheap digital Computers, there has been renewed emphasis o the analysis and design of digital system which represent a major class of engineering systems. However, it is a mistake to believe that the mathematical basis of this area of work is of such recent vintage. The first comprehensive text in English dealing with difference equations was the treatise of the calculus of Finite Differences due to George Boole and published in 1860. Much of early impetus for the finite calculus was due to the need to carry out interpolation and to approximate derivatives and integrals. Later, numerical methods for the solution of difference equations were devised, many of which were based on finite difference methods, involving the approximation of the derivative terms to produce a difference equation.

The classical methods of solution of initial and boundary value problems in physics and engineering sciences have their roots in Fourier's pioneering work. An alternative approach through integral transform methods emerged primarily through Heavisides efforts on operational techniques. In addition to being of great theoretical interest to mathematicians, integral transform methods have been found to provide easy and effective ways of solving a variety of problems arising in engineering and physical sciences. The use of an integral transform is somewhat analogous to that of logarithms. That is a problem involving multiplication or division can be reduced to one involving the simpler processes of addition or subtraction by taking logarithms. After the solution has been obtained in the logarithm domain, the original solution can be recovered by finding an antilogarithm, In the same way, a problem involving derivatives can be reduced to a simpler problem involving only multiplication by polynomials in the transform variable by taking an integral transform, solving the problem in the transform domain and then finding an inverse transform. Integral transforms arise in a natural way through the principal of linear superposition in constructing in integral representations of solutions of linear differential equations.

By an integral transform, we mean a relation of the form

$$I\{f(t)\} = \int_{-\infty}^{\infty} k(s,t)\ f(t)\ d\ \ = F(s)$$

Such that a given function $f(t)$ is transformed into another function F(s) by means of an integral. The new function F(s) is said to be the transform of $f(t)$ and k(s, t) is called the kernel of the transformation. Both k(s, t) and $f(t)$ must satisfy certain condition to ensure existence of the integral and a unique transform function F(s). also generally speaking, not more than one function $f(t)$ should yield the same transform F(s). When both of the limits of the integration in the defining integral are finite, we have what is called a finite transform.
Within the above guidelines there are a variety a kernels that may be used to define particular integral transforms for a wide class of functions $f(t)$. If the kernel is defined by

$$k(s,t) = \begin{Bmatrix} 0, & t < 0 \\ e^{-s} & t \geq 0 \end{Bmatrix}$$

the resulting transform

$$\int_{0}^{\infty} e^{-s}\ f(t) d\ \ = F(s)$$

is called the Laplace transform. When

$$k(s,t) = \frac{1}{\sqrt{2\pi}} e^{ist}$$

we generate the Fourier transform

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t) dt = F(s) \qquad (1.1)$$

which, when t is restricted to the positive real line, leads to the Fourier sine and Fourier cosine transforms

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin st \, dt = F(s)$$

and

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos st \, dt = F(s)$$

The Laplace and Fourier transforms are by far the most prominent in applications. Many other transforms have seen developed, but most have limited applicability. In addition to the Laplace and Fourier transforms the next most useful transforms are perhaps the Hankel transform of order ν

$$\int_0^{\infty} t J_v(s,t) f(t) dt = F(s)$$

where J_v is the Bessel function of the first kind and the Mellin transform

$$\int_0^{\infty} t^{s-1} f(t) dt = F(s)$$

The Hankel transform arises naturally in solving value problems formulated in cylindrical coordinates while the Mellin transforms useful in the solution of certain potential problems formulated in wedge shape regions.

The integral transform mentioned thus far are applicable to problems involving either semi-infinite or infinite domains. However in applying the method of integral transforms to problems formulated on finite domains it is necessary to introduce finite intervals on the transform integral. Transforms of this nature are called finite integral transforms.

A basic problem in the use of integral transform is to determine the function $f(t)$ when its transform $F(s)$ is known. We refer to this as the inverse problem. In many cases the solution of the inverse problem is another integral transform relation of the type

$$\int_D H(s,t) F(s) \, ds = f(t)$$

where $H(s,t)$ is another kernel and D is the domain of s..Such a result is called an inversion formula for the particular transform. For example, the inversion formula for the Fourier transform takes the form

$$\int_0^{\infty} e^{-ist} F(s) ds = f(t)$$

which is very much like the transform itself in Eq (1.1). This means that the problems of evaluating transforms or inverse transforms are essentially the same for Fourier transforms. This is not the necessarily the case for other transforms like the Laplace transform, however where the inversion formula is quite distinct from that of the transform integral. Also in the case of finite transforms, the inverse transform is in the form of an infinite series.

The basic aim of the transform method is to transform a given problem into one that is easier to solve. In the case of ordinary differential equation with constant coefficient, the transformed problems are algebraic. The effect of applying an integral transform to a partial differential equation is to reduce it to a partial differential equation in one less variable. The solution of the transformed problem in either case will be a function of the transformed variable and any remaining independent variables. Inversion of this solution produces the solution of the original problem.

The exponential Fourier transform does not incorporate any boundary conditions in transforming the derivatives. Thus, it is best suited for solving differential equations on infinite domains where the boundary conditions usually only require bounded solutions. On the other hand, the Fourier cosine and sine transforms are well suited for solving certain problems on semi infinite domains where the governing differential equations involves only even-order derivatives. It can be seen that the Fourier transform leads itself nicely to solving boundary-value problems associated with the following partial differential equations.

The heat equation: $\nabla^2 u = a^{-2} u_t - q(x, y, z, t)$

The wave equation: $\nabla^2 u = c^{-2} u_t - q(x, y, z, t)$

The potential equation: $\nabla^2 u = 0$

In addition it is useful in the solution of linear integral equations of the form

$$f(x) = u(x) - \lambda \int_0^{\infty} k(s, t) u(t) d$$

While the Fourier transform is suited for boundary-value problems, the Laplace transform is suited for initial value problems. However, there are other situations for which the Laplace transform can also be used, such as in the evaluation of certain integrals and in the solution of certain integral equations of convolution type like

$$\int_0^t u(\tau)\, k(t - \tau)\, d \quad = f(t), \quad t > 0$$

The main objective of this study was to familiarize the topics, the integral transform which consists of many of the type. Out of them in this study only two Laplace transform and Hankel transform has been considered. Laplace transform has its great application in solving ordinary differential equations especially in the field of electrical engineering. It has been seen though there are different method of obtaining inverse Laplace transform of them complex inversion formula is the most unused me. But it has great importance when the function is not in the common form. Hence it has been tried to utilize this method in solving differential equations arising in certain electronic circuits, this has been presented in chapter-v. In the same chapter an example has also been presented that utilized the Hankel transform in solving a problem of fluid dynamics having circular symmetry. As usual chapter-I contains the introduction, the knowledge regarding the topics those are required for this study are presented in Chapter-II. Chapter-III contains the Laplace transform, which includes the inverse Laplace transform also. In Chapter-IV Hankel transformation related information has been presented. As this is a study and as there is no new findings but some concluding remarks has been passed under the headline “Concluding remarks” the end of the writer as Chapter-VI. Finally references are listed.

Chapter II

Some Special Functions and Related Topics

In many engineering applications, functions other than trigonometric and hyperbolic functions are used. A few of them which will be handled in this thesis including related topics are discussed below:

2.1 THE GAMMA FUNCTION

Many important functions in applied sciences are defined via improper integrals. Among them, one of the simplest but very important special functions is the gamma function.
The gamma function is defined for all complex numbers except the non-positive real part, it is defined via a convergent integral:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

This integral is also known as the Euler integral of the second kind.

Basic Properties of $\Gamma(z)$

When n is a positive integer, then the gamma function is related to the factorial function:

$$\Gamma(n+1) = n! \qquad n = 0, 1, 2, \cdots$$

One of the most important formulas satisfied by the Gamma function is

$$\Gamma(x+1) = x\Gamma(x)$$

for any $x > 0$

Other important functional equations for the gamma function are Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}$$

2.2 BESSEL FUNCTIONS

Bessel functions are closely associated with problems possessing circular or cylindrical summitry, such as the study of free vibrations of a circular membrane and finding the temperature distribution in a circular cylinder.
Bessel functions of the first kind are defined by the series,

$$J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)} \tag{2.1}$$

where the parameter v denotes the order of the given Bessel functions.

When $\nu = n(n = 0, 1, 2, \cdots)$, Eq. (2.1) defines the Bessel function of integer order

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+n}}{k!(k+n)!}, n = 0, 1, 2, \cdots \tag{2.2}$$

The simplest representative of which is

$$J_0(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{(k!)^2} \tag{2.3}$$

The parameter v in (2.1) may also take on negative values, For example, when $v=-n(n=0,1,2,\cdots)$ we get

$$J_{-n}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k-n}}{k!(k-n)!}$$

$$=\sum_{k=n}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k-n}}{k!(k-n)!}$$

where we have used the fact that $1/(k-n)!=0(k=0,1,\cdots\cdots,n-1)$.

Finally, the change of index $k=m+n$ yields

$$J_{-n}(z)=\sum_{m=0}^{\infty}\frac{(-1)^{m+n}\left(\frac{z}{2}\right)^{2m+n}}{m!(m+n)!}$$

From which we deduce

$$J_{-n}(z)=(-1)^n J_n(z),\quad n=0,1,2,\cdots \tag{2.4}$$

However, this last relation applies only to integral-order Bessel functions.
Rewriting (2.1) in the form,

$$J_v(z)=\left(\frac{z}{2}\right)^v\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{k!\,\Gamma(k+v+1)}$$

It can be shown that the series m the right converges in the whole z-plane. Therefore, the function $\left(\frac{2}{z}\right)^v J_v(z)$ is an entire function (The function which has only singularities at infinity) of z. If $v<0$ and non-integral, then clearly $J_v(z)$ has an infinite discontinuity at $z=0$ and hence cannot represent an entire function. But if $v=\pm n, n=0,1,2,\cdots\cdots,$ then it can be shown that $J_v(x)$ is entire.

It has been found that $J_v(x)$ and $J_{-v}(x)$ both satisfy the following differential equation named after the great mathematician F.W. Bessel:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-v^2)y=0$$

Which is clearly a second order differential equation and as per theory, it should have to linearly independent solutions that are considered as $J_v(x)$ and $J_{-v}(x)$. As they are linearly independent so we may consider the general solutions as

$$y = c_1 J_v(x) + c_2 J_{-v}(x)$$

But when v is integer as we have seen $J_{-n}(x) = (-1)^n J_n(x)$.

So $J_n(x)$ and $J_{-n}(x)$ are not linearly independent. Hence it is required to obtain another solution which will be linearly independent to $J_n(x)$.

In literature, this solution is labeled as $Y_n(x)$ the Bessel function of second kind of order n with the mathematical definition as

$$Y_n(x) = \frac{2}{\circledR}\left(\yen + \log\frac{x}{2}\right) J_n(x) - \frac{1}{\circledR}\sum_{p=0}^{n-1}\frac{\Gamma(n-p)}{p!}\left(\frac{2}{x}\right)^{n-2p} - \sum_{p=0}^{\infty}(-1)^p$$

$$\frac{\left(\frac{1}{2}x\right)^{n+2p}}{p!\,\Gamma(n+p+1)}\left\{1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{p}+1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n+p}\right\}$$

where n is integer.

The Bessel function of second kind of order v (non-integer) is also defined as

$$Y_v(x) = \frac{J_v(x)\cos v\circledR - J_{-v}(x)}{\sin v\circledR}$$

Hence the general solution of the Bessel differential equation with integral values (n) of v is, $$Y = c_1 J_n(x) + c_2 Y_n(x)$$

The following recurrence relations for Bessel function of first kind are of importance:

(i) $x J_v' = v J_v - x J_{v+1} = x J_{v-1} - v J_v$

(ii) $\dfrac{d}{dx}\left(x^v J_v\right) = x^v J_{v-1}$

(iii) $\dfrac{d}{dx}\left(x^{-v} J_v\right) = -x^{-v} J_{v+1}$

From (ii), it can be written as

$$\int_0^a x J_0(\sim x)\,dx = \frac{a}{\sim} J_1(a\sim)$$

It way be note worthy that

$$J_{1/2}(x) = \sqrt{\frac{2}{\circledR x}}\sin x$$

and $$J_{-1/2}(x) = \sqrt{\frac{2}{\circledR x}}\cos x$$

Orthogonality of Bessel Functions

Since Bessel functions often appear in solutions of partial differential equation (PDE), it is necessary to be able to compute coefficients of series whose terms include Bessel functions. Therefore, we need to understand their orthogonality properties.

Consider the Bessel equation,

$$\ell^2 \frac{d^2 J_v(k\ell)}{d\ell^2} + \ell \frac{dJ_v(k\ell)}{d\ell} + \left(k^2\ell^2 - v^2\right) J_v(k\ell) = 0,$$

Where $v \geq -1$. Rearranging yields

$$-\left(\frac{d^2}{d\ell^2} + \frac{1}{\ell}\frac{d}{d\ell} - \frac{v^2}{\ell^2}\right) J_v(k\ell) = k^2 J_v(k\ell)$$

Thus $J_v(k\ell)$ is an eigen function of the linear differential operator,

$\sim\; = -\left(\frac{d^2}{d\ell^2} + \frac{1}{\ell}\frac{d}{d\ell} - \frac{v^2}{\ell^2}\right)$ with eigen value k^2.

The operator $\sim$ is not self-adjoint with respect to the standard scalar product, as the coefficients $p_0(\ell) = -1$ and $p_1(\ell) = -\frac{1}{\ell}$ do not satisfy the condition $p_1'(p) = p_0(\ell)$, so we use the weight function

$$w(\ell) = \frac{1}{p_0(\ell)} e^{\int \frac{p_1(\ell)}{p_0(\ell)} d\ell} e^{\int \frac{1}{\ell} d\ell} = e^{\ln \ell} = -\ell$$

It follows from the relation

$$(\text{a}_u - \text{a}_v)\int_a^b u*(x)u(x)w(x)\,dx = \left[w(x)\,p_0(x)u*(x)u'(x) - (v*)'(x)u(x)\right]_a^b$$

that $$\int_{a0}^{a} \ell J_v(k\ell) J_v(k'\ell)\,d\ell = \frac{a\left[k'J_v(ka)J_v'(k'a) - kJ_v(k'a)J_v'(ka)\right]}{k^2 - k'^2}$$

Therefore, in order to ensure orthogonality, we must have ka and $k'a$ be zeros of J_v. Thus we have the orthogonality relation

$$\int_0^a \ell J_v\left(\sim_{vi}\frac{\ell}{a}\right) J_v\left(\sim_{vj}\frac{\ell}{a}\right) d\ell = 0, i \neq j$$

where $\sim_{vj}$ is the j-th zero of J_v.

It is worth noting that because of the weight function ℓ being the Jacobian of the change of variable to polar coordinates, Bessel functions that are scaled as in the above orthogonality relation are also orthogonal with respect to the un-weighted scalar product over a circle of radius a.

2.3 Error Function:

In mathematics, the error function is a special function of sigmoid shape that occurs in probability, statistics and partial differential equations describing diffuse. It is defined as,

$$erf(x) = \frac{1}{\sqrt{\text{®}}}\int_{-x}^{x} e^{-t^2 dt}$$

$$= \frac{2}{\sqrt{\text{®}}}\int_0^x e^{-t^2}\,dt$$

In statistics for non-negative values of x, the error function has the following interpretation: for a random variable x that is normally distributed with mean o and variance $\frac{1}{2}$, $erf\,(x)$ describes the probability of x falling in the range $[-x, x]$

Complementary error function:

The complementary error function, denoted $erfc$, is defined as

$$erfc(x) = 1 - erf\,(x)$$

$$= \frac{2}{\sqrt{®}} \int_x^{\infty} e^{-t^2} dt$$

Which also define $erfc(x)$, the scaled complement error function.

2.4 The Unit Step function (Heaviside Function):

In engineering applications, we frequently encounter functions whose values change abruptly at specified values of time t. One common example is when a voltage is switched on or off in an electrical circuit at a specified value of time t.

The value of $t = 0$ is usually taken as a convenient time to switch on or off to supply the given voltage. The switching process can be described mathematically by the function called the unit step function (otherwise known as the Heaviside function after Oliver Heaviside.)

Definition: The unit step function, $u(t)$ is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

That is, u is a function of time t and u has value zero when time is negative (before we flip the switch) and value one when time is positive (form when we flip the switch)

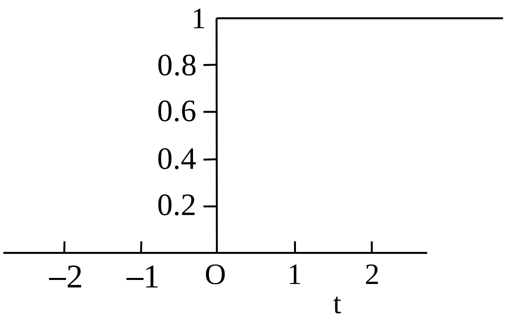

Fig: 2.1

2.5 The Unit Impulse function or Dirac delta function:

Unit impulse function or Dirac delta function is one of the special functions which is widely used in the field of signal processing. It has nice properties that helps in some situations specially its sifting property. But this depends on the fact of its integral is equal to one.

Consider the function,

$$F_t(t) = \begin{cases} \dfrac{1}{t} & 0 \le t \le t \\ 0 & t > t \end{cases}$$

Where $t > 0$, whose graph appers in Fig 1-4

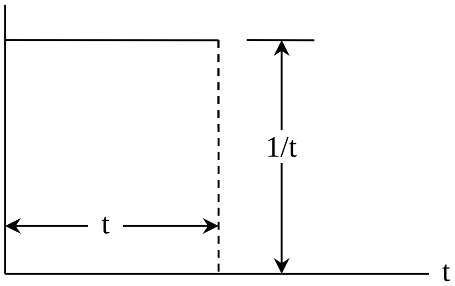

Fig: 2.2

It is geometrically evident that as $t \to 0$ the height of the rectangular shaded region increases indefinitely and the width decreases in such a way that the area is always equal to 1, i.e. $\int_0^\infty F_t(t)\,dt = 1.$

This idea has led some engineers and physicists to think of a limiting function, denoted by $¢\,(t)$, approached by $F_t(t)$ as $t \to 0$. This limiting function they have called the unit impulse function or Dirac delta function. Some of its properties are

1. $\int_0^\infty ¢\,(t)\,dt = 1$
2. $\int_0^\infty ¢\,(t)G(t)\,dt = G(0)$ for any continuous function $F(t)$.
3. $\int_0^\infty ¢\,(t-a)G(t)\,dt = G(a)$ for any continuous function $G(t)$.

To find the inverse Laplace transform of a function out of all the methods partial function method is used mostly. To split a function into partial fractions it is not mandatory to check whether the fraction is proper or improper. But in the cases of inverse Laplace transform the fraction is always proper. As the matters of interests are the engineering applications where most of the equations are of second order so in the following the only tennis will be discussed those may arise in engineering applications.

2.6 Concepts of Some Partial Fractions: The following forms will be handled

(i) $\dfrac{1}{(s-a)(s-b)(s-c)}$

(ii) $\dfrac{1}{(s-a)^2(s-b)}$

(iii) $\dfrac{1}{\{(s-a)^2+b^2\}(s-c)}$

(iv) $\dfrac{(s-a)^2+k}{\{(s-a)^2+b^2\}\{(s-a)^2+c^2\}}$

The cases will be discussed with the help of some examples.

Example-1: Let $\dfrac{1}{(s-a)(s-b)(s-c)}=\dfrac{A}{s-a}+\dfrac{B}{s-b}+\dfrac{c}{s-c}$

then
$$A=\left[(s-a)\frac{1}{(s-a)(s-b)(s-c)}\right]_{s=a}$$
$$B=\left[(s-b)\frac{1}{(s-a)(s-b)(s-c)}\right]_{s=b}$$
$$C=\left[(s-c)\frac{1}{(s-a)(s-b)(s-c)}\right]_{s=c}$$

Example-2: Let $\dfrac{1}{(s-a)^2(s-b)}=\dfrac{A_1}{(s-a)}+\dfrac{A_2}{(s-a)^2}+\dfrac{B}{(s-b)}$ (1)

then
$$A_2=\left[(s-a)^2\frac{1}{(s-a)^2(s-b)}\right]_{s=a}$$
$$B=\left[(s-b)\frac{1}{(s-a)^2(s-b)}\right]_{s=b}$$

To get A_1 multiply Eq(1) by s and then consider $s\to\infty$.

Example-3: Let $\dfrac{1}{\{(s-a)^2+b^2\}(s-c)}=\dfrac{A}{s-c}+\dfrac{Bs+C}{(s-a)^2+b^2}$ (2)

then
$$A=\left[(s-c)\frac{1}{\{(s-a)^2+b^2\}(s-c)}\right]_{s=c}$$

C can be evaluated putting the value of $s=k$ in Eq(2).
To get B, multiply Eq(2) by s and then consider $s\to\infty$.

Example-4: Let $\dfrac{(s-a)^2+k}{\{(s-a)^2+b^2\}\{(s-a)^2+c^2\}}=\dfrac{A\ +B}{\{(s-a)^2+b^2\}}+\dfrac{C\ +D}{\{(s-a)^2+c^2\}}$

Cross multiplication and then coefficient equating provides
$$A+C=0,\quad B+D=1,\quad Bc^2+Db^2=k,\quad Ac^2-Ab^2=0$$
Thus we get $A=0$, $B=\dfrac{b^2-k}{b^2-c^2}$, $C=0$, $D=\dfrac{k-c^2}{b^2-c^2}$

To find the inverse Laplace transform complex inverse formula will be used which requires some knowledge on complex variable. The subject matters that will be required for the understanding of the complex inversion formula are discussed below:

2.7 Analytic Functions of a Complex variable

Definition 1: A function $f(z)$ is said to be analytic in a region R of the complex plane if $f(z)$ has a derivative at each point of R and if $f(z)$ is single valued.

Definition 2: A function $f(z)$ is said to be analytic at a point z if z is an interior point of some region where $f(z)$ is analytic.

Hence the concept of analytic function at a point implies that the function is analytic in some circle with center at this point.

From the definition, it is asserted that if $f(z)$ is analytic at a point z, then $f(z)$ has continuous derivatives of all order at the point z.

Different Types of Singularity:

A point at which an analytic function $f(z)$ is failed to be analytic i.e. at which $f'(z)$ fails to exist, is called a singular point or singularity of the function.

There are different types of singular points:

1. **Poles:** If $f(z)$ has an isolated singular point at $z=a$, i.e. $f(z)$ is not finite at $z=a$ and if in addition there exists an integer n such that the product

$$(z-a)^n f(z)$$

is analytic at $z=a$, then $f(z)$ has a pole of order n at $z=a$ if n is the smallest such integer.

Example: $f(z)=\frac{1}{z^2}$ has a pole of order 2 at $z=0$.

2. **Removable singular point:** An isolated singular point z_0 such that f can be defined or redefined at z_0 in such a way as to be analytic at z_0. A singular point z_0 is removable if

$$\lim_{z\to z_0} f(z) \text{ exists.}$$

3. **Essential singular point:** A singular point that is not a pole or removable singularity is called an essential singular point.

Example: $f(z)=\sin\left(\frac{1}{z}\right)$ has an essential singularity at $z=0$.

4. **Branch point:** When $f(z)$ is a multiple valued function, any point which cannot be an interior point of the region of definition of a single-valued branch of $f(z)$ is a singular branch point.

Example: $f(z)=\sqrt{z-a}$ has a branch point at $z=a$.

5. **Singular points at infinity:** The type of singularity of $f(z)$ at $z=\infty$ is the same as that of $f\left(\frac{1}{w}\right)$ at $w=0$.

On the basis of the location of the singular points another classification is also sometimes used which describes whether the singular points are too close or not.

6. **Isolated singularity:** Let $z = z_0$ is a singular point such that there exists a circle of radius ε centered at z_0 such that there is no other singular point. Then the singular point is termed as isolated singular point.

2.7 **Residue:** Let f be analytic everywhere within and on a closed curve c except possibly at a point z_0 in the interior of c where f may have an isolated singularity.

Then the residue of f at z_0 is defined as

$$\operatorname{Re}s_{z0} f = \frac{1}{2\circledR i}\oint_c f(z)\,dz$$

An alternate definition given below is also used.

The constant a_{-1} in the Laurent series $\quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$

of $f(z)$ about a point z_0 is called the residue of $f(z)$.

If z_0 is a non-singular point, then $\quad \operatorname{Re}s_{z0} f = 0$

otherwise, $\operatorname{Re}s_{z0} f$ may be $\neq 0$ The alternate definition presented above suggests that if there is a multiple pole of order m at $z = z_0$ then the residue at that point will be

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z - z_0)^m f(z)\right]_{z=z_0}$$

It is important to note here that in case of essential singularity the residue is calculated via the alternate definition i.e. the residue is given by the coefficient of z^{-1} in the Laurent expansion.

Residue Theorem: Let c be closed path within and on which f is analytic except for m isolated singularities. Then

$$\oint_c f(z)\,dz = 2\circledR i \sum_{j=1}^{m} \operatorname{Re}s_{z_j} f$$

Example 1: Let us compute the residue at the singularity of the function $f(z) = \dfrac{1-z}{(1-2z)^2}$

Evidently the function has pole at $z = \dfrac{1}{2}$ of order 2.

$$\frac{1-z}{(1-2z)^2} = \frac{1}{8}\frac{1}{\left(z - \frac{1}{2}\right)^2} - \underbrace{\frac{1}{4}}_{\text{residue}}\frac{1}{z - \frac{1}{2}}$$

$$\Rightarrow \operatorname{Re}s_{z=\frac{1}{2}} f = -\frac{1}{4}$$

Example 2: Let us compute the residue at the singularity of the function $f(z) = e^{\frac{1}{z^2}}$

Clearly the function has singularity at $z = 0$ which is an essential singularity

Now $\quad e^{\frac{1}{z^2}} = 1 + \dfrac{1}{z^2} + \dfrac{1}{2!}\dfrac{1}{z^4} + \cdots\cdots$

$\Rightarrow \operatorname{Re}s_{z=0} f = 0$ (as the coefficient of z^{-1} is zero here).

Chapter III

The Laplace Transform

3.1 THE LAPLACE TRANSFORM: The Laplace transform in an integral transform perhaps second only to the Fourier transform in its utility in solving physical problems. The Laplace transform is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

Let $F(t)$ be a function of t specified for $t > 0$. Then the Laplace transform of $F(t)$ denoted by $\mathrm{L}\{F(t)\}$, is defined by

$$\mathrm{L}\{F(t)\} = f(s) = \int_0^\infty \bar{e}^{st} F(t)\,dt$$

where we assume at present that the parameter s is real. Later it will be found useful to consider s complex.

A table of Laplace transform of some important but common functions are given below:

List of Laplace transforms of some common function

$F(t)$	$\mathrm{L}\{F(t)\} = f(s)$	Conditions
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$n \in Z \geq 0$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$R[a] > -1$
e^{at}	$\frac{1}{s-a}$	$S > a$
$\cos(wt)$	$\frac{s}{s^2+w^2}$	$w \in R$
$\sin(wt)$	$\frac{w}{s^2+w^2}$	$s > \lvert I[w] \rvert$
$\cosh(wt)$	$\frac{s}{s^2-w^2}$	$s > \lvert R[w] \rvert$
$\sin\mathrm{h}(wt)$	$\frac{w}{s^2-w^2}$	$s > \lvert I[w] \rvert$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a + \lvert I[b] \rvert$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$b \in \sqcup$

3.2 PROPERITES OF LAPLACE TRANSFORM:

The Laplace transform has many important properties. A few of them are given below:

(i) Linearity : If C_1 and C_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transforms $f_1(s)$ and $f_2(s)$ respectively, then

$$\mathrm{L}\{C_1F_1(t)+C_2F_2(t)\}=C_1\mathrm{L}\{F_1(t)\}+C_2\mathrm{L}\{F_2(t)\}=C_1f_1(s)+c_2f_2(s)$$

(ii) First translation or shifting : It $\mathrm{L}\{F(t)\}=f(s)$ then

$$\mathrm{L}\{e^{at}F(t)\}=f(s-a)$$

(iii) Second translation or shifting : If $\mathrm{L}\{F(t)\}=f(s)$ and $G(t)=\begin{cases}F(t-a) & t>a\\ 0 & t<a\end{cases}$,

Then $$\mathrm{L}\{G(t)\}=\bar{e}^{as}f(s)$$

(iv) Change of scale :It $\sim\{F(t)\}=f(s)$ then

$$\mathrm{L}\{F(at)\}=\frac{1}{a}f\left(\frac{s}{a}\right)$$

(v) Laplace transform of derivatives : It $\mathrm{L}\{F(t)\}=f(s)$ then

$$\mathrm{L}\{F'(t)\}=sf(s)-F(0)$$

(vi) Laplace transform of integrals: If $\mathrm{L}\{F(t)\}=f(s)$ then

$$\mathrm{L}\left\{\int_0^t F(u)\,du\right\}=\frac{f(s)}{s}$$

(vii) Multiplication by t^n : If $\mathrm{L}\{F(t)\}=f(s)$ then

$$\mathrm{L}\{t^nF(t)\}=(-1)^n\frac{d^n}{ds^n}f(s)=(-1)^n f^{(n)}(s)$$

(viii) Division by t : If $\mathrm{L}\{F(t)\}=f(s)$ then

$$\mathrm{L}\left\{\frac{F(t)}{t}\right\}=\int_0^\infty f(u)\,du$$

(ix) Behavior of $f(s)$ as $s\to\infty$: It $\mathrm{L}\{F(t)\}=f(s)$ then

$$\lim_{s\to\infty}f(s)=0$$

(x) Initial value theorem : It the indicated limits exist, then

$$\lim_{t\to 0}F(t)=\lim_{s\to\infty}sf(s)$$

(xi) Final value theorem : It the indicated limits exist, then

$$\lim_{t\to\infty}F(t)=\lim_{s\to 0}sf(s)$$

3.3 LAPLACE TRANSFORMS OF SOME SPECIAL FUNCTIONS:

In the following table we have listed Laplace transforms of various special functions.

Table of Laplace transforms of special function :

	$F(t)$	$f(s) = \mathrm{L}\{F(t)\}$
1.	$J_0(at)$	$\dfrac{1}{\sqrt{s^2+a^2}}$
2.	$J_n(at)$	$\dfrac{\left(\sqrt{s^2+a^2-s}\right)^n}{a^n\sqrt{s^2+a^2}}$
3.	$erf(t)$	$\dfrac{e^{\frac{s^2}{4}}}{s} erfc\left(\dfrac{s}{2}\right)$
4.	$erf\left(\sqrt{t}\right)$	$\dfrac{1}{s\sqrt{s+1}}$
5.	$si(t)$	$\dfrac{1}{s}\tan^{-1}\dfrac{1}{s}$
6.	$ci(t)$	$\dfrac{\ln\left(s^2+1\right)}{2s}$
7.	$Ei(t)$	$\dfrac{\ln(s+1)}{s}$
8.	$u(t-a)$	$\dfrac{\overline{e}^{as}}{s}$
9.	¢ (t)	1
10.	¢ $(t-a)$	$\overline{e}^{as}$

3.4 LAPLACE TRANSFORM OF PERIODIC FUNCTION: Let $F(t)$ is a periodic function having period $T > 0$ so that $F(t+T) = F(t)$, then

$$\mathrm{L}\{F(t)\} = \frac{\int_0^T \overline{e}^{st} F(t)\,dt}{1-\overline{e}^{st}}$$

3.5 THE INVERSE LAPLACE TRANSFORM:

Definition: It the Laplace transform of a function $F(t)$ is $f(s)$, i.e. if $\mathrm{L}\{F(t)\} = f(s)$, then $F(t)$ is called an inverse Laplace transform of $F(s)$ and we write symbolically

$$F(t) = \mathrm{L}^{-1}\{f(s)\}$$

where L^{-1} is called the inverse Laplace transformation operator.

Example: Since $\mathrm{L}\{e^{-3t}\} = \dfrac{1}{s+3}$

We can write, $\mathrm{L}^{-1}\left\{\dfrac{1}{s+3}\right\} = e^{-3t}$

The idea discussed above is not the mathematical definition of Inverse Laplace transform, which will be provided in the methods of finding inverse Laplace transform by Complex inversion formula.

Table of Inverse Laplace Transforms

	$F(s)$	$\mathrm{L}^{-1}\{f(s)\} = F(t)$
1.	$\dfrac{1}{s^{n+1}} \quad n = 0,1,2$	$\dfrac{t^n}{n!}$
2.	$\dfrac{1}{s-a}$	e^{at}

3.6 METHODS TO FIND INVERSE LAPLACE TRANSFORMS:

Various means are available to determine inverse Laplace transforms as indicated in the following list.

3.6.1. Method of Partial fractions: The main idea of this method is splitting the function in several forms of which the inverse transform can be easily sought from the table.

Any rational function $\dfrac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials, with the degree of $P(s)$ less than that of $Q(s)$ can be written as the sum of rational functions [called partial fractions] having the form,

$$\frac{A}{(as+b)^r}, \frac{As+B}{(as^2+bs+c)^2} \text{ where } r = 1,2,3,\cdots$$

By finding the inverse Laplace transform of each of the partial fractions, we can find $\mathrm{L}^{-1}\left\{\dfrac{P(s)}{Q(s)}\right\}$

Example-1: To find the inverse transform of the following function $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}$

According to the sprite of this method, we should split the function in the following form $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3} = \dfrac{A}{s+1} + \dfrac{B}{(s-2)^3} + \dfrac{C}{(s-2)^2} + \dfrac{D}{s-2}$

Using the rules discussed in the previous chapter, the values of A,B,C,D can be determined and to use the table/the known forms.

We can write, $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}=\dfrac{-\frac{1}{3}}{s+1}+\dfrac{-7}{(s-2)^3}+\dfrac{4}{(s-2)^2}+\dfrac{\frac{1}{3}}{s-2}$

Hence the required inverse transform is

$$-\frac{1}{3}\overline{e}^{t}-\frac{7}{2}t^2e^{2t}+4te^{2t}+\frac{1}{3}e^{2t}$$

Example-2: It we want to find the inverse Laplace transform of $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ we should have to split the function in the following form

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}=\frac{As+B}{s^2+2s+2}+\frac{Cs+D}{s^2+2s+5}$$

The values of A,B,C,D can be calculated through the ideas presented in previous chapter.

Hence we will have $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}=\dfrac{\frac{1}{3}}{(s+1)^2+1}+\dfrac{\frac{2}{3}}{(s+1)^2+4}$

which can be readily used to find the inverse Laplace transform and we will get,

$$\mathrm{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}=\frac{1}{3}e^{-t}(\sin t+\sin 2t)$$

3.6.2. Series Method: The main idea of this method is to expand the function, of which the inverse Laplace transform is to be find out is expanded in a Maclauren series or a Tylor series with the negative powers. The linearity property provides the term by term inverse transforms which may be accumulated to produce a known function.

It $f(s)$ has a series expansion in inverse power of s given by

$$f(s)=\frac{a_0}{s}+\frac{a_1}{s^2}+\frac{a_2}{s^3}+\frac{a_3}{s^4}+\cdots$$

Then under suitable conditions we can invert term by term to obtain

$$F(t)=a_0+a_1t+\frac{a_2t^2}{2!}+\frac{a_3t^3}{3!}+\cdots\cdots$$

Example: Let us find $\mathrm{L}^{-1}\left\{\dfrac{\overline{e}^{\frac{1}{s}}}{s}\right\}$

Using infinite series, we find $\dfrac{1}{s}e^{-\frac{1}{s}}=\dfrac{1}{s}\left\{1-\dfrac{1}{s}+\dfrac{1}{2!s^2}-\dfrac{1}{3!s^3}+\cdots\cdots\right\}$

$$=\frac{1}{s}-\frac{1}{s^2}+\frac{1}{2!s^3}-\frac{1}{3!s^4}+\cdots\cdots$$

Inverting term by term, $\mathrm{L}^{-1}\left\{\dfrac{1}{s}e^{-\frac{1}{s}}\right\}=1-t+\dfrac{t^2}{(2!)^2}-\dfrac{t^3}{(3!)^3}+\cdots\cdots$

$$= 1 - \frac{\left(2t^{\frac{1}{2}}\right)^2}{2^2} + \frac{\left(2t^{\frac{1}{2}}\right)^4}{2^2 \cdot 4^2} - \frac{\left(2t^{\frac{1}{2}}\right)^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\cdots$$

$$= J_0\left(2\sqrt{t}\right)$$

3.6.3. Methods of differential equations: In this method, firstly we try to obtain the differential equation of which the given function, of which inverse transform is to be find out, is a solution. Using the idea of multiplication effect and other related properties, can ring the inverse transform of the given function as dependent variable, new deferential equation is generated whose solution provides the required inverse Laplace transform to calculate the value of the arbitrary constant present in the solution the initial and final value theorems are required to be used.

Example: Let us find $\mathrm{L}^{-1}\left\{e^{-\sqrt{s}}\right\}$

Let $y = e^{-\sqrt{s}}$ .The differential equation of which y is a solution is

$$4sy'' + 2y' - y = 0 \tag{1}$$

Let $\qquad \mathrm{L}^{-1}\{y\} = Y$

$\therefore \qquad \mathrm{L}\{Y\} = y$

$\therefore \qquad \mathrm{L}\left\{t^2 Y\right\} = y''$

So that, $\qquad sy'' = \mathrm{L}\left\{\frac{d}{dt}\left[t^2 Y\right]\right\} = \mathrm{L}\left\{t^2 Y' + 2tY\right\}$ and also $y' = \mathrm{L}\{-tY\}$

Thus (1) can be written as $\qquad 4\mathrm{L}\left\{t^2 Y' + 2tY\right\} - 2\mathrm{L}\{tY\} - \mathrm{L}\{Y\} = 0$

or, $4t^2 Y^2 + (6t - 1)Y = 0$

Whose solution is, $\qquad Y = \frac{c}{t^{\frac{3}{2}}} e^{-\frac{1}{4t}}$

Now $\quad tY = \frac{c}{t^{\frac{1}{2}}} e^{-\frac{1}{4t}}$ thus $\qquad \mathrm{L}\{tY\} = -\frac{d}{ds}\mathrm{L}\{Y\} = -\frac{d}{ds}\left(e^{-\sqrt{s}}\right) = \frac{e^{-\sqrt{s}}}{2\sqrt{s}}$

For large $t, tY \sim \frac{c}{t^{\frac{1}{2}}}$ and $\mathrm{L}\{tY\} \sim \frac{c\sqrt{\text{®}}}{s^{\frac{1}{2}}}$

For small $s, \frac{\overline{e}^{\sqrt{s}}}{2\sqrt{s}} \sim \frac{1}{2s^{\frac{1}{2}}}$

Hence by the final value theorem, $\quad c\sqrt{\text{®}} = \frac{1}{2} \qquad$ or, $c = \frac{1}{2}\sqrt{\text{®}}$

It follows that $\qquad \mathrm{L}^{-1}\left\{\overline{e}^{\sqrt{s}}\right\} = \frac{1}{2\sqrt{\text{®}}\; t^{\frac{3}{2}}} e^{-\frac{1}{4t}}$

3.6.4. Differentiation with respect to a parameter: The basic idea of this method comes from the differentiation under sine of integration and interchangeability of operators. The multiplication effect properly of Laplace transform plays an important role in this method.

Example: Let us find $\mathrm{L}^{-1}\left\{\dfrac{s}{\left(s^2+a^2\right)^2}\right\}$

Let us differentiate a known function with respect to the parameter a, we find

$$\frac{d}{da}\left(\frac{s}{s^2+a^2}\right)=\frac{-2as}{\left(s^2+a^2\right)^2}$$

Hence $\mathrm{L}^{-1}\left\{\dfrac{d}{da}\left(\dfrac{s}{s^2+a^2}\right)\right\}=\mathrm{L}^{-1}\left\{\dfrac{-2as}{\left(s^2+a^2\right)^2}\right\}$

or $$\frac{d}{da}\left\{\mathrm{L}^{-1}\left(\frac{s}{s^2+a^2}\right)\right\}=-2a\,\mathrm{L}^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}$$

i.e. $$\mathrm{L}^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{1}{2a}\frac{d}{da}\left(\cos at\right)=-\frac{1}{2a}\left(-t\sin at\right)=\frac{t\sin at}{2a}$$

so $$\mathrm{L}^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{t\sin at}{2a}$$

3.6.5. The Complex Inversion formula: If $f(s)=\mathrm{L}\left\{F(t)\right\}$, then $\mathrm{L}^{-1}\left\{f(s)\right\}$ is given by

$$F(t)=\frac{1}{2®i}\int_{¥-i\infty}^{¥+i\infty}e^{st}f(s)\,ds,\ t>0 \qquad (3.1)$$

and $F(t)=0$ for $t<0$.

This result is called the complex inversion integral or formula. It is also known as Bromwich's integral formula. The result provides a direct means for obtaining the inverse Laplace transform of a given function $f(s)$.

The integration in (3.1) is to be performed along a line $s=¥$ in the complex plane where $s=x+iy$. The real number ¥ is chosen so that $s=¥$ Lies to the right of all the singularities (poles, branch points or essential singularities) but is otherwise arbitrary.

3.6.5.1 The Bromwice Contour: In practice, the integral in (1) is evaluated by considering the contour integral $\dfrac{1}{2®i}\oint_c e^{st}f(s)\,ds$ (3.2)

where C is the contour of Fig.3-1. This contour, sometimes called the Bromwich contour, is composed of line AB and the are BJKLA of a circle of radius R with center at the origin O

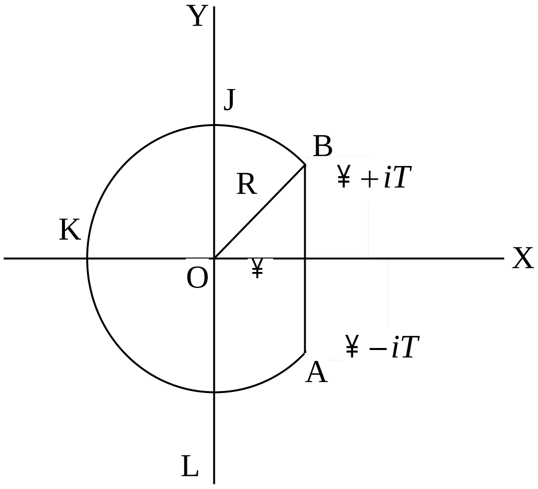

Figure- 3.1

If we represent are BJKLA by Γ, it follows from (3.1) that since

$T = \sqrt{R^2 - ¥^2}$

$$F(t) = \lim_{R \to \infty} \frac{1}{2®i} \int_{¥-iT}^{¥+iT} e^{st} f(s)\,ds \tag{3.3}$$

$$= \lim_{R \to \infty} \left\{ \frac{1}{2®i} \oint_C e^{st} f(s)\,ds - \frac{1}{2®i} \int_\Gamma e^{st} f(s)\,ds \right\}$$

3.6.5.2 Use of Residue theorem in finding Inverse Laplace Transform: Suppose that the only singularities of $f(s)$ are poles all of which lie to the left of the line $s = ¥$ for some real constant $¥$.

Suppose further that the integral around Γ in (3.3) approaches zero as $R \to \infty$, Then by the residue theorem we can write (3.3) as

$$F(t) = \text{ sum of residues of } e^{st} f(s) \text{ at poles of } f(s)$$

$$= \sum \text{residues of } e^{st} f(s) \text{ at poles of } f(s)$$

In earlier chapter we have used the complex inversion formula to find the inverse L.T, so we are not giving additional example here.

3.7 THE CONVOLUTION THEOREM:

3.7.1 Introduction: In this sub section we introduce the convolution of two functions $f(t), g(t)$ which we denote by $(f * g)(t)$. The convolution is an important construct, because the convolution theorem which gives the inverse Laplace transform of a product of two transformed functions:

$$\mathrm{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

3.7.2 The Convolution : Let $f(t)$ and $g(t)$ be two functions of t. The convolution of $f(t)$ and $g(t)$ is also a function of t, denoted by $(f*g)(t)$ and is defined by the relation $(f*g)(t)=\int_{-\infty}^{\infty} f(t-x)g(x)dx$

However if f and g are both causal functions then (strictly) $f(t),g(t)$ are written $f(t)u(t)$ and $g(t)u(t)$ respectively,

so that, $(f*g)(t)=\int_{-\infty}^{\infty} f(t-x)u(t-x)g(x)u(x)dx=\int_0^t f(t-x)g(x)dx$ because of the properties of the step functions $(u(t-x)=0$ if $x>t$ and $u(x)=0$ if $x<0)$

Hence, It $f(t)$ and $g(t)$ are causal functions then their convolution is defined by:

$$(f*g)(t)=\int_0^t f(t-x)g(x)dx$$

This is an odd looking definition but it turns out to have considerable use in Laplace transform theory.

3.7.3 The Convolution Theorem : Let $f(t)$ and $g(t)$ be causal functions with Laplace transforms $F(s)$ and $G(s)$ respectively, i.e.

$$\mathrm{L}\{f(t)\}=F(s) \quad \text{and} \quad \mathrm{L}\{g(t)\}=G(s)$$

Then it can be shown that $\mathrm{L}^{-1}\{F(s)G(s)\}=(f*g)(t)$

or equivalently $\mathrm{L}\{(f*g)(t)\}=F(s)G(s)$

Example: In the following convolution theorem is used to find the inverse Laplace transform of $\dfrac{6}{s(s^2+9)}$.

In this case we can, of course find the inverse transform by using partial fractions and then using the table of transforms. That is $\dfrac{6}{s(s^2+9)}=\dfrac{\left(\frac{2}{3}\right)}{s}-\dfrac{\left(\frac{2}{3}\right)s}{s^2+9}$

and so $$\mathrm{L}^{-1}\left\{\frac{6}{s(s^2+9)}\right\}=\frac{2}{3}\mathrm{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{2}{3}\mathrm{L}^{-1}\left\{\frac{s}{s^2+9}\right\}=\frac{2}{3}u(t)-\frac{2}{3}\cos 3t\cdot u(t)$$

However, we can alternatively use the Convolution Theorem. Let us choose

$$F(s)=\frac{2}{s} \qquad \text{and} \qquad G(s)=\frac{3}{s^2+9}$$

Then $f(t)=\mathrm{L}^{-1}\{F(s)\}=2u(t)$ and $g(t)=\mathrm{L}^{-1}\{G(s)\}=\sin 3t\cdot u(t)$

$$\mathrm{L}^{-1}\{F(s)G(s)\}=(f*g)(t)=\int_0^t 2u(t-x)\sin 3x\cdot u(x)dx$$

Now the variable t can take any value from $-\infty$ to $+\infty$. If $t<0$ then the variable of integration, x is negative and so $u(x)=0.$ We conclude that

$$(f*g)(t)=0 \text{ if } t<0$$

That is $(f*g)(t)$ is a causal function.

Let us now consider the other possibility for $t,$ that is the range $t \geq 0$.

Now in the range of integration $0 \leq x \leq t$ and so $u(t-x)=1, u(x)=1$

Since both $t-x$ and x are non-negative.

Therefore, $$\mathrm{L}^{-1}\{F(s)G(s)\}=\int_0^t 2\sin 3x dx$$

$$=\left[-\frac{2}{3}\cos 3x\right]_0^t=-\frac{2}{3}(\cos 3t-1) t \geq 0$$

$$\mathrm{L}^{-1}\left\{\frac{6}{s\left(s^2+9\right)}\right\}=-\frac{2}{3}(\cos 3t-1)u(t)$$

which agrees with the values obtained above using the partial fraction approach. In that we generally consider $\mathrm{L}^{-1}\left\{\frac{6}{\partial\left(s^2+9\right)}\right\}=(1-\cos 3t)$

Chapter IV

The Hankel Transform

Hankel transforms are integral transformations whose kernels are Bessel functions. They are sometimes referred to as Bessel transforms. When we are dealing with problems that show circular symmetry. Hankel transforms may be useful. Laplace's partial differential equation in cylindrical coordination can be transform into an ordinary differential equation by using the Hankel transform. Because the Hankel transform is the two dimensional Fourier transform of a circularly symmetric function, it plays an important role in optical data processing.

4.1 Hankel Transform: Let $f(r)$ be a function defined for $r \geq 0$. The $\neg$ th order Hankel transform of $f(r)$ is defined as $F_\neg(s) = \mathcal{H}_\neg\{f(r)\} \equiv \int_0^\infty rf(r)J_\neg(sr)\,dr$ (4.1)

If $\neg > -\dfrac{1}{2}$, Hankel's repeated integral immediately gives the inversion formula

$$f(r) = \mathcal{H}_\neg^{-1}\{F_\neg(s)\} = \int_0^\infty sF_\neg(s)\,J_\neg(sr)\,ds \tag{4.2}$$

The most important special cases of the Hankel transform correspond to $\neg = 0$ and $\neg = 1$. Sufficient but not necessary conditions for the validity of (4.1) and (4.2) are

1. $f(r) = 0(r^{-k}), r \to \infty$ where $k > \dfrac{3}{2}$
2. $f'(r)$ is piecewise continuous over each bounded subinterval of $[0, \infty)$
3. $f(r)$ is defined as $\left[f(r+) + f(r-)\right]/2$.

The inversion formula that is the inverse Hankel transform is defined as

$$f(r) = \int_0^\infty sF_\neg(s)\,J_\neg(sr)\,ds$$

4.2 Finite Hankle Transform: If $f(r)$ be a function satisfying Dirichlet's conditions:

(i) $f(x)$ is bounded in (a,b) i.e., there exists an upper bound M such that $|f(x)| \leq M$ for the values of x in (a,b) and the interval (a,b) can be divided into a finite number open sub-intervals in each of which the function $f(x)$ is monotonic.

(ii) $f(x)$ has a finite number of points of infinite discontinuity in the ineterval (a,b) but when the arbitrary small neighbourhoods of these points are excluded then $f(x)$ remains bounded in the deleted interval and the interval can be divided into a finite integral $\int_0^b f(x)\,dx$ is to be absolutely convergent.] in the interval $(0,a)$,

then its finite Hankel transform is defined as

$$\hat{f}(p_i) = \int_0^a r\, f(r)\, J_n(p_i r)\,dr$$

where p_i is a positive root of the transcendental equation $J_n(p_i a) = 0$

with the inversion formula, $f(r)=\dfrac{2}{a^2}\sum_i \tilde{f}(p_i)\dfrac{J_n(p_i r)}{\{J_n'(p_i a)\}^2}$

Other forms of Hankel Transform:

[1] When the range of variation does not included the origin, i.e. $f(r)$ satisfies Dirichlet's condition in $0<b\le r\le a$, then the finite Hankel transform is defined as

$$\tilde{f}(p_i)=\int_a^b rf(r)\left[J_n(p_i r)Y_n(p_i a)-Y_n(p_i r)J_n(p_i a)\right]dr \qquad (1)$$

where Y_n is the Bessel function of order n of second kind and p_i is a root of the equation $J_n(p_i a)Y_a(p_i b)-J_n(p_i b)Y_n(p_i a)=0$. Also then at each point of (a,b) where $f(r)$ is continuous we have the inversion formula,

$$f(r)=\sum_r \frac{2p_i^2 J_n^2(p_i a)\tilde{f}(p_i)}{J_n^2(p_i b)-J_n^2(p_i a)}\left[J_n(p_i x)Y_n(p_i b)-J_n(p_i b)Y_n(p_i a)\right]$$

where the summation extends over all the positive roots of (1).

[2] If $f(r)$ satisfies Dirichlet's conditions in the closed interval $[0,1]$, i.e. $0\le r\le 1$ and its finite Hankel transform is defined as

$$\tilde{f}(p_i)=\int_{¢}^1 rf(r)J_n(p_i r)dr$$

in which p_i is a root of the transcendental equation $pJ_n'(p)+hJ_n(p)=0$ (1)

then at each point of $[0,1]$ where $f(r)$ is continuous, we have the inversion formula

$$f(r)=2\sum_p \frac{p^2\tilde{f}(p)}{h^2+p^2-n^2}\,\frac{J_n(pr)}{J_n^2(p)}$$

where the summation extends over all the positive roots of (1).

4.3 Linearity Property: If $f(x)$ and $g(x)$ are two functions and C_1,C_2 two constants, then $\mathcal{H}\{C_1 f(x)+C_2 g(x)\}=C_1\mathcal{H}\{f(x)\}+C_2\mathcal{H}\{g(x)\}$

We have, $\mathcal{H}\{C_1 f(x)+C_2 g(x)\}=\int_0^\infty \{C_1 f(x)+C_2 g(x)\}\cdot xJ_n(px)dx$

$$=C_1\int_0^\infty f(x)\cdot xJ_n(px)dx+C_2\int_0^\infty g(fx)\cdot xJ_n(px)dx$$

$$=C_1\mathcal{H}\{f(x)\}+C_2\mathcal{H}\{g(x)\}$$

This result can be extended to any number of functions.

4.4 Parseval's Theorem for Hankel Transform: If $\tilde{f}(p)$ and $\tilde{g}(p)$ be respectively the Hanke's Transform of $f(x)$ and $g(x)$ then,

$$\int_0^\infty xf(x)g(x)dx=\int_0^\infty p\tilde{f}(p)\tilde{g}(p)dp$$

We have, $\int_0^\infty p\tilde{f}(p)\tilde{g}(p)dp=\int_0^\infty p\tilde{f}(p)dp\int_0^\infty xg(x)J_n(px)dx$

$$= \int_0^\infty xg(x)\,dx \int_0^\infty p\tilde{f}(p) J_n(px)\,dx \quad \left[\text{on substituting } \tilde{g}(p) = \int_0^\infty g(x)\, xJ_n(px)\,dx, n \geq -\frac{1}{2}\right]$$

$$= \int_0^\infty xg(x) f(x) \qquad \text{(changing the order of integration)}$$

4.5 Hankle Transform of the derivatives of a function: If $\tilde{f}_n(p)$ be the Hankel transforms of order n of the function $f(x)$, i.e. $\tilde{f}_n(p) = \int_0^\infty xf(x) J_n(px)$, then the Hankel transform of $\frac{df}{dx}$ is

$$\tilde{f}_n'(p) = \int_0^\infty x \frac{df}{dx} J_n(px)\,dx$$

$$= \left[xf(x) J_n(px)\right]_0^\infty - \int_0^\infty f(x) \frac{d}{dx}\{xJ_n(px)\}\,dx$$

$$= -\int_0^\infty f(x)\{J_n(px) + xJ_n'(px)\}\,dx$$

under the assumption that $xf(x) \to 0$ and $x \to 0$ or $x \to \infty$

$$= -\int_0^\infty f(x)\{(1-n) J_n(px)\}\,dx - \int_0^\infty pxf(x) J_{n-1}(px)\,dx$$

$$= (n-1)\int_0^\infty f(x) J_n(px)\,dx - p\tilde{f}_{n-1}(p) \qquad \left[\because xJ_n(px) = pxJ_n(px) - nJ_n(px)\right]$$

$$= (n-1) I - pf_{n-1}(p) \quad \text{(say)}$$

where $I \equiv \int_0^\infty f(x) J_n(px)\,dx$

The recurrence relation, $\quad J_{n-1}(x) - \frac{2n}{x} J_n(x) + J_{n+1}(x) = 0$

gives $\quad J_{n-1}(px) - \frac{2n}{px} J_n(px) + J_{n+1}(px) = 0 \quad$ [on replacing x by px]

i.e. $\quad 2nJ_n(x) = pxJ_{n-1}(x) + pxJ_{n+1}(x)$

So that $\quad 2nI = 2n\int_0^\infty f(x) J_n(px)\,dx$

$$= P\left[\int_0^\infty xf(x) J_{n-1}(px)\,dx + \int_0^\infty xf(x)J_{n+1}(px)\,dx\right] = pf_{n-1}(p) + pf_{n+1}(p) \tag{1}$$

Hence (1) reduce to $\quad \tilde{f}_n(p) = \frac{n-1}{2n} p\tilde{f}_{n-1}(p) + \frac{n-1}{2n} pf_{n+1}(p) - pf_{n-1}(p)$

$$= -p\left[\frac{n+1}{2n} f_{n-1}(p) - \frac{n-1}{2n} f_{n+1}(p)\right] \tag{2}$$

hich gives the required Hankel transform of $\frac{df}{dx}$.

Replacing n by $(n-1)$ and $(n+1)$ in succession (2) yields,

$$\tilde{f}_{n-1}'(p) = -p\left[\frac{n}{2(n-1)} \tilde{f}_{n-2}(p) - \frac{n-2}{2(n-1)} \tilde{f}_n(p)\right]$$

and $$\tilde{f}'_{n+1}(p) = -p\left[\frac{n+2}{2(n+1)}\tilde{f}_n(p) - \frac{n}{2(n+1)}\tilde{f}_{n+2}(p)\right]$$

Using these results and replacing f by f' in (2) we find,

$$\tilde{f}''_{n+1}(p) = -p\left[\frac{n+1}{2n}\tilde{f}'_{n-1}(p) - \frac{n-1}{2n}\tilde{f}_{n+2}(p)\right]$$

$$= \frac{p^2}{4}\left[\frac{n+1}{n-1}\tilde{f}_{n-2}(p) - 2\frac{n^2-3}{n^2-1}\tilde{f}_n(p) + \frac{n-1}{n+1}\tilde{f}_{n+2}(p)\right]$$

Corollary 1: Putting $n = 1, 2, 3,$ successively in (2) we have

$$\tilde{f}_1(p) = -p\tilde{f}_0(p)$$

$$\tilde{f}_2(p) = -p\left[\frac{3}{4}\tilde{f}_1(p) - \frac{1}{4}\tilde{f}_3(p)\right]$$

$$\tilde{f}'_3(p) = -p\left[\frac{2}{3}\tilde{f}_2(p) - \frac{1}{3}\tilde{f}_4(p)\right]$$

4.6 Hankle Transforms of $\frac{d^2f}{dx^2}, \frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}$ and $\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f$ under certain condition:

We have, $\mathcal{H}\left\{\frac{d^2f}{dx^2}\right\} = \int_0^\infty \frac{d^2f}{dx^2}\cdot xJ_n(px)\,dx$

$$= \left[\frac{df}{dx}xJ_n(px)\right]_0^\infty - \int_0^\infty \frac{df}{dx}\cdot\frac{d}{dx}\{xJ_n(px)\}\,dx$$

Assuming that $xf(x) \to 0$ as $x \to 0$, $x \to \infty,$ we have

$$\mathcal{H}\left\{\frac{d^2f}{dx^2}\right\} = -\int_0^\infty \frac{df}{dx}\left[J_n(px) + pxJ'_n(px)\right]dx$$

$$= \int_0^\infty x\frac{d^2f}{dx^2}\cdot J_n(px)\,dx$$

or, $\int_0^\infty x\left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right)J_n(px)\,dx = -p\int_0^\infty \frac{df}{dx}\cdot xJ'_n(px)\,dx$

$$= -p\left[f(x)xJ'_n(px)\right]_0^\infty - \int_0^\infty f(x)\frac{d}{dx}\{xJ'_n(px)\}\,dx$$

$$= p\int_0^\infty f(x)\frac{d}{dx}\{xJ'_n(px)\}\,dx \quad \left(\because x\,f(x) \to 0 \text{ as } x \to 0 \text{ or } \infty\right) \quad (1)$$

But $J_n(px)$ satisfies Bessel's differential equation, i.e.

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(1 - \frac{n^2}{x^2}\right)xy = 0$$

$$\therefore \frac{d}{dx}\left(x\frac{dy}{dx}J_n(x)\right) + \left(1 - \frac{n^2}{x^2}\right)x\cdot J_n(x) = 0$$

or, $$\frac{1}{p}\frac{d}{dx}\left[px\cdot J_n'(px)\right]=-\left(p^2-\frac{n^2}{x^2}\right)\frac{x}{p^2}J_n(px) \qquad \text{(on replacing } x \text{ by } px\text{)}$$

or, $$\frac{d}{dx}\left[xJ_n'(px)\right]=-\left[p^2-\frac{n^2}{x^2}\right]\frac{x}{p}J_n(px)$$

As such (1) reduces to

$$\int_0^\infty\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right)\cdot xJ_n(px)dx=-p\int_0^\infty f(x)\left(p^2-\frac{n^2}{x^2}\right)\frac{x}{p}J_n(px)dx$$

or, $$\int_0^\infty\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{n^2}{x^2}f\right)\cdot xJ_n(px)dx=-p^2\int_0^\infty f(x)\cdot xJ_n(px)dx$$

$$=-p^2\tilde{f}_n(p) \tag{2}$$

where $\tilde{f}_n(p)$ is the Hankel Transform of order n of $f(x)$.

Corollary-1: If we put $n=0$ in (2), we find

$$\int_0^\infty\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right)xJ_0(px)=-p^2\tilde{f}_0(p)$$

where $\tilde{f}_0(p)$ is the Hankel Transform of zeroth order.

Corollary-2: If we put $n=1$ in (2), we find

$$\int_0^\infty\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{f}{x^2}\right)xJ_1(px)=-p^2\tilde{f}_1(p)$$

or, $$\int_0^\infty x\frac{df}{dx}J_1(px)=-p\tilde{f}(p)$$

where $$\tilde{f}(p)=\int_0^\infty x\ f(x)J_0(px)dx$$

4.7 Finite Hankel Transform of $\frac{df}{dx}$, $\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}$

$$H_n\left\{\frac{df}{dx}\right\}=\int_0^1\frac{df}{dx}\cdot xJ_n(px)dx=\left[f(x)\cdot xJ_n(px)\right]_0^1-\int_0^1 f(x)\frac{d}{dx}\{xJ_n(px)\}dx$$

$$=-\int_0^1 f(x)\frac{d}{dx}\{xJ_n(px)\}dx \tag{1}$$

Now we have the recurrence relations $2J_n'(x)=J_{n-1}(x)-J_{n+1}(x)$

and $\quad 2nJ_n(x)=x\left[J_{n-1}(px)+J_{n+1}(x)\right]$

which on replacing x by px, become $\quad J_n'(px)=\frac{1}{2}\left[J_{n-1}(px)-J_{n+1}(px)\right]$

and $\quad J_n(px)=\frac{px}{2n}\left[J_{n-1}(px)+J_{n+1}(px)\right]$

Substituting them in $\quad \frac{d}{dx}\{xJ_n(px)\}=xpJ_n'(px)+J_n(px)$

we get, $$\frac{d}{dx}\{xJ_n(px)\}=\frac{px}{2}\left[J_{n-1}(px)-J_{n+1}(px)\right]+\frac{px}{2n}\left[J_{n-1}(px)+J_{n+1}(px)\right]$$

$$=\frac{p}{2n}\left[(n+1)xJ_{n-1}(px)-(n-1)J_{n+1}(px)\right]$$

So that (1) yields $$\mathcal{H}_n\left\{\frac{df}{dx}\right\}=\frac{p}{2n}\int_0^1\left[(n-1)xf(x)J_{n+1}-(n+1)xf(x)J_{n-1}\right]dx$$

$$=\frac{p}{2n}\left[(n-1)\mathcal{H}_{n+1}\{f\}-(n+1)\mathcal{H}_{n-1}\{f\}\right] \tag{2}$$

where $H_n(f)$ denotes the Hankel transform of order n.

Transform of $\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}$

Case-I: When p is a root of $J_n(p)=0$

We have, $$\mathcal{H}_n\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}=\int_0^1 x\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}J_n(px)dx$$

$$=\int_0^1 x\frac{d^2f}{dx^2}J_n(px)dx+\int_0^1\frac{df}{dx}J_n(px)dx$$

$$=\mathcal{H}_n\left\{\frac{d^2f}{dx^2}\right\}+\int_0^1\frac{df}{dx}J_n(px)dx$$

Now, $$\mathcal{H}_n\left\{\frac{d^2f}{dx^2}\right\}=\frac{p}{2n}\left[(n-1)\mathcal{H}_{n+1}\left\{\frac{df}{dx}\right\}-(n+1)\mathcal{H}_{n-1}\left\{\frac{df}{dx}\right\}\right]$$

Now replacing x by px in the recurrence relation, $2nJ_n(x)=x\left[J_{n-1}(x)+J_{n+1}(x)\right]$

We have, $$J_n(px)=\frac{px}{2n}\left[J_{n-1}(x)+J_{n+1}(px)\right]$$

So that, $$\int_0^1\frac{df}{dx}J_n(px)dx=\int_0^1\frac{df}{dx}\cdot\frac{px}{2n}\left[J_{n-1}(x)+J_{n+1}(px)\right]dx$$

$$=\frac{p}{2n}\int_0^1 x\frac{df}{dx}J_{n-1}(px)dx+\frac{p}{2n}\int_0^1 x\frac{df}{dx}J_{n+1}(px)dx=\frac{p}{2n}\left[\mathcal{H}_{n-1}\left\{\frac{df}{dx}\right\}+\mathcal{H}_{n+1}\left\{\frac{df}{dx}\right\}\right]$$

With the help of (2) and (3); (1) yields, $$\mathcal{H}_n\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}=\frac{p}{2}\left[-\mathcal{H}_{n-1}\left\{\frac{df}{dx}\right\}+\mathcal{H}_{n+1}\left\{\frac{df}{dx}\right\}\right] \tag{4}$$

Case-II: When p is a root of $pJ_n'(p)+hJ_n(p)=0$ (5)

Taking $n=0$, (5) becomes $pJ_0'(p)+hJ_0(p)=0$

We have $$H_0\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}=\int_0^1\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}xJ_0(px)dx$$

$$=\int_0^1\frac{d^2f}{dx^2}xJ_0(px)dx+\int_0^1\frac{df}{dx}\cdot J_0(px)dx$$

$$=\left[\frac{df}{dx}\cdot xJ_0(px)\right]_0^1-\int_0^1\frac{df}{dx}\cdot\frac{d}{dx}\{xJ_0(px)\}dx+\int_0^1\frac{df}{dx}\cdot J_0(px)dx$$

$$=\left[\frac{df}{dx}\right]_{x=1} J_0(p)-\int_0^1 \frac{df}{dx}\cdot\left\{xPJ_0'(px)+J_0(px)\right\}dx+\int_0^1 \frac{df}{dx} J_0(px)dx$$

$$=\left[\frac{df}{dx}\right]_{x=1} J_0(p)-p\int_0^1 \frac{df}{dx} xJ_0'\ (px)dx$$

$$=\left[\frac{df}{dx}\right]_{x=1} J_0(p)-p\left[f(x)\cdot xJ_0'(px)\right]_0^1+p\int_0^1 f(x)\cdot\frac{d}{dx}\left\{xJ_0'(px)\right\}dx$$

$$=\left[\frac{df}{dx}\right]_{x=1} J_0(p)-pf(1)J_0'(p)+\int_0^1 f(x)\cdot\{xPJ_0''(x)+J_0'(px)\}dx$$

But $J_0(x)$ being the solution of Bessel's equation on

$$x^2\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0\ ,$$

we have $xJ_0''(x)+J_0'(x)+xJ_0(x)=0$

Replacing x by px, $\quad PxJ_0''(px)+J_0'(px)=-pxJ_0(px)$

$$\therefore \mathcal{H}_0\left\{\frac{d^2}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}=\left[\frac{df}{dx}\right]_{x=1} J_0(p)-pf(1)\left\{-\frac{hJ_0(p)}{p}\right\}-p^2\int_0^1 xf(x)J_0(px)dx \quad \text{[by (6)]}$$

$$=\left\{\left[\frac{df}{dx}\right]_{x=1}+hf(1)\right\}J_0(p)-P^2\mathcal{H}_0\left\{f(x)\right\}$$

$$=\left[\frac{df}{dx}+hf(x)\right]_{x=1} J_0(p)-P^2\mathcal{H}_0\left\{f(x)\right\} \tag{7}$$

If boundary condition be such that

$$\frac{df}{dx}+hf(x)=0$$

when $x=1$, then we get $\quad \mathcal{H}_0\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}=-P^2\mathcal{H}_0\left\{f(x)\right\} \quad (8)$

Case-III : When P is a root of $J_n(pa)Y_n(pb)-J_n(pb)Y_n(pa)=0 \quad (9)$

Integrating by parts, the following, we have

$$\int_a^b x\left[J_n(px)Y_n(bp)-J_n(bp)Y_n(xp)\right]\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right)dx$$

$$=-p\left\{xf(x)\left[J_n'(xp)Y_n(bp)-J_n(bp)Y'(xp)\right]\right.$$

$$+p\int_a^b xf(x)\frac{d}{dx}\left[J_n'(xp)Y_n(bp)-J_n(bp)Y_n'(xp)\right]$$

$$+f(x)\left[J_n'(xp)Y_n(bp)-J_n(bp)Y_n'(xp)\right]dx \tag{10}$$

It is easy to show that

$$J_n'(pa)Y_n(bp)-Y_n'(ap)J_n(pb)=\frac{1}{pa}\frac{J_n(pb)}{J_n(pa)} \tag{11}$$

With the help of (11), (10) yields,

$$\mathcal{H}_n\left\{\frac{d^2 f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}=\int_0^1\left(\frac{n^2}{x^2}-p^2\right)x\,f(x)\left[J_n(px)Y_n(bp)-J_n(bp)Y_n(xp)\right]dx$$

or, $$\mathcal{H}_n\left\{\frac{d^2 f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{n^2}{x^2}\right\}=\frac{J_n(pb)}{J_n(pa)}f(a)-f(b)-p^2\mathcal{H}_n\{f(x)\} \tag{12}$$

Chapter-V

Application of Transforms

A periodic voltage $E(t)$ in the form of a “Rectified sine wave function” as shown in Fig 5-1 is applied to the electric circuit of Fig 5-2. Assuming that the current is zero at time $t=0$, find it at any later time,

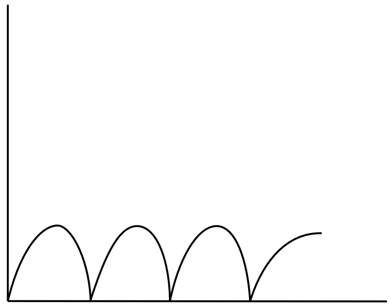

Fig: 5-1

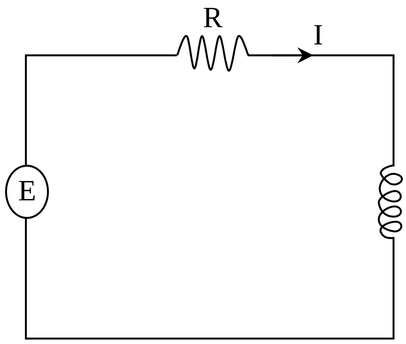

Fig: 5-2

The differential equation for the current $I(t)$ in the circuit is

$$L\frac{dI}{dt}+RI=E(t), \qquad \text{where} \qquad I(0)=0 \tag{5.1}$$

Taking Laplace Transforms and using table we find,

$$Ls\tilde{I}+R\tilde{I}=E_0\frac{\pi a}{a^2s^2+\pi^2}\ \coth\left(\frac{as}{2}\right)$$

or, $$\tilde{I}(s)=\frac{\pi a\ E_0}{(Ls+R)(a^2s^2+\pi^2)}\coth\left(\frac{as}{2}\right)$$

where $\tilde{I}(s)=\mathcal{L}\{I(t)\}$, *Thus*

$$I(t)=\frac{\pi aE_0}{La^2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+R/L\right)\left(s^2+\pi^2/a^2\right)}\ \coth\frac{as}{2}\right\}$$

$$\therefore I(t)=\frac{\pi E_0}{La}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+R/L\right)\left(s^2+\pi^2/a^2\right)}\ \coth\frac{as}{2}\right\} \tag{5.2}$$

To obtain the inverse Laplace transform, we will use residue method, which requires to find the residues of $e^{st}\ f(s)$ at the singularities.

The function $f(s)=\dfrac{1}{\left(s+R/L\right)\left(s^2+\pi^2/a^2\right)}\ \coth\dfrac{as}{2}$

has simple poles at $s=-R/L,\pm i\pi/a$ and simple poles at $s=s_k=\dfrac{2\pi ik}{a}, k=0,\pm1,.....$ where $\sin h\left(as/2\right)=0$.

Now we obtain the residues of $e^{st} f(s)$ at the poles. The residue for the pole at $s = -\, {}^{R}\!/_{L}$ is

given by, $$\lim_{S \to -R/L} \left(s + {}^{R}\!/_{L}\right) \left\{ \frac{e^{st}}{\left(s + {}^{R}\!/_{L}\right)\left(s^2 + {}^{®^2}\!/_{a^2}\right)} \ \coth \frac{as}{2} \right\}$$

$$= \frac{e^{-Rt/2}}{\left({}^{R^2}\!/_{L^2} + {}^{®^2}\!/_{a^2}\right)} \ \coth\left(\frac{aR}{2L}\right) \ = \frac{a^2 L^2 \ e^{-Rt/L}}{\left(a^2R^2 + ®^2 L^2\right)} \ \coth\left(\frac{aR}{2L}\right)$$

Residue at $s = i® \,/\, a$ is

$$\lim_{S \to i®/a} \left(s - {}^{i®}\!/_{a}\right) \left\{ \frac{e^{st}}{\left(s + {}^{R}\!/_{L}\right)\left(s + {}^{i®}\!/_{a}\right)\left(s - {}^{i®}\!/_{a}\right)} \coth \frac{as}{2} \right\}$$

$$= \frac{e^{i®t/a}}{\left({}^{i®}\!/_{a} + {}^{R}\!/_{L}\right) \cdot 2\, {}^{i®}\!/_{a}} \coth\left(\frac{i®}{2}\right) \ = \frac{a e^{i®t/a}}{2i® \left({}^{i®}\!/_{a} + {}^{R}\!/_{L}\right)} \cdot 0 \ = 0$$

Hence, Residue at $s = -\, {}^{i®}\!/_{a}$ be 0

Residue at $s = s_k = \dfrac{2i®k}{a}$ is

$$\lim_{S \to S_k} (s - s_k) \left\{ \frac{e^{st}}{\left(s + {}^{R}\!/_{L}\right)\left(s^2 + {}^{®^2}\!/_{a^2}\right)} \coth \frac{as}{2} \right\} = \left\{ \lim_{S \to S_k} \frac{s - s_k}{\sin h\left({}^{as}\!/_{2}\right)} \right\} \left\{ \lim_{s \to s_k} \frac{e^{st} \cos h\left({}^{as}\!/_{2}\right)}{\left(s + {}^{R}\!/_{L}\right)\left(s^2 + {}^{®^2}\!/_{a^2}\right)} \right\}$$

$$= \left\{ \frac{1}{\left({}^{a}\!/_{2}\right) \cos h \left(a s_k / 2\right)} \right\} \left\{ \frac{e^{s_k t} \cos \mathrm{h}\left(a s_k / 2\right)}{\left(s_k + {}^{R}\!/_{L}\right)\left(s^2{}_k + {}^{®^2}\!/_{a^2}\right)} \right\} \ = \frac{2\, e^{2i®kt/a}}{a\left(\dfrac{2i®k}{a} + \dfrac{R}{L}\right)\left(-\dfrac{4®^2 k^2}{a^2} + \dfrac{®^2}{a^2}\right)}$$

$$= \frac{2a^2 L \ \ e^{2i®kt/a}}{\left(aR + 2i®kL\right)\left(®^2 - 4®^2 k^2\right)}$$

Then the sum of the residues is

$$\frac{a^2 L^2 e^{-Rt/L}}{\left(a^2R^2 + ®^2 L^2\right)} \coth\left(\frac{aR}{2L}\right) + \sum_{k=-\infty}^{\infty} \frac{2a^2 L e^{2i®kt/a}}{\left(aR + 2i®kL\right)\left(1 - 4k^2\right)}$$

$$= \frac{a^2 L^2 e^{-Rt/L}}{\left(a^2R^2 + ®^2 L^2\right)} \coth\left(\frac{aR}{2L}\right) + \frac{2a^2 L}{®^2 aR} - \frac{4a^2 L}{®^2} \sum_{n=1}^{\infty} \frac{aR \cos 2n®t/a + 2n®L \sin 2n®t/a}{\left(4n^2 - 1\right)\left(a^2R^2 + 4n^2 ®^2 L^2\right)}$$

Thus from (2) we have the required result,

$$I(t) = \frac{®E_0}{La} \left[\frac{a^2 L^2 e^{-\mathrm{R}t/L}}{\left(a^2R^2 + ®^2 L^2\right)} \coth\left(\frac{aR}{2L}\right) - \frac{4a^2 L}{®^2} \times \sum_{n=1}^{\infty} \frac{aR \cos 2n®t/a + 2n®L \sin 2n®t/a}{\left(4n^2 - 1\right)\left(a^2R^2 + 4n^2 ®^2 L^2\right)} + \frac{2a^2 L}{®^2 aR} \right]$$

$$= \frac{®aLE_0 e^{-\mathrm{R}t/L}}{\left(a^2R^2 + ®^2 L^2\right)} \coth\left(\frac{aR}{2L}\right) + \frac{2®E_0}{®^2 R} - \frac{4aE_0}{®} \sum_{n=1}^{\infty} \frac{aR \cos 2n®t/a + 2n®L \sin 2n®t/a}{\left(4n^2 - 1\right)\left(a^2R^2 + 4n^2 ®^2 L^2\right)}$$

$$\therefore I(t)=\frac{®aLE_0e^{-Rt/L}}{\left(a^2R^2+®^2L^2\right)}\coth\left(\frac{aR}{2L}\right)+\frac{2E_0}{®R}-\frac{4aE_0}{®}\sum_{n=1}^{\infty}\frac{aR\cos 2n®t/a+2n®L\sin 2n®t/a}{\left(4n^2-1\right)\left(a^2R^2+4n^2®^2L^2\right)}$$

A periodic voltage $E(t)$ in the form of a "Half rectified sine wave function" as shown in Fig. 5-3 is applied to the electric circuit of Fig 5-4. Assuming that the charge is zero at time $t=0$, find it at any later time.

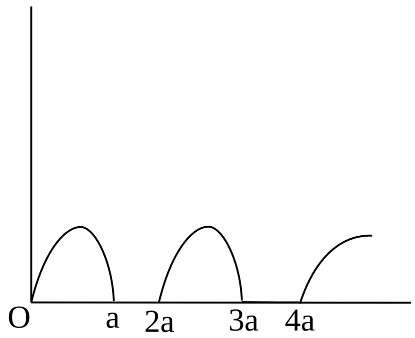

Fig: 5-3

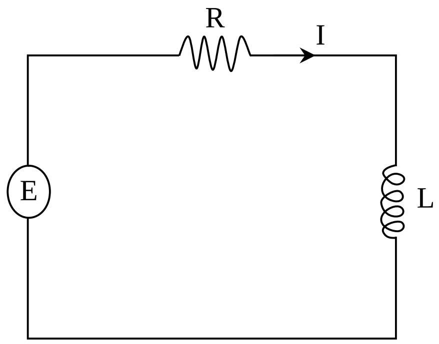

Fig: 5-4

The differential equation for the current $I(t)$ in the circuit is,

$$\frac{q}{c}+IR=E(t)\text{, where } q(0)=0 \tag{1}$$

or, $$R\frac{dq}{dt}+\frac{q}{c}=E(t)$$

Taking Laplace transforms, we find $RsQ+\frac{1}{c}Q=E_0\cdot\frac{®a}{\left(a^2s^2+®^2\right)\left(1-e^{-as}\right)}$

or, $$Q(s)=\frac{®aE_0}{\left(Rs+\frac{1}{c}\right)\left(a^2s^2+®^2\right)\left(1-e^{-as}\right)}$$

where $Q(s)=\mathrm{L}\{q(t)\}$.

$$\therefore q(t)=\frac{®aE_0}{Ra^2}\mathrm{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{cR}\right)\left(s^2+\frac{®^2}{a^2}\right)\left(1-e^{-as}\right)}\right\}$$

or, $$q(t)=\frac{®E_0}{Ra}\mathrm{L}^{-1}\left\{\frac{e^{as/2}}{\left(s+\frac{1}{cR}\right)\left(s^2+\frac{®^2}{a^2}\right)\left(e^{as/2}-e^{-as/2}\right)}\right\}$$

or, $$q(t)=\frac{®E_0}{aR}\mathrm{L}^{-1}\left\{\frac{e^{as/2}}{2\left(s+\frac{1}{cR}\right)\left(s^2+\frac{®^2}{a^2}\right)\sin\text{h}\,\frac{as}{2}}\right\} \tag{2}$$

To obtain the inverse Laplace transform we will use residue method, which requires to find the residues of $e^{st} f(s)$ at the singularities.

The function $f(s) = \dfrac{e^{as/2}}{2\left(s+\dfrac{1}{cR}\right)\left(s^2+\dfrac{\circledR^2}{a^2}\right)\sin h\dfrac{as}{2}}$ has simple poles at $s = -\frac{1}{cR}, \pm\frac{i\circledR}{a}$ and simple poles at $s = s_k = \dfrac{2i\circledR k}{a}, k = 0, \pm l, \cdots$ where $\sin \text{h}\left(\dfrac{as}{2}\right) = 0$

Now we obtain the residues of $e^{st} f(s)$ at the poles. The residue for the pole at $s = -\dfrac{1}{cR}$ is given by

$$\lim_{s\to -\frac{1}{cR}} \left(s+\frac{1}{cR}\right)\left\{\frac{e^{st+\frac{as}{2}}}{2\left(s+\dfrac{1}{cR}\right)\left(s^2+\dfrac{\circledR^2}{a^2}\right)\sin h\dfrac{as}{2}}\right\} = -\frac{e^{t/cR-a/2cR}}{2\left(\dfrac{1}{c^2R^2+\dfrac{\circledR^2}{a^2}}\right)\sin h\left(\dfrac{a}{2cR}\right)}$$

Residue at $s = \frac{i\circledR}{a}$ is $\displaystyle\lim_{s\to \frac{i\circledR}{a}} \left(s-\frac{i\circledR}{a}\right)\left\{\frac{e^{st+\frac{as}{2}}}{2\left(s+\dfrac{1}{cR}\right)\left(s+\dfrac{i\circledR}{a}\right)\left(s-\dfrac{i\circledR}{a}\right)\sin h\dfrac{as}{2}}\right\}$

$$= \frac{e^{i\circledR t/a+i\circledR/2}}{2\left(\frac{i\circledR}{a}+\frac{1}{cR}\right)\cdot\dfrac{2i\circledR}{a}\sin\left(\dfrac{i\circledR}{2}\right)} = \frac{ae^{i\circledR\left(\frac{t}{a}+\frac{1}{2}\right)}}{4i\circledR\left(\dfrac{i\circledR}{a}+\dfrac{1}{cR}\right)\cdot i} = -\frac{a^2c\,\text{R}e^{i\circledR\left(\frac{t}{a}+\frac{1}{2}\right)}}{4\circledR\,(a+i\circledR cR)}$$

Hence, Residue at $s = -\frac{i\circledR}{a}$ be $-\dfrac{a^2c\,\text{R}e^{-i\circledR\left(\frac{t}{a}+\frac{1}{2}\right)}}{4\circledR\,(a-i\circledR cR)}$

Residue at $s = s_k = \dfrac{2i\circledR k}{a}$ is $\displaystyle\lim_{s\to s_k}(s-s_k)\left\{\frac{e^{st+\frac{as}{2}}}{2\left(s+\frac{1}{cR}\right)\left(s^2+\frac{\circledR^2}{a^2}\right)\sin h\dfrac{as}{2}}\right\}$

$$= \left\{\lim_{s\to s_k}\frac{s-s_k}{2\sin h\dfrac{as}{2}}\right\}\left\{\lim_{s\to s_k}\frac{e^{st+\frac{as}{2}}}{\left(s+\frac{a}{cR}\right)\left(s^2+\dfrac{\circledR^2}{a^2}\right)}\right\} = \left\{\frac{1}{2\left(\frac{a}{2}\right)\cos h\left(\dfrac{as_k}{2}\right)}\right\}\left\{\frac{e^{s_k\left(t+\frac{a}{2}\right)}}{\left(s_k+\frac{1}{cR}\right)\left(s_k{}^2+\dfrac{\circledR^2}{a^2}\right)}\right\}$$

$$= \frac{e^{2i\circledR k(t+a/2)/a}}{a\left(\dfrac{2i\circledR k}{a}+\dfrac{1}{cR}\right)\left(\dfrac{-4\circledR^2k^2}{a^2}+\dfrac{\circledR^2}{a^2}\right)\cos h\left(\dfrac{2i\circledR k}{2}\right)} = \frac{a^2cR\cdot e^{2i\circledR k(t+a/2)/a}}{\circledR^2(a+2i\circledR kcR)(1-4k^2)\cos h(i\circledR k)}$$

Then the sum of the residue is

$$-\frac{e^{-\frac{t}{cR}-\frac{a}{2cR}}}{2\left(\frac{1}{c^2R^2}+\frac{\circledR^2}{a^2}\right)\sin h\left(\frac{a}{2cR}\right)} - \frac{a^2cR}{4\circledR}\left[\frac{e^{i\circledR\left(\frac{t}{a}+\frac{1}{2}\right)}}{(a+i\circledR cR)}+\frac{e^{i\circledR\left(\frac{t}{a}+\frac{1}{2}\right)}}{(a-i\circledR cR)}\right]$$

$$+\sum_{k=-\infty}^{\infty}\frac{a^2cR\,e^{2i\circledR j\left(t+a/2\right)/a}}{\circledR^2(a+2i\circledR kcR)(1-4k^2)\cos h(i\circledR k)}$$

$$=-\frac{a^2c^2R^2e^{-t/cR-a/2cR}}{2(a^2+\circledR^2c^2R^2)\sin h\left(\frac{a}{2cR}\right)}-\frac{a^2cR}{2\circledR}\left[\frac{a\cos\circledR\left(\frac{t}{a}+\frac{1}{2}\right)+\circledR cR\sin\circledR\left(\frac{t}{a}+\frac{1}{2}\right)}{(a^2+4\circledR^2c^2R^2)}\right]$$

$$+\frac{acR}{\circledR}-(-1)^n\cdot\frac{2a^2cR}{\circledR^2}\sum_{n=1}^{\infty}\frac{a\cos 2n\circledR\left(t/a+1/2\right)+2n\circledR cR\sin 2n\circledR\left(t/a+1/2\right)}{(4n^2-1)\left\{a^2+(2n^2)\circledR^2c^2R^2\right\}}$$

Thus from (2) we have the required result,

$$q(t)=-\frac{\circledR\,acRE_0e^{-1/cR\left(t+a/2\right)}}{2(a^2+\circledR^2c^2R^2)\sin h\left(a/2cR\right)}$$

$$-\frac{acE_0}{2}\left[\frac{a\cos\circledR\left(t/a+1/2\right)+\circledR cR\sin\left(t/a+1/2\right)}{(a^2+4\circledR^2c^2R^2)}\right]$$

$$+cE_0-(-1)^n\frac{2acE_0}{\circledR}\sum_{n=1}^{\infty}\frac{a\cos 2n\circledR\left(\frac{t}{a}+\frac{1}{2}\right)+2n\circledR cR\sin 2n\circledR\left(\frac{t}{a}+\frac{1}{2}\right)}{(4n^2-1)\left\{a^2+(2n)^2\circledR^2c^2R^2\right\}}$$

A periodic voltage $E(t)$ in the form of a "Half rectified sine wave function" as shown in Fig. 5-5 is applied to the electric circuit of Fig. 5-6. Assuming that the charge on the capacitor and current are zero at time $t=0$, find the charge on the capacitor at any later time.

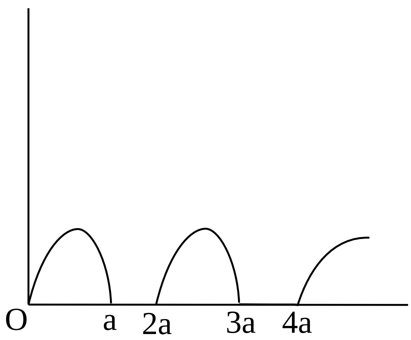

Fig: 5-5

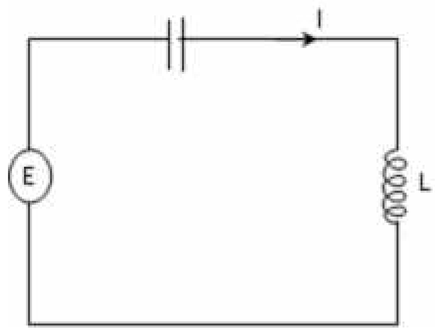

Fig: 5-6

The differential equation that govern the current $I(t)$ in the circuit is

$$L\frac{dI}{dt}+\frac{q}{c}=E(t)\quad\text{where } I(0)=0 \text{ and } q(0)=0$$

We know, $\frac{dq}{dt}=I$

Now taking Laplace transform in both equations, we find,

$$Ls\tilde{I}+\frac{1}{c}Q=E_0\frac{2\text{®}}{T\left(s^2+\frac{4\text{®}^2}{T^2}\right)\left(1-e^{-\frac{sT}{2}}\right)}$$

and $sQ=\tilde{I}$

$$\therefore Ls^2Q+\frac{1}{c}Q=\frac{2\text{®}E_0}{T\left(s^2+\frac{4\text{®}^2}{T^2}\right)\left(1-e^{-\frac{st}{2}}\right)}$$

$$\therefore Q(s)=\frac{2\text{®}E_0}{T\left(Ls^2+\frac{1}{c}\right)\left(s^2+\frac{4\text{®}^2}{T^2}\right)\left(1-e^{-\frac{st}{2}}\right)}$$

$$\therefore Q(s)=\frac{2\text{®}E_0}{LT\left(s^2+w^2\right)\left(s^2+a^2\right)\left(1-e^{-\frac{st}{2}}\right)}$$

where $Q(s)=\sim\{q(t)\}$

$$\therefore q(t)=\frac{2\text{®}E_0}{LT}\mathrm{L}^{-1}\left\{\frac{e^{\frac{sT}{4}}}{2\left(s^2+w^2\right)\left(s^2+a^2\right)\sin\mathrm{h}\frac{sT}{4}}\right\}$$

$$\therefore q(t)=\frac{\text{®}E_0}{LT}\mathrm{L}^{-1}\left\{\frac{e^{\frac{st}{4}}}{\left(s^2+w^2\right)\left(s^2+a^2\right)\sin h\frac{st}{4}}\right\} \tag{2}$$

To obtain the inverse Laplace transform, we will use residue method which requires to find the residues of $e^{st}f(s)$ at the singularities.

The function $f(s)=\frac{e^{\frac{sT}{4}}}{\left(s^2+w^2\right)\left(s^2+a^2\right)\sin h\frac{sT}{4}}$ has simple poles at $s=\pm iw,\pm ia$ and simple poles at $s=s_k=\frac{4i\text{®}k}{T}=2iak, k=0\pm1,\cdots\cdots$ where $\sin h\left(\frac{sT}{4}\right)=0$

Now we obtain the residues of $e^{st}f(s)$ at the poles. The residue for the pole at $s=i\mu$ is given by

$$\lim_{s\to i\mu}(s-i\mu)\left\{\frac{e^{s\left(t+\frac{T}{4}\right)}}{(s+iw)(s-iw)\left(s^2+a^2\right)\sin h\frac{sT}{4}}\right\}=\frac{e^{i\mu\left(t+\frac{T}{4}\right)}}{2i\mu\left(-\mu^2+a^2\right)\sin h\left(\frac{-i\mu T}{4}\right)}$$

$$=\frac{e^{i\mu\left(t+\frac{T}{4}\right)}}{2i\mu\left(-\mu^2+a^2\right)i\sin\frac{\mu T}{4}}=\frac{e^{i\mu\left(t+\frac{T}{4}\right)}}{2\mu\left(\mu^2-a^2\right)\sin\frac{\mu T}{4}}$$

Hence, Residue at $s=-i\mu$ be $\qquad \dfrac{e^{-i\mu\left(t+\frac{T}{4}\right)}}{2\mu\left(\mu^2-a^2\right)\sin\frac{\mu T}{4}}$

Residue at $s=ia$ is $\qquad \lim\limits_{s\to ia}(s-ia)\left\{\dfrac{e^{s\left(t+\frac{T}{4}\right)}}{\left(s^2+\mu^2\right)(s+ia)(s-ia)\sin h\frac{sT}{4}}\right\}$

$$=\frac{e^{ia\left(t+\frac{T}{4}\right)}}{2ia\left(-a^2+\mu^2\right)\sin h\left(\frac{iaT}{4}\right)}=\frac{e^{ia\left(t+\frac{T}{4}\right)}}{2ia\left(-a^2+\mu^2\right)i\sin\frac{aT}{4}}=\frac{e^{ia\left(t+\frac{T}{4}\right)}}{2a\left(a^2-\mu^2\right)\sin\frac{aT}{4}}$$

Hence Residue at $s=-ia$ be $\qquad \dfrac{e^{-ia\left(t+\frac{T}{4}\right)}}{2a\left(a^2-\mu^2\right)\sin\frac{aT}{4}}$

Residue at $s=s_k=2iak$ is $\qquad \lim\limits_{s\to s_k}(s-s_k)\left\{\dfrac{e^{s\left(t+\frac{T}{4}\right)}}{\left(s^2+\mu^2\right)\left(s^2+a^2\right)\sin h\frac{sT}{4}}\right\}$

$$=\left\{\lim_{s\to s_k}\frac{s-s_k}{\sin h\frac{sT}{4}}\right\}\left\{\lim_{s\to s_k}\frac{e^{s\left(t+\frac{T}{4}\right)}}{\left(s^2+\mu^2\right)\left(s^2+a^2\right)}\right\}=\left\{\frac{1}{\left(\frac{T}{4}\right)\cos h\frac{s_kT}{4}}\right\}\left\{\frac{e^{s_k\left(t+\frac{T}{4}\right)}}{\left(s_k{}^2+\mu^2\right)\left(s_k{}^2+a^2\right)}\right\}$$

$$=\frac{4}{T\cos h\left(\frac{iakT}{2}\right)}\cdot\frac{e^{2iak\left(t+\frac{T}{4}\right)}}{\left(-4a^2k^2+\mu^2\right)\left(-4a^2k^2+a^2\right)}=\frac{4e^{2iakt}\cdot e^{\frac{iakt}{2}}}{T\left(\mu^2-4a^2k^2\right)\left(a^2-4a^2k^2\right)\cos\left(\frac{akT}{2}\right)}$$

$$=\frac{4\cdot(-1)^k\cdot e^{2iakt}}{T\left(\mu^2-4a^2k^2\right)\left(a^2-4a^2k^2\right)(-1)^k}=\frac{4e^{2iakt}}{T\left(\mu^2-4a^2k^2\right)\left(a^2-4a^2k^2\right)}$$

Then the sum of the residue is-

$$\frac{2\cos\mu\left(t+\frac{T}{4}\right)}{2\mu\left(\mu^2-a^2\right)\sin\frac{\mu T}{4}}+\frac{2\cos a\left(t+\frac{T}{4}\right)}{2a\left(a^2-\mu^2\right)\sin\frac{aT}{4}}+\sum_{k=-\infty}^{\infty}\frac{4e^{2iakt}}{T\left(\mu^2-4a^2k^2\right)\left(a^2-4a^2k^2\right)}$$

$$=\frac{\cos\mu\left(t+\frac{T}{4}\right)}{\mu\left(\mu^2-a^2\right)\sin\frac{\mu T}{4}}+\frac{\cos a\left(t+\frac{T}{4}\right)}{a\left(a^2-\mu^2\right)\sin\frac{aT}{4}}+\frac{4}{T\mu^2a^2}+\frac{8}{T}\sum_{n=1}^{\infty}\frac{\cos 2ant}{\left(\mu^2-4a^2n^2\right)\left(a^2-4a^2n^2\right)}$$

$$=\frac{2\cos\mu\left(t+\frac{T}{4}\right)\cos\frac{\mu T}{4}}{\mu\left(\mu^2-a^2\right)\sin\frac{\mu T}{2}}+\frac{2\cos a\left(t+\frac{T}{4}\right)\cos\frac{aT}{4}}{a\left(a^2-\mu^2\right)\sin\frac{aT}{2}}+\frac{4}{T\mu^2a^2}+\frac{8}{T}\sum_{n=1}^{\infty}\frac{\cos 2ant}{\left(\mu^2-4a^2n^2\right)\left(a^2-4a^2n^2\right)}$$

$$=\frac{\cos\left(\mu t+\frac{\mu t}{4}+\frac{\mu t}{4}\right)+\cos\left(\mu t+\frac{\mu t}{4}-\frac{\mu t}{4}\right)}{\mu\left(\mu^2-a^2\right)\sin\frac{\mu T}{2}}+\frac{\cos\left(at+\frac{aT}{4}+\frac{aT}{4}\right)+\cos\left(at+\frac{aT}{4}-\frac{aT}{4}\right)}{a\left(a^2-\mu^2\sin\frac{aT}{2}\right)}$$

$$+\frac{4}{T\mu^2a^2}+\frac{8}{T}\sum_{n=1}^{\infty}\frac{\cos 2ant}{\left(\mu^2-4a^2n^2\right)\left(a^2-4a^2n^2\right)}$$

$$=\frac{\cos\left(\mu T+\frac{\mu T}{2}\right)+\cos\left(\mu T\right)}{\mu\left(\mu^2-a^2\right)\sin\frac{\mu T}{2}}+\frac{\cos\left(at+\frac{aT}{2}\right)+\cos\left(at\right)}{a\left(a^2-\mu^2\right)\sin\frac{aT}{2}}$$

$$+\frac{4}{T\mu^2a^2}+\frac{8}{T}\sum_{n=1}^{\infty}\frac{\cos 2ant}{\left(\mu^2-4a^2n^2\right)\left(a^2-4a^2n^2\right)}$$

$$=\frac{\sin\frac{\mu T}{2}\cos\left(\mu T+\frac{\mu T}{2}\right)+\cos\left(\mu T\right)\sin\left(\frac{\mu T}{2}\right)}{\mu(\mu^2-a^2)\left(1-\cos\mu T\right)}+\frac{\sin\frac{aT}{2}\cos\left(at+\frac{aT}{2}\right)+\cos\left(at\right)\sin\frac{aT}{2}}{a\left(a^2-\mu^2\right)\left(1-\cos aT\right)}$$

$$+\frac{4}{T\mu^2a^2}+\frac{8}{T}\sum_{n=1}^{\infty}\frac{\cos 2ant}{\left(\mu^2-4a^2n^2\right)\left(a^2-4a^2n^2\right)}$$

$$=\frac{\sin\left(\mu T+\mu t\right)+\sin\left(-\mu t\right)+\sin\left(\frac{\mu T}{2}+\mu t\right)+\sin\left(\frac{\mu T}{2}+\mu t\right)}{2\mu\left(\mu^2-a^2\right)\left(1-\cos\mu T\right)}$$

$$+\frac{\sin\left(aT+at\right)+\sin\left(-at\right)+\sin\left(\frac{aT}{2}+at\right)+\sin\left(\frac{aT}{2}-at\right)}{2a\left(a^2-\mu^2\right)\left(1-\cos aT\right)}$$

$$+\frac{4}{T\mu^2a^2}+\frac{8}{T}\sum_{n=1}^{\infty}\frac{\cos 2ant}{\left(\mu^2-4a^2n^2\right)\left(a^2-4a^2n^2\right)}$$

$$=\frac{\sin\mu\left(t+T\right)-\sin\mu t+\sin\left(\mu t+\frac{\mu T}{2}\right)-\sin\left(\mu t-\frac{\mu t}{2}\right)}{2\mu\left(\mu^2-a^2\right)\left(1-\cos\mu T\right)}$$

$$+\frac{\sin a\left(t+T\right)-\sin a+\sin\left(at+\frac{aT}{2}\right)-\sin\left(at-\frac{aT}{2}\right)}{2a\left(a^2-\mu^2\right)\left(1-\cos aT\right)}$$

$$+\frac{4}{t\mu^2a^2}+\frac{8}{T}\sum_{n=1}^{\infty}\frac{\cos 2ant}{\left(\mu^2-4a^2x^2\right)\left(a^2-4a^2n^2\right)}$$

$$=\frac{\sin\mu t-\sin\mu\left(t+T\right)}{2\mu\left(a^2-\mu^2\right)\left(1-\cos\mu T\right)}+\frac{\sin at-\sin a\left(t+T\right)}{2a\left(\mu^2-a^2\right)\left(1-\cos aT\right)}$$

$$+\frac{4}{T\mu^2a^2}+\frac{8}{T}\sum_{n=1}^{\infty}\frac{\cos 2ant}{\left(\mu^2-4a^2n^2\right)\left(a^2-4a^2n^2\right)} \qquad \text{[Using phase shift]}$$

Thus from (2) we have the required result,

$$L\left(t\right)=\frac{4\circledR E_0}{LT^2\mu^2a^2}+\frac{\circledR E}{2LT}\left\{\frac{\sin\mu t-\sin\mu\left(t+T\right)}{\mu\left(a^2-\mu^2\right)\left(1-\cos\mu T\right)}+\frac{\sin at-\sin a\left(t+1\right)}{a\left(\mu^2-a^2\right)\left(1-\cos aT\right)}\right\}$$

$$+\frac{8\circledR E_0}{LT^2}\sum_{n=1}^{\infty}\frac{\cos 2ant}{\left(\mu^2-4a^2n^2\right)\left(a^2-4a^2n^2\right)}$$

where $\mu^2=\dfrac{1}{Lc}$, $a^2=\dfrac{4\circledR^2}{T^2}$ and $\mu\neq\infty$.

Viscous fluid is contained between two infinitely long concentric circular cylinders of radii a and b. The inner cylinder is kept at rest and outer cylinder suddenly starts with uniform angular velocity w. Find the velocity v of the fluid if the equation of motion is $\dfrac{\partial^2 v}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial v}{\partial r}-\dfrac{v}{r^2}=\dfrac{1}{c}\dfrac{\partial v}{\partial t}, a<r<b, t>0$

We have, $\qquad \mathcal{H}_1\left\{\dfrac{\partial^2 v}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial v}{\partial r}-\dfrac{v}{r^2}\right\}=\displaystyle\int_a^b\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{1}{r^2}v\right]rB_1\left(pr\right)dr$

[where $B_1\left(pr\right)=J_1\left(pr\right)Y_1\left(pa\right)-Y_1\left(pr\right)J_1\left(pa\right), Y_1\left(pr\right)$ being Bessel's function of second kind of order one, and p is a positive root of $J_1\left(pb\right)Y_1\left(pa\right)=Y_1\left(pb\right)J_1\left(pa\right)$]

$$=\int_a^b\frac{\partial^2 v}{\partial r^2}\cdot rB_1\left(pr\right)dr+\int_a^b\frac{\partial v}{\partial r}B_1\left(pr\right)dr-\int_a^b\frac{1}{r}vB_1\left(pr\right)dr$$

$$=\left[\frac{\partial v}{\partial r}\cdot rB_1\left(pr\right)\right]_a^b-\int_a^b\frac{\partial v}{\partial r}\cdot\frac{d}{dr}\left\{rB_1\left(pr\right)\right\}dr+\int_a^b\frac{\partial v}{\partial r}\cdot B_1\left(pr\right)dr-\int_a^b vB_1\left(pr\right)dr$$

$$=0-0-p\int_a^b\frac{\partial v}{\partial r}\cdot rB_1'\left(pr\right)dr-\int_a^b\frac{1}{r}vB_1\left(pr\right)dr$$

$$=-p\left[v(r)rB_1'\left(pr\right)\right]_a^b+p\int_a^b v(r)\cdot\frac{d}{dr}\left\{dB_1'\left(pr\right)\right\}dr-\int_a^b\frac{1}{r}vB_1\left(pr\right)dr$$

$$=-p\left[v\left(b\right)\cdot bB_1'\left(pb\right)-v\left(a\right)\cdot aB_1'\left(pa\right)\right]+p\int_a^b v\left(r\right)\left\{prB_1''\left(pr\right)+B_1'\left(pr\right)\right\}dr-\int_a^b\frac{1}{r}vB_1\left(pr\right)dr$$

$$=pbv\left(b\right)B_1'\left(pb\right)+p\cdot 0+p\int_a^b v\left(r\right)\left\{prB_1''\left(pr\right)+B_1'\left(pr\right)\right\}dr-\int_a^b\frac{1}{r}vB_1\left(pr\right)dr$$

$$=-pbv\left(b\right)B_1'\left(pb\right)+p\int_a^b v\left(r\right)\left\{prB_1''\left(pr\right)+B_1'\left(pr\right)\right\}dr-\int_a^b\frac{1}{r}bB_1\left(pr\right)dr$$

But $B_1\left(r\right)$ being the solution of Bessel's equation,

$$r^2\frac{d^2y}{dr^2}+r\frac{dy}{dr}+\left(r^2-1\right)y=0$$

We have $\qquad r^2 B_1''(r) + rB_1'(r) + (r^2 - 1) B_1(r) = 0$

Replacing r by pr, this gives $p^2 r^2 B_1''(pr) + prB_1'(pr) + (p^2 r^2 - 1) B_1(pr) = 0$

i.e., $\qquad prB_1''(pr) + B_1'(pr) = pr\left(\dfrac{1}{p^2 r^2} - 1\right) B_1(pr)$

$$\therefore \mathcal{H}_1\left\{\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right\}$$

$$= pbv(b) B_1'(pb) + p\int_a^b v(r)\left\{pr\left(\frac{1}{p^2 r^2} - 1\right) B_1(pr)\right\} dr - \int_a^b \frac{1}{r} vB_1(pr)\, dr$$

$$= -pbv(b) B_1'(pb) - p^2 \int_a^b v(r) rB_1(pr)\, dr = -pbv(b) B_1'(pb) - p^2 H_1\{v(r)\}$$

Given, $\qquad \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} - \dfrac{v}{r^2} = \dfrac{1}{c}\dfrac{\partial v}{\partial t}$

or, $\qquad \mathcal{H}_1\left\{\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} - \dfrac{v}{r^2}\right\} = \dfrac{1}{c}\dfrac{\partial}{\partial t}\mathcal{H}_1\{v\}$

or, $\qquad -pbv(b) B_1'(pb) - p^2 \tilde{v} = \dfrac{1}{v}\dfrac{d\tilde{v}}{dt} \cdot \left[\text{let}\, \mathcal{H}_1\{v(r)\} = \tilde{v}\right]$

or, $\qquad \dfrac{d\tilde{v}}{dt} + cp^2 \tilde{v} = -cpb^2 wB_1'(pb) [\because v(b) = wb]$

or, $\qquad \dfrac{d\tilde{v}}{dt} + cp^2 \tilde{v} = -cpb^2 wB_1'(pb) = -k$ (say)

$\therefore I.F \quad e^{cp^2 t}$

$$\therefore \tilde{v} e^{cp^2 t} = -k\int e^{cp^2 t} dt + c_1 = \frac{-k}{\text{¥} p^2} e^{\text{¥} p^2 t} + c_1$$

$$\therefore \tilde{v} = \frac{-k}{cp^2} + c_1 e^{-cp^2 t}$$

But $\qquad \tilde{v} = 0$ when $t = 0$

$$\therefore 0 = \frac{-k}{cp^2} + c_1 \qquad \therefore c_1 = \frac{k}{cp^2}$$

Now, $\tilde{v} = \dfrac{-k}{cp^2}\left(1 - e^{-cp^2 t}\right)$ is its solution.

Applying inversion formula, we find

$$v = \sum -\frac{k\left(1 - e^{-cp^2 t}\right)}{cp^2} \cdot \frac{2p^2 J_n^2(pa) B_1(pr)}{J_n^2(pb) - J_n^2(pa)}$$

$$= \sum \frac{-\text{¥}\, pb^2 wB_1'(pb)\left(1 - e^{-cp^2 t}\right)}{cp^2} \cdot \frac{2p^2 J_n^2(pa) B_1(pr)}{J_n^2(pb) - J_n^2(pa)}$$

$$= 2b^2 w \sum \frac{1 - e^{-cp^2 t}}{J_n^2(pa) - J_n^2(pb)} \cdot pB_1'(pb) J_n^2(pa) B_1(pr)$$

Chapter-VI

Concluding Remarks

After the introduction of operational methods in the engineering problems by Haviside integral transforms are playing important role though the pioneering work was due to Joseph Fourier. During the last few decades the ideas of integral transforms has be generalized and newer transforms are imagine, but the older has their own importance. In the field of digital signal processing which has many different applications, wavelet transform is widely applied along with Discreet Fourier transform and Fast Fourier transform.

It has been found that Laplace transform is suitable in solving ordinary differential equations especially when the applied forces have some special character. The most commonly used method of finding inverse Laplace transform is not suitable in those cases rather the complex inversion formula is better suited for the purpose. Thus whenever possible this method should be addressed.

Laplace's equation has important role in engineering problems while it has been referred to cylindrical problems the transformation procedure that is required is the Hankel transform. Thus it can be said that the boundary value problems involving cylindrical shape can be easily solved using the Hankel transform.

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