Analytical Technique for Solving Second Order Generalized Strongly Nonlinear Duffing Equation with Varying Coefficients in Presence of Small Damping

by

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Dedication

I dedicate this thesis work to my beloved parents

RATAN KUNDU & ANJALI KUNDU

And

My Elder Brothers **UJJAL KUNDU** & **SANJAY KUNDU**

Whose affection, encouragement and pray makes me able to get such success and honor.

Along with all of my respectable

Teachers

Declaration

This is to certify that the thesis work entitled "**Analytical Technique for Solving Second Order Generalized Strongly Nonlinear Duffing Equation with Varying Coefficients in Presence of Small Damping**" has been carried out by **Dipa Kundu**, Roll No: **1751551**, in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna 9203, Bangladesh. The above thesis work or any part of the thesis work has not been submitted anywhere for the award of any degree or diploma.

05.08.19

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Dipa kundu

Abstract

In this thesis, we have extended He's homotopy perturbation method for obtaining the approximate analytical solution of second order generalized strongly nonlinear Duffing equation with varying coefficients in presence of significant small damping based on the extended form of the Krylov-Bogoliubov-Mitropolskii (KBM) method. Accuracy and validity of the solutions obtained by the present method are compared with the corresponding numerical solutions obtained by the well-known fourth order **Runge-Kutta** method in graphically. The method has been illustrated by examples. In this study, the present technique gives acceptable results avoiding any numerical complexity. The results presented through figures show that the approximations are of extreme accuracy with significant damping. The proposed method is simple and suitable for solving the above mentioned nonlinear damped systems.

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Nomenclature

ε	Small positive parameter
ω	Angular Frequency
a, b	Amplitudes
φ	Phase
T	Period
t	Time
ψ	Phase
$\dot{x}$	First derivative of x with respect to time t
$\ddot{x}$	Second derivative of x with respect to time t
Ω	Domain
Γ	The boundary of the domain Ω
$f(r)$	Analytical function
k	Constant
$f^{(k-1)}$	The periodic functions of ψ with period 2π
$(u_k)_\psi$	Partial derivative of functions u_k with respect to ψ
λ	Unknown function which can be evaluated by eliminating the secular terms from particular solution.

CHAPTER I

Introduction

Over the past decades, the study of nonlinear problems has been the interest of many researchers. Since most of the phenomena in our world are nonlinear and are described by nonlinear ordinary or partial differential equations, so the study of nonlinear oscillators is of great importance not only in all areas of physics but also in applied mathematics, engineering and other disciplines.

In general, the physicists and engineers use the linear solutions of the nonlinear problems imposing some proper restrictions. But such linearization is not always easy to apply to the researchers and sometimes, it does not able to give the physically acceptable approximations. Consequently, the original nonlinear differential equations itself must be considered. The exact solutions of those nonlinear differential equations are usually unobtainable. Therefore, researchers have focused on analytical approximation methods or numerical methods to solve those nonlinear differential equations. But the positions of the particles/objects are desired by the numerical methods only where as the positions as well as the amplitudes and phases of the particles/objects are obtained by the analytical methods. So, the analytical techniques are important and powerful tools to solve these original nonlinear differential equations. Usually, it is too much difficult to solve nonlinear differential equations with generalized nonlinearities and varying coefficients in presence of damping effects for their complexity and tedious work.

To solve these nonlinear differential equations, various perturbation methods have been widely used [2-27]. The traditional perturbation techniques are based on the assumption that a small parameter and linear term must exist in the equations, which are too over-strict to find wide application of these techniques. The limitation of the perturbation methods is that the approximate solutions are required to be expressed in a set of power series associated with small parameters. However, the solutions of some nonlinear problems cannot meet this requirement. In order to overcome the limitation of the perturbation methods, several analytical approximation methods such as Homotopy perturbation method (HPM) [28-35], Harmonic balance method (HBM) [36] and Variational iteration method (VIM) [37] have been developed.

In general, it is often more difficult to find an approximate solution than a numerical one of a given nonlinear oscillatory system. There are many analytical approaches to construct the approximate periodic solutions of nonlinear differential equations. Perturbations methods are the most common and widely used technique for solving nonlinear differential equations, whereby the solution is expanded in the power series of a small parameter. The perturbation methods provide accurate results for weak nonlinearities but it is unable to give the desire results when nonlinearities becomes strong.

The homotopy perturbation method is a one kind of asymptotic method. The basic assumption of the homotopy perturbation method is that the solution of the given nonlinear equations can be written as power series of homotopy parameter. In contrast to the traditional perturbation methods, this technique does not require a small parameter and linear term in the equations. In accordance to the homotopy perturbation technique, a homotopy with an imbedding parameter $p \in [0,1]$ is constructed and the imbedding parameter is considered as a "small parameter". The results at the first approximations obtained by the homotopy perturbation technique are more accurate than the solutions obtained by the traditional perturbation techniques at the second and higher approximations.

At first van der Pol [1] paid attention to the new (self-excited) oscillations and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. Thus, this nonlinearity appears as the very essence of these phenomena and by linearizing the differential equations in the sense of small oscillations, one simply eliminates the possibility of investigating such problems. Thus, it is necessary to deal with the nonlinear differential equations directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential equations, there exist some methods such as perturbation method [2-27], homotopy perturbation method [28-35], harmonic balance method [36], variational iterative method [37], etc. Among these methods, the method of perturbations, i.e., asymptotic expansions in terms of a small parameter are first and more frequently used.

A perturbation method known as "the asymptotic averaging method" in the theory of nonlinear oscillations was first introduced by Russian famous scientists Krylov and Bogoliubov (KB) [2] in 1947. Primarily, the method was developed only for obtaining the periodic solutions of second order weakly conservative nonlinear differential systems. Later, the method of KB has been improved and justified by Bogoliubov and Mitropolskii [3] in 1961. In literature, this method is known as the Krylov-Bogoliubov-Mitropolskii (KBM) [2, 3] method.

At the beginning, The KBM [2, 3] method was developed for obtaining only the periodic solutions of second order weakly nonlinear differential systems without damping. Now a days, this method is used for obtaining the solutions of second, third and fourth order weakly nonlinear differential systems for oscillatory, damped, over damped, and critically damped cases by imposing some special restrictions with quadratic and cubic nonlinearities. Several authors [4-27] have investigated and developed many significant results concerning the solutions of the weakly nonlinear differential systems based on the **KBM** method.

The KB [2] method is an asymptotic method in the sense that $\varepsilon \to 0$. An asymptotic series itself may not be convergent, but for a fixed number of terms, the approximate solution approaches toward the exact solution. Averaging asymptotic KBM [2, 3] method and the multiple-time scale method [25, 26] are frequently used two techniques in the theory of nonlinear oscillations in literature. Particularly the KBM method is convenient and extensively used technique for determining the approximate solutions among the methods used for studying the weakly nonlinear differential systems with cubic nonlinearity. The KBM method starts with the solution of linear equation (sometimes called the generating solution of the linear equation) assuming that in the nonlinear cases, the amplitude and phase variables in the solution of the linear differential equation are time dependent functions instead of constants. This method introduces an additional condition on the first derivative of the assumed solution for determining the solution of second order nonlinear differential systems. The KBM method demands that the asymptotic solutions are free from secular terms. These assumptions are mainly valid for second and third order nonlinear differential systems. But, for the fourth order differential equations, the correction terms sometimes contain secular terms, although the solution is generated by the classical KBM asymptotic method. For this reason, sometimes the traditional solutions fail to explain the proper situation of the nonlinear systems. One needs to impose some

special conditions for removing the presence of secular terms to get the desired results.
He [28-31] has developed a homotopy perturbation technique for solving second order strongly nonlinear differential systems. In this method the solution is considered as the summation of an infinite series which converges rapidly to the exact solutions. This technique has been employed to solve a large variety of nonlinear differential equations. Uddin *et al.* [32], Uddin and Sattar [33], Dey *et al.* [34] have been extended the homotopy perturbation method to damped nonlinear differential systems. Recently, Uddin and Saiful [35] have developed an analytical technique for solving strongly nonlinear damped systems with fractional power restoring force. Rahman and Lee [36] have developed new modified multi-level residue harmonic balance method for solving nonlinear vibrating double-beam problem.

In this research, an analytical technique is extended to find the approximate solutions of second order generalized strongly nonlinear Duffing equation with varying coefficients in presence of small damping effects based on the He's homotopy perturbation [28-36] and the extended form of the KBM [2,3] method. On the other hand, the proposed technique can take full advantage of the traditional perturbation techniques. Analytical approximate techniques are more appealing because of their analytical expressions which are inherent in physical meaning and more suitable for parametric study. In contrast, the numerical methods are comparatively easy to program but they need heavy computational effort and proper initial guess values. To justify the proposed method, the analytical approximate solution is compared to those numerical solution obtained by the fourth order **Runge-Kutta** method for different values of the parameters in graphically.

In Chapter II, the review of literature is presented. An analytical technique has been extended for solving second order generalized strongly nonlinear Duffing equation with varying coefficients in presence of small damping in **Chapter III,.** Finally, **in Chapter IV**, the concluding remarks are given.

CHAPTER II

Literature Review

Few issues occurring in different field of applied sciences and in various engineering problems are linear whereas as a large number of oscillation problems are nonlinear. So, the nonlinear oscillations in applied mathematics, physical sciences, mechanical structures, engineering, medical science and economics have been directed towards a topic to intensive research for many years. In various engineering problems like long span bridges, aerospace vehicles, robot arms, the beam/plate modeling and the corresponding vibrations analyses are quiet important. The mathematical formations of various engineering problems are governed by nonlinear ordinary or partial differential equations. Practically, most of all differential equations involving physical phenomena and in various engineering problems are nonlinear. In general, the physicists and engineers use the linear solutions of those problems imposing some proper restrictions. But such linearization is not always easy to apply to the researchers and sometimes, it does not able to give the physically acceptable approximations. Consequently, the original nonlinear differential equations itself must be considered. The exact solutions of those nonlinear differential equations are usually unobtainable. Therefore, some researchers have focused on numerical methods or analytical methods to solve these problems. But the positions of the particles/objects can be obtained by the numerical methods with proper guess values only where as the positions as well as the amplitudes and phases of the particles/objects are obtained by the analytical methods with ignoring the noise. So, the analytical techniques are important and powerful tools to solve these original nonlinear differential equations. Usually, it is too much difficult to solve nonlinear differential equations with generalized strong nonlinearities and varying coefficients in presence of damping effects for their complexity and tedious work. In general, it is often more difficult to find approximate solution than a numerical one of a given nonlinear oscillatory system. There are many analytical approaches to construct the approximate periodic solutions of nonlinear differential equations. Perturbations methods are the most common and widely used technique for solving nonlinear differential equations, whereby the solution is expanded in the power series of a small parameter. Among the perturbations methods, Krylov-Bogoliubov-Mitropolskii (KBM) method [2-4], Lindstedt-Poincare (LP) method [5] and multiple time scale method [4] are frequently used. The perturbation methods provide accurate results for weak nonlinearities but it unable to give the desired results when nonlinearity becomes strong.

During last several decades in the $20th$ century, some famous Russian scientists like Krylov and Bogoliubov [2], Bogoliubov and Mitropolskii [3], Mitropolskii [4], have investigated the nonlinear dynamics. For solving nonlinear differential equations, there exist some methods. Among the methods, the method of perturbations, i.e., an asymptotic expansion in terms of small parameter is foremost. In 1947, Krylov and Bogoliubov (KB) [2] considered the equation of the form

$$\ddot{x} + \omega^2 x = \varepsilon \, f(x, \dot{x}, t, \varepsilon), \tag{2.1}$$

where $\ddot{x}$ denotes the second order derivative with respect to t, ε is a small positive parameter and f is a power series in ε, whose coefficients are polynomials in $x, \dot{x}, \sin t$ and $\cos t$. In general, f does not contain either ε or t explicitly. In KBM [2, 3] method, it is assumed that the amplitude and phase variables in the solution of the linear equations are time dependent functions instead of constants in nonlinear differential systems. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the desired results. To describe the behavior of nonlinear oscillations by the solutions obtained by the perturbation method, Poincare [5] discussed only periodic solutions. Duffing [6] has investigated many significant results for the periodic solutions of the following damped nonlinear differential systems

$$\ddot{x} + 2k\,\dot{x} + \omega^2 x = -\varepsilon\, x^3. \tag{2.2}$$

Sometimes different types of nonlinear phenomena occur, when the amplitude of a dynamic system is less than or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation having these phenomena is

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0. \tag{2.3}$$

In literature, this equation is known as van der Pol [1] equation and is used in electrical circuit theory. Kruskal [7] has extended the KB [2] method to solve the fully nonlinear differential equation of the following form

$$\ddot{x} = F(x, \dot{x}, \varepsilon). \tag{2.4a}$$

Cap [8] has studied nonlinear differential system of the form

$$\ddot{x} + \omega^2 x = \varepsilon F(x, \dot{x}). \tag{2.4b}$$

Generally, F does not contain ε or t explicitly, so the equation (2.1) leads to

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}). \tag{2.5}$$

In the treatment of nonlinear oscillations by the perturbation methods, only periodic solutions are discussed, transients are not considered by different investigators, where as KB [2] have discussed transient response.

When $\varepsilon = 0,$ the equation (2.5) reduces to linear equation and its solution can be obtained as

$$x = a\cos(\omega t + \varphi). \tag{2.6}$$

where a and φ are arbitrary constants and the values of these arbitrary constants are determined by using the given initial conditions.

When $\varepsilon \neq 0,$ but is sufficiently small, then in KB [2] it is assumed that the solution of equation (2.5) is still given by equation (2.6) together with the derivative of the form

$$\dot{x} = -a\omega\sin(\omega t + \varphi). \tag{2.7}$$

where a and φ are functions of *t*, rather than being constants. In this case, the solution of equation (2.5) is

$$x = a(t)\cos(\omega t + \varphi(t)) \tag{2.8}$$

and the derivative of the solution is

$$\dot{x} = -a(t)\omega\sin(\omega t + \varphi(t)). \tag{2.9}$$

Differentiating the assumed solution equation (2.8) with respect to time *t*, we obtain

$$\dot{x} = \dot{a}\cos\psi - a\omega\sin\psi - a\dot{\varphi}\sin\psi, \;\; \psi = \omega t + \varphi(t). \tag{2.10}$$

Using the equations (2.7) and (2.10), we get

$$\dot{a}\cos\psi = a\dot{\varphi}\sin\psi. \tag{2.11}$$

Again, differentiating equation (2.9) with respect to *t*, we have

$$\ddot{x} = -\dot{a}\omega\sin\psi - a\omega^2\cos\psi - a\omega\dot{\varphi}\cos\psi. \tag{2.12}$$

Putting the value of $\ddot{x}$ from equation (2.12) into the equation (2.5) and using equations (2.8) and (2.9), we obtain

$$\dot{a}\,\omega\sin\psi + a\omega\dot{\varphi}\cos\psi = -\varepsilon f(a\cos\psi, -a\omega\sin\psi). \tag{2.13}$$

Solving equations (2.11) and (2.13), we have

$$\dot{a} = -\frac{\varepsilon}{\omega}\sin\psi \, f(a\cos\psi, -a\omega\sin\psi), \tag{2.14}$$

$$\dot{\varphi} = -\frac{\varepsilon}{a\omega}\cos\psi \, f(a\cos\psi, -a\omega\sin\psi). \tag{2.15}$$

It is observed that, a basic differential equation (2.5) of the second order in the unknown x, leads to two first order differential equations (2.14) and (2.15) in the unknowns a and φ to get its solutions.

Moreover, $\dot{a}$ and $\dot{\varphi}$ are proportional to ε and they are slowly varying functions of the time t with period $T = \frac{2\pi}{\omega}$. It is noted that these first order differential equations are now written in terms of the amplitude a and phase φ as dependent variables. Therefore, the right sides of equations (2.14) and (2.15) show that both a and $\dot{\varphi}$ are periodic functions of period T. In this case, the right-hand terms of these equations contain a small parameter ε and also contain both a and φ, which are slowly varying functions of the time t with period $T = \frac{2\pi}{\omega}$. We can transform the equations (2.14) and (2.15) into more convenient form. Now, expanding $\sin\psi \, f(a\cos\psi, -a\omega\sin\psi)$ and $\cos\psi \, f(a\cos\psi, -a\omega\sin\psi)$ in a Fourier series with phase ψ, the first approximate solution of equation (2.5) is obtained by averaging equations (2.14) and (2.15) with period $T = \frac{2\pi}{\omega}$ in the following form

$$\begin{aligned} \langle \dot{a} \rangle &= -\frac{\varepsilon}{2\pi\,\omega}\int_0^{2\pi} \sin\psi \, f(a\cos\psi, -a\omega\,\sin\psi)\, d\psi, \\ \langle \dot{\varphi} \rangle &= -\frac{\varepsilon}{2\pi\,\omega\, a}\int_0^{2\pi} \cos\psi \, f(a\cos\psi, -a\,\omega\,\sin\psi)\, d\psi, \end{aligned} \tag{2.16}$$

where a and φ are independent of time t under the integral signs.

Later, KB method has been extended mathematically by Bogoliubov and Mitropolskii [3], and has been extended to non-stationary vibrations by Mitropolskii [4]. They have assumed the solution of equation (2.5) in the following form

$$x = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \dots\dots\dots\dots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{2.17}$$

where u_k, $(k = 1, 2, ..., n)$ are periodic functions of ψ with period 2π, and the terms a and ψ are functions of time t and is obtained by solving the following first order ordinary differential equations

$$\dot{a} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + ... + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \quad (2.18\text{ a})$$

$$\dot{\psi} = \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + ... + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}). \quad (2.18\text{ b})$$

The functions $u_{k,}$ A_k and B_k, $(k = 1, 2, ..., n)$ are to be chosen in such a way that the equation (2.17), after replacing a and ψ by the functions defined in equation (2.18), is a solution of equation (2.5). Since there are no restrictions in choosing functions A_k and B_k, it generates the arbitrariness in the definitions of the functions u_k (Bogoliubov and Mitropolskii [3]). To remove this arbitrariness, the following additional conditions are imposed

$$\int_0^{2\pi} u_k(a,\psi)\cos\psi \, d\psi = 0, \quad (2.19\text{a})$$

$$\int_0^{2\pi} u_k(a,\psi)\sin\psi \, d\psi = 0. \quad (2.19\text{b})$$

Secular terms are removed by using these conditions in all successive approximations. Differentiating equation (2.17) two times with respect to t, and then substituting the values of $\ddot{x}, \dot{x}$ and x into equation (2.5), and using the relations equation (2.18) and equating the coefficients of ε^k, $(k = 1, 2, ..., n)$, we obtain

$$\omega^2((u_k)_{\psi\psi} + u_k) = f^{(k-1)}(a,\psi) + 2\omega(a B_k \cos\psi + A_k \sin\psi), \quad (2.20)$$

and

$$f^{(0)}(a,\psi) = f(a\cos\psi, -a\omega\sin\psi), \quad (2.21\text{a})$$

$$\begin{aligned} f^{(1)}(a,\psi) &= u_1 f_x(a\cos\psi, -a\omega\sin\psi) + (A_1\cos\psi - a B_1 \sin\psi + \omega(u_1)_\psi) \\ &\times f_{\dot{x}}(\cos\psi, -a\omega\sin\psi) + (a B_1^2 - A_1 \frac{dA_1}{da})\cos\psi \\ &+ (2A_1B_1 - a A_1 \frac{dB_1}{da})\sin\psi - 2\omega(A_1(u_1)_{a\psi} + B_1(u_1)_{\psi\psi}). \end{aligned} \quad (2.21\text{b})$$

where $(u_k)_\psi$ denotes partial derivative with respect to ψ and $f^{(k-1)}$ is a periodic function of ψ with period 2π which depends also on the amplitude a. Therefore, $f^{(k-1)}$ and u_k can be expanded in a Fourier series in the following form

$$f^{(k-1)}(a,\psi) = g_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi), \tag{2.22a}$$

$$u_k(a,\psi) = v_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi), \tag{2.22 b}$$

where

$$g_0^{(k-1)}(a) = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\,d\psi. \tag{2.23}$$

Here, $v_1^{(k-1)} = \omega_1^{(k-1)} = 0$ for all values of k, since both integrals of equation (2.19) are vanished. Substituting these values into the equation (2.20), we obtain

$$\begin{aligned}&\omega^2 v_0^{(k-1)}(a) + \sum_{n=2}^{\alpha}\omega^2(1-n^2)[v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi]\\ &= g_0^{(k-1)}(a) + (g_1^{(k-1)}(a) + 2\omega a B_k)\cos n\psi + (h_1^{(k-1)}(a) + 2\omega A_k)\sin n\psi\\ &+ \sum_{n=2}^{\alpha}[g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi].\end{aligned} \tag{2.24}$$

Now, equating the coefficients of the harmonics of the same order, we get

$$\begin{aligned}&g_1^{(k-1)}(a) + 2\omega a B_k = 0, \quad h_1^{(k-1)}(a) + 2\omega A_k = 0, \quad v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2},\\ &v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad \omega_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad n \geq 1.\end{aligned} \tag{2.25}$$

These are the sufficient conditions to obtain the desired order of approximations. For the first approximation, we have

$$A_1 = -\frac{h_1^{(0)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\sin\psi\,d\psi, \tag{2.26a}$$

$$B_1 = -\frac{g_1^{(0)}(a)}{2a\omega} = -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\cos\psi\,d\psi. \tag{2.26b}$$

Thus, the variational equations in equation (2.18) become

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\,d\psi, \tag{2.27a}$$

$$\dot{\psi} = \omega - \frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\,d\psi. \tag{2.27b}$$

It is seen that the equation (2.27) are similar to the equation (2.16). Thus, the first approximate solution obtained by Bogoliubov and Mitropolskii [3] is identical to the original solution obtained by KB [2]. The correction term u_1 is obtained from equation (2.22) by using equation (2.25) as

$$u_1 = \frac{g_0^{(0)}(a)}{\omega^2} + \sum_{n=2}^{\alpha} \frac{g_n^{(0)}(a)\cos n\psi + h_n^{(0)}(a)\sin n\psi}{\omega^2(1-n^2)} \tag{2.28}$$

The solution equation (2.17) together with u_1 is known as the first order improved solution in which a and ψ are obtained from equation (2.27). If the values of the functions A_1 and B_1 are substituted from equation (2.26) into the second relation of equation (2.21b), the function $f^{(1)}$ is determined. In the similar way, the functions A_2, B_2 and u_2 can be found. Therefore, the determination of the second order approximation is completed. The KB [2] method is very similar to that of van der Pol [1] method and related to it. van der Pol [1] has applied the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of $\ddot{x} + \omega^2 x = 0,$ on the other hand KB [2] has applied the same method to the basic solution $x = a\cos(\omega t + \varphi)$ of the same equation. Thus, in the KB [2] method the varied constants are a and $\varphi,$ while in the van der Pol's method the constants are a and b. The KB [2] method seems more interesting from the point of view of physical applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillations. The solution of the equation (2.4a) is based on recurrent relations and is given as the power series of the small parameter. Cap [8] has solved the equation (2.4b) by using elliptical functions in the sense of KB [2]. The KB [2] method has been extended by Popov [9] to damped nonlinear differential systems represented by the following equation

$$\ddot{x} + 2k\,\dot{x} + \omega^2 x = \varepsilon\, f(\dot{x}, x), \tag{2.29}$$

where $2k\,\dot{x}$ is the linear damping force and $0 < k < \omega.$ It is noteworthy that, because of the importance of the Popov's method in the physical nonlinear differential systems, involving damping force, Mendelson [10] and Bojadziev [11] have retrieved Popov's [9] results. In case of damped nonlinear differential systems, the first equation of equation (2.18a) has been replaced by

$$\dot{a} = -k\,a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots\dots\dots\dots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}). \tag{2.18a}$$

Murty and Deekshatulu [12] have developed a simple analytical method to obtain the time response of second order over damped nonlinear differential systems with small nonlinearity represented by the equation (2.29), based on the KB [2] method. In accordance to the KBM [2, 3] method, Murty *et al.* [13] have found a hyperbolic type asymptotic solution of an over damped system represented by the nonlinear differential equation (2.29), i.e., in the case $k > \omega$. They have used hyperbolic functions, $\cosh\varphi$ and $\sinh\varphi$ instead of their circular counterpart, which are used by KBM [2, 3], Popov [9] and Mendelson [10]. Murty [14] has presented a unified KBM method for solving the nonlinear differential systems represented by the equation (2.29), which cover the undamped, damped and over-damped cases. Bojadziev and Edwards [15] have investigated solutions of oscillatory and non-oscillatory systems represented by equation (2.29) when k and ω are slowly varying functions of time *t*. Initial conditions may be used arbitrarily for the case of oscillatory or damped oscillatory process. But, in case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions (Murty *et al.* [13], Murty [14], Bojadziev and Edwards [15]). Arya and Bojadziev [16, 17] have examined damped oscillatory systems and time dependent oscillating systems with slowly varying parameters and delay. Sattar [18] has developed an asymptotic method to solve a second order critically damped nonlinear differential system represented by equation (2.29). He has assumed the asymptotic solution of the equation (2.29) in the following form

$$x = a(1+\psi) + \varepsilon u_1(a,\psi) + \dots\dots\dots. + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{2.30}$$

where a is defined by the equation (2.18a) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \varepsilon^2 C_2(a) + \dots\dots\dots. + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}) \tag{2.18b}$$

Also Sattar [19] has extended the KBM method for three dimensional over damped nonlinear systems. Osiniskii [20] has extended the KBM method to the following third order nonlinear differential equation

$$\dddot{x} + c_1 \ddot{x} + c_2 \dot{x} + c_3 x = \varepsilon f(\ddot{x}, \dot{x}, x), \tag{2.31}$$

where ε is a small positive parameter and *f* is a given nonlinear function. He has assumed the asymptotic solution of equation (2.31) in the form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \dots\dots\dots. + \varepsilon^n u_n(a,b,\psi) + O(\varepsilon^{n+1}), \tag{2.32}$$

where each u_k $(k = 1, 2, ..., n)$ is a periodic function of ψ with period 2π and a, b and ψ are functions of time *t*, and they are given by

$$\dot{a} = -\lambda a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + ... + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \tag{2.33a}$$

$$\dot{b} = -\mu b + \varepsilon B_1(b) + \varepsilon^2 B_2(b) + ... + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}), \tag{2.33b}$$

$$\dot{\psi} = \omega + \varepsilon C_1(b) + \varepsilon^2 C_2(b) + ... + \varepsilon^n C_n(b) + O(\varepsilon^{n+1}), \tag{2.33c}$$

where $-\lambda, -\mu \pm i\omega$ are the eigen values of the equation (2.31) when $\varepsilon = 0$.

Lin and Khan [21] have also used the KBM method for solving biological problems. Proskurjakov [22] has investigated periodic solutions of nonlinear systems by using the Poincare and KBM methods, and compared the two solutions.

Alam and Sttar [23] have investigated a unified KBM method for solving *nth* $(n = 2,3)$ order nonlinear differential equation with varying coefficients. Nayfeh [24, 25] and Murdock [26] have developed perturbation methods and theory for obtaining the solutions of weakly nonlinear differential systems. Sachs *et al.* [27] have developed a simple ODE model of tumor growth and anti-angiogenic or radiation treatment.

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji Huan He [28]. The essential idea of this method is to introduce a homotopy parameter, say p, which varies from 0 to 1. At $p = 0$, the system of equations usually has been reduced to a simplified form which normally admits a rather simple solution. As p gradually increases continuously toward 1, the system goes through a sequence of deformations, and the solution at each stage is closed to that at the previous stage of the deformation. Eventually at $p = 1$ the system takes the original form of the equation and the final stage of the deformation give the desired solution.

He [28] has investigated a novel homotopy perturbation technique for finding a periodic solution of a general nonlinear oscillator for conservative systems. He [30] has considered the following nonlinear differential equation in the form

$$A(u) - f(r) = 0, \; r \in \Omega, \tag{2.34}$$

with the boundary conditions

$$B(u, \frac{\partial u}{\partial t}) = 0, \quad r \in \Gamma, \tag{2.35}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of the domain Ω. Then He [28] has written Eq. (2.34) in the following form

$$L(u) + N(u) - f(r) = 0, \tag{2.36}$$

where L is linear part, while N is nonlinear part. He [28] has constructed a homotopy $v(r, p) : \Omega \times [0,1] \to \Re$ which satisfies

$$\mathrm{H}(v, p) = (1-p)[L(v) - L(u_0)] + p[A(u) - f(r)] = 0, \;\; p \in [0,1], \;\; r \in \Omega \tag{2.37a}$$

or

$$\mathrm{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{2.37b}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of equation (2.34), which satisfies the boundary conditions. Obviously, from equation (2.37), it becomes

$$\mathrm{H}(v, 0) = L(v) - L(u_0) = 0, \tag{2.38}$$

$$\mathrm{H}(v, 1) = A(v) - f(r) = 0. \tag{2.39}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. He [28] has assumed the solution of Eq. (2.37) as a power series of p in the following form

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{2.40}$$

The approximate solution of Eq. (2.34) is given by setting $p = 1$ in the form

$$u = v_0 + v_1 + v_2 + \cdots. \tag{2.41}$$

The series (2.41) is convergent for most of the cases, and also the rate of convergence depends on how one chooses $A(u)$.

He [29] has developed some new approaches to solve Duffing equation with strongly and high order nonlinearity without damping. He [30] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities.

Also, He [31] has presented a new interpretation of homotopy perturbation method. Uddin *et al.* [32] and Uddin and Sattar [33] have presented an approximate technique for solving second order strongly nonlinear differential systems with damping by combining the He's [28-31] homotopy perturbation and the extended form of the

KBM [2-4] methods. Recently, Dey *et al.* [34] have also developed an approximate solutions of second order strongly and high order nonlinear Duffing equation with slowly varying coefficients in presence of small damping based on He's [28-31] homotopy perturbation and the extended form of the KBM [2-4] methods. Recently, Uddin and Saiful [35] have developed an analytical technique for solving strongly nonlinear damped systems with fractional power restoring force by combining He's HPM and the extended form of the KBM method. Rahman and Lee [36] have developed new modified multi-level residue harmonic balance method for solving nonlinear vibrating double-beam problem. He [37] has presented a variational iteration method for solving strongly nonlinear differential systems.

.CHAPTER III

An Analytical Technique for Solving Second Order Generalized Strongly Nonlinear Duffing Equation with Varying Coefficients in Presence of Small Damping

3.1 Introduction

The development of analytical techniques for solving strongly nonlinear damped differential systems is a subject of considerable interest that arises in all areas of applied mathematics, physics, engineering, medical science, economics and other disciplines, since most of the phenomena in the real world are essentially nonlinear and described by the nonlinear differential systems in presence of small damping. In general, it is often very difficult to get an approximate analytical solution for strongly generalized nonlinear differential systems with varying coefficients in presence of small damping than a numerical one. The most well-known common methods for constructing the approximate analytical solutions to the nonlinear oscillators are the perturbation techniques. Among these techniques the Krylov-Bogoliubov-Mitropolskii (KBM) [2-4] method, the Lindstedt-Poincare (LP) method [24- 26], and the multiple time scales [24] method are frequently used. Perturbation methods are based on an assumption that small parameters must exist in the equations, which is too strict to find wide application of the classical perturbation techniques. It determines not only the accuracy of the perturbation approximations, but also the validity of the perturbation methods itself. However, in science and engineering, there exist many nonlinear oscillatory problems which do not contain any small parameter, especially those appear in nature with strong nonlinearities. Therefore, many new techniques have been proposed to eliminate the “small parameter” assumption. Among these, the homotopy perturbation method (HPM) is a popular one. The method is a coupling of the traditional perturbation method and homotopy in topology. This method, which does not require a small parameter in an equation, has a significant advantage in that, it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences.

Arya and Bojadziev [17] have presented time dependent oscillating systems with damping, slowly varying parameters and delay. In recent years, He [29] has developed some new approaches to Duffing equation with strongly and high order nonlinearity. In another paper, He [30] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities. He [31] has presented a new interpretation of homotopy perturbation method. Uddin *et al.* [32] have presented an approximate technique for solving strongly nonlinear differential systems with cubic nonlinearity in presence of small damping. Uddin and Sattar [33] have also presented an approximate technique for solving Duffing` equation with slowly varying coefficients and cubic nonlinearity in presence of small damping. Dey *et al.* [34] have presented an approximate solutions of second order strongly and high order nonlinear Duffing equation with slowly varying coefficients in presence of small damping.

From our study, it has been seen that the author [28-31] have studied nonlinear differential systems without considering any damping effects. But most of the physical and engineering problems occur in nature as nonlinear differential systems with varying coefficients in presence of small damping. The aim of this research is to extend an approximate analytical technique combining the He's homotopy perturbation technique and the extended form of the KBM method for solving the second order generalized strongly nonlinear Duffing equation with varying coefficients in presence of small damping. The presented procedure can eliminate the limitations of classical perturbation and He's homotopy perturbation techniques, and the solution procedure is very simple and lead to high accurate solutions which are valid for the whole solution domain.

3.2. The Method

Let us assume the strongly nonlinear generalized Duffing oscillator with slowly varying coefficients in presence of small damping modeling in the form

$$\ddot{x} + 2k\dot{x} + e^{\tau} x = -\varepsilon_1 \{\alpha_3 f_3(x,\dot{x}) + \alpha_5 f_5(x,\dot{x}) + \alpha_7 f_7(x,\dot{x}) + \cdots + \alpha_n f_n(x,\dot{x})\}, \qquad (3.1)$$

with the initial conditions

$$x(0) = b_0, \quad \dot{x}(0) = 0, \qquad (3.2)$$

where over dots denote differentiations with respect to time t, $\tau = \varepsilon t$ is slow time, ε is a parameter small, ε_1 is a parameter but not necessarily small $2k$ is the linear damping coefficient, b_0 is an initial amplitude, $\alpha_i's$ are constants and $f_i(x,\dot{x}),\ i = 3,5,7\cdots n$ are given nonlinear functions which satisfies the following condition

$$f_i(-x,-\dot{x}) = -f_i(x,\dot{x}). \qquad (3.3)$$

To solve the equation (3.1), we are going to use the following transformation [32, 33]

$$x = y(t)e^{-kt}. \tag{3.4}$$

Now differentiating equation (3.4) twice with respect to time t and substituting $\ddot{x}, \dot{x}$ together with x into equation (3.1), we obtain

$$\ddot{y} + (e^{\tau} - k^2)y = -\varepsilon_1 e^{kt} \sum_{i=1,3,5}^{n} \alpha_i \, f_i\left(ye^{-kt}, (\dot{y} - ky)e^{-kt}\right). \tag{3.5}$$

According to the homotopy perturbation [28-35] method, equation (3.5) can be re-written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon_1 e^{kt} \sum_{i=1,3,5}^{n} \alpha_i \, f_i\left(ye^{-kt}, (\dot{y} - ky)e^{-kt}\right), \tag{3.6}$$

where

$$\omega = \sqrt{e^{\tau} - k^2 + \lambda}. \tag{3.7}$$

Herein ω is known as the frequency of the nonlinear differential systems and λ is an unknown function which can be evaluated by eliminating the secular terms. In the case of the damped nonlinear differential systems ω is a time dependent function and it varies slowly with time t. To handle this situation, we are interested to use the extended form [4] of the KBM [2,3] method. According to this method, we choose the solution of equation (3.6) in the following form

$$y = b\cos\varphi, \tag{3.8}$$

where b and φ vary slowly with time t. In literature b and φ are known as the amplitude and phase variables respectively and they keep an important role to nonlinear physical systems. The amplitude b and phase variable φ satisfy the following first order ordinary differential equations

$$\begin{aligned} \dot{b} &= \varepsilon B_1(b,\tau) + \varepsilon^2 B_2(b,\tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + \varepsilon C_1(b,\tau) + \varepsilon^2 C_2(b,\tau) + \cdots, \end{aligned} \tag{3.9}$$

Now differentiating equation (3.8) twice with respect to time t, utilizing the relations equation (9) and substituting $\ddot{y}$, y into equation (3.6) and then equating the coefficients of $\sin\varphi$, $\cos\varphi$ and neglecting $O(\varepsilon^2)$,we obtain

$$B_1 = -\omega' b/(2\omega), \; C_1 = 0, \tag{3.10}$$

where prime denotes differentiation with respect to slow time τ . Now putting equation (3.8) into equation (3.4) and equation (3.10) into equation (3.9), we obtain the following equations

$$x = be^{-kt}\cos\varphi, \tag{3.11}$$

$$\dot{b} = -k\omega' b/(2\omega),$$
$$\dot{\varphi} = \omega(\tau). \tag{3.12}$$

Thus, the first approximate analytical solution of equation (3.1) is given by equation (3.11) with the help of equations (3.7) and (3.12). Usually the integration of equation (3.12) is performed by well-known techniques of calculus [24-26], but sometimes they are calculated by a numerical procedure [14-19].

3.3. Examples

3.3.1 To justify the validity of the presented method, let us consider the strongly generalized nonlinear Duffing equation with a linear damping effects for $i = 3,5$ in the following form

$$\ddot{x} + 2k\dot{x} + e^{\tau} x = -\varepsilon_1(\alpha_3 x^3 + \alpha_5 x^5), \tag{3.13}$$

where $f_3(x,\dot{x}) = x^3$, $f_5(x,\dot{x}) = x^5$. Now using the transformation equation (3.4) into equation (3.13) and then simplifying them, we obtain

$$\ddot{y} + (e^{\tau} - k^2)y = -\varepsilon_1(\alpha_3 y^3 e^{-2kt} + \alpha_5 y^5 e^{-4kt}). \tag{3.14}$$

According to the homotopy perturbation [28-35] technique, equation (3.14) can be written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon_1(\alpha_3 y^3 e^{-2kt} + \alpha_5 y^5 e^{-4kt}), \tag{3.15}$$

where ω is given by equation (3.7). According to the extended form [4] of the KBM [2,3] method, the solution of equation (3.15) is obtained from equation (3.8).
By the trigonometric identity, we know

$$\cos^n \varphi = \frac{1}{2^{n-1}} \begin{bmatrix} \cos n\varphi + n\cos(n-2)\varphi + \dfrac{n(n-1)}{2!}\cos(n-4)\varphi \\ + \dfrac{n(n-1)(n-2)}{3!}\cos(n-6)\varphi + \cdots \end{bmatrix}, \tag{3.16),}$$

for all odd n. Now using the value of y from equation (3.8) into the right hand side of equation (3.15) and using the trigonometric identity equation (3.16) for $n = 3,5$ and then rearranging, we obtain

$$\ddot{y} + \omega^2 y = \left(\lambda b - \frac{3\varepsilon_1 \alpha_3 b^3 e^{-2kt}}{4} - \frac{5\varepsilon_1 \alpha_5 b^5 e^{-4kt}}{16} \right)\cos\varphi$$
$$- \varepsilon_1 \left(\frac{\alpha_3 b^3 e^{-2kt}}{4} + \frac{5\alpha_5 b^5 e^{-4kt}}{16} \right)\cos 3\varphi + \cdots. \tag{3.17}$$

The requirement of no secular terms in particular solution of equation (3.15) implies that the coefficients of the $\cos\varphi$ terms are zero. Setting these terms to zero, we obtain

$$\lambda b - \frac{3\varepsilon_1 \alpha_3 b^3 e^{-2kt}}{4} - \frac{5\varepsilon_1 \alpha_5 b^5 e^{-4kt}}{16} = 0, \tag{3.18}$$

which leads to (for non trivial solution $b \neq 0$)

$$\lambda = \frac{3\varepsilon_1 \alpha_3 b^2 e^{-2kt}}{4} + \frac{5\varepsilon_1 \alpha_5 b^4 e^{-4kt}}{16}. \tag{3.19}$$

Putting the value of λ from equation (3.19) into equation (3.6), then it leads to

$$\omega = \sqrt{e^{\tau} - k^2 + \frac{3\varepsilon_1 \alpha_3 b^2 e^{-2kt}}{4} + \frac{5\varepsilon_1 \alpha_5 b^4 e^{-4kt}}{16}}. \tag{3.20}$$

From equation (3.20) it is clear that, the frequency of the damped nonlinear differential systems depends on both amplitude b and time *t*. When $t \to 0$ then equation (3.20) yields

$$\omega_0 = \omega(0) = \sqrt{1 - k^2 + \frac{3\varepsilon_1 \alpha_3 b_0^2}{4} + \frac{5\varepsilon_1 \alpha_5 b_0^4}{16}}. \tag{3.21}$$

Integrating equation (3.12), we get

$$b = b_0 \sqrt{\frac{\omega_0}{\omega}}, \qquad \varphi = \varphi_0 + \int_0^t \omega(\tau)\, d\tau, \tag{3.22}$$

where b_0 and φ_0 are constants of integration and is known as the initial amplitude and phase variable of the systems respectively. Now putting equation (3.22) into equation (3.20), we obtain a biquadratic algebraic equation in ω in the following form

$$\omega^4 + p\omega^2 + q\omega + r = 0, \tag{3.23}$$

where

$$p = k^2 - e^{\tau},\ q = -\frac{3\varepsilon_1 \alpha_3 \omega_0 b_0^2 e^{-2kt}}{4}\ ,\ r = -\frac{5\varepsilon_1 \alpha_5 {\omega_0}^2 b_0^4 e^{-4kt}}{16}. \tag{3.24}$$

The solution of equation (3.23) is computed by using the well-known **Newton-Raphson** method. Thus, the first order analytical approximate solution of equation (3.13) is given by

$$x = b\, e^{-kt} \cos\varphi, \tag{3.25}$$

where ω_0 is obtained by equation (3.21), ω is calculated from equation (3.23), b and φ are carried out by equation (3.22).

3.3.2 As a second example, let us consider the strongly generalized nonlinear Duffing oscillator with a linear damping effects modeling in the following form

$$\ddot{x}+2k\,\dot{x}+e^{\tau}x=-\varepsilon_1(\alpha_3\,x^3+\alpha_5\,x^5+\alpha_7\,x^7). \tag{3.26}$$

Now by using the transformation equation (3.4) into equation (3.26) and then simplifying them, we obtain

$$\ddot{y}+(e^{\tau}-k^2)y=-\varepsilon_1(\alpha_3\,y^3e^{-2\,k\,t}+\alpha_5\,y^5e^{-4\,k\,t}+\alpha_7\,y^7e^{-6\,k\,t}). \tag{3.27}$$

According to the homotopy perturbation [28-35] method, equation (3.27) yields

$$\ddot{y}+\omega^2y=\lambda\,y-\varepsilon_1(\alpha_3\,y^3e^{-2k\,t}+\alpha_5\,y^5e^{-4k\,t}+\alpha_7\,y^7e^{-6kt}), \tag{3.28}$$

where ω is obtained by equation (3.7). By the extended form of the KBM [2-4] method, the solution of equation (3.28) is performed by equation (3.8).

For $n=7$, the trigonometric identity equation (3.16) leads to

$$\cos^7\varphi=(\cos7\varphi+7\cos5\varphi+21\cos3\varphi+35\cos\varphi)/64. \tag{3.29}$$

The requirement of no secular terms in particular solution of equation (3.28) implies that the coefficients of the $\cos\varphi$ terms are zero. Setting these terms to zero, we obtain

$$\lambda\,b-\frac{3\varepsilon_1\,\alpha_3\,b^3e^{-2\,k\,t}}{4}-\frac{5\varepsilon_1\,\alpha_5\,b^5e^{-4\,k\,t}}{16}-\frac{35\varepsilon_1\,\alpha_7\,b^7e^{-6\,k\,t}}{64}=0, \tag{3.30}$$

which leads to

$$\lambda=\frac{3\varepsilon_1\,\alpha_3\,b^2e^{-2\,k\,t}}{4}+\frac{5\varepsilon_1\,\alpha_5\,b^4e^{-4\,k\,t}}{16}+\frac{35\varepsilon_1\alpha_7\,b^6e^{-6\,k\,t}}{64}. \tag{3.31}$$

Putting the value of λ from equation (3.31) into equation (3.7), yields

$$\omega=\sqrt{e^{\tau}-k^2+\frac{3\varepsilon_1\,\alpha_3\,b^2e^{-2\,k\,t}}{4}+\frac{5\varepsilon_1\,\alpha_5\,b^4e^{-4\,k\,t}}{16}+\frac{35\varepsilon_1\,\alpha_7\,b^6e^{-6\,k\,t}}{64}}. \tag{3.32}$$

From equation (3.32), we obtain (as $t\to0$)

$$\omega_0=\omega(0)=\sqrt{1-k^2+\frac{3\varepsilon_1\alpha_3\,b_0^2}{4}+\frac{5\varepsilon_1\,\alpha_5\,b_0^4}{16}+\frac{35\varepsilon_1\alpha_7\,b_0^6}{64}}. \tag{3.33}$$

By integrating the first equation of equation (3.12) and using it into equation (3.32), we obtain a fifth degree algebraic equation in ω in the following form

$$\omega^5+p\omega^3+q\omega^2+r\omega+s=0, \tag{3.34}$$

where

$$p = k^2 - e^{\tau}, \quad q = -\frac{3\varepsilon_1 \alpha_3 \omega_0 b_0^2 e^{-2kt}}{4},$$
$$r = -\frac{5\varepsilon_1 \alpha_5 \omega_0{}^2 b_0^4 e^{-4kt}}{16}, \; s = -\frac{35\varepsilon_1 \alpha_7 \omega_0{}^3 b_0{}^6 e^{-6kt}}{64}. \qquad (3.35)$$

The solution of equation (3.34) is obtained by using the well-known **Newton-Raphson** method.

Thus, the first order analytical approximate solution of equation (3.26) is given by

$$x = b e^{-kt} \cos\varphi, \qquad (3.36)$$

$$b = b_0 \sqrt{\frac{\omega_0}{\omega}}, \qquad \varphi = \varphi_0 + \int_0^t \omega(\tau) d\tau, \qquad (3.37)$$

where ω_0 is obtained by equation (3.33), ω is calculated from equation (3.32), b and φ are given by equation (3.37).

CHAPTER IV

Results and Discussion

In this research, we have extended He's homotopy perturbation method to solve the second order strongly generalized nonlinear Duffing oscillators with significant small damping effects. The classical perturbation methods [1-5] are failed to solve the generalized strongly nonlinear Duffing type problems in presence of damping and He's homotopy perturbation method is failed to handle the strongly generalized nonlinear Duffing type problems in presence of damping. But the suggested method has been successfully applied to solve second order strongly as well as weakly generalized nonlinear differential systems with significant small damping effects. The first order approximate solutions of equation (3.13) and equation (3.26) are computed with small damping by equations. (3.25) and (3.36) respectively and the corresponding numerical solutions are obtained by using fourth order ***Runge-Kutta*** method. The variational equations for the amplitude and phase variable appeared in a set of first order differential equations. The integration of these variational equations is carried out by the well-known techniques of calculus [24-26]. In a lack of analytical solutions, they are solved by numerical procedure [23, 32-35]. The amplitude and phase variable change slowly with time t. The behavior of amplitude and phase variable characterize the oscillating processes and amplitude tends to zero in presence of small damping for large time t $(i.e., t \to \infty)$. On the other hand, our proposed technique can take full advantages of the classical perturbation methods. The solutions obtained by the present method show a good agreement with those solutions obtained by the numerical procedure [23, 32-35] with several significant small damping effects.

It is noticed that the present method is suitable for second order strongly as well as weakly generalized nonlinear Duffing oscillators with significant small damping effects while the classical perturbation and He's homotopy perturbation methods are not suitable for such cases. Comparisons are made between the solutions obtained by the present technique and those obtained by the numerical procedure in **Figs. 4.1-4.3** for both strongly $(\varepsilon_1 = 1.0)$ as well as weakly $(\varepsilon_1 = 0.1)$ generalized nonlinear differential systems with significant small damping effects in graphically. Also the solution of the Duffing equation for cubic nonlinearity is obtained from equation (3.13) and equation (3.26) by setting $\alpha_5 = 0, \alpha_7 = 0$ with significant small damping (**Figs. 4.3)** which agrees to the results of [32]. Thus, the present method is proved to be a powerful mathematical tool to find the approximate solutions of strongly as well weakly generalized nonlinear differential systems in presence of significant small damping effects.

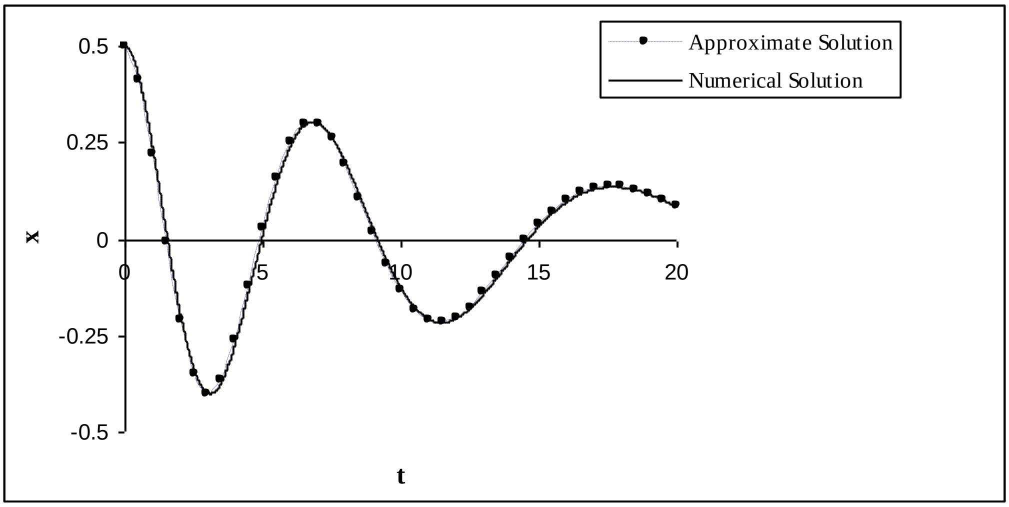

Fig. 4.1 (a) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dash-dots lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.00989]$ with $k = 0.1,\ \varepsilon = 0.1, \varepsilon_1 = 1.0, \alpha_3 = 1.0, \alpha_5 = 1.0$ and $f_3 = x^3,\quad f_5 = x^5$. Corresponding numerical solution is denoted by - (solid line).

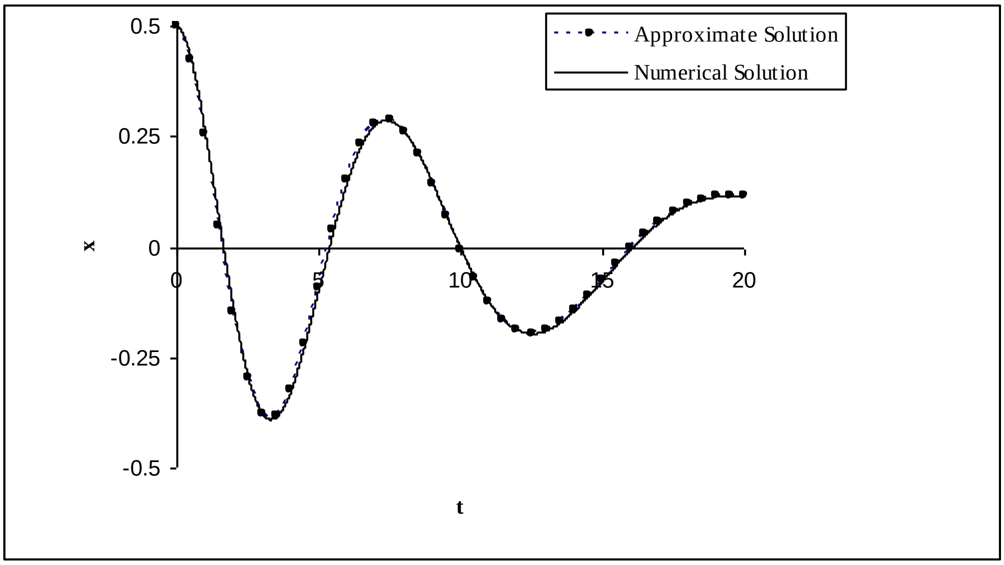

Fig. 4.1 (b) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dash-dots lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.03327]$ with $k = 0.1,\ \varepsilon = 0.1, \varepsilon_1 = 0.1, \alpha_3 = 1.0, \alpha_5 = 1.0$ and $f_3 = x^3,\quad f_5 = x^5$. Corresponding numerical solution is denoted by - (solid line).

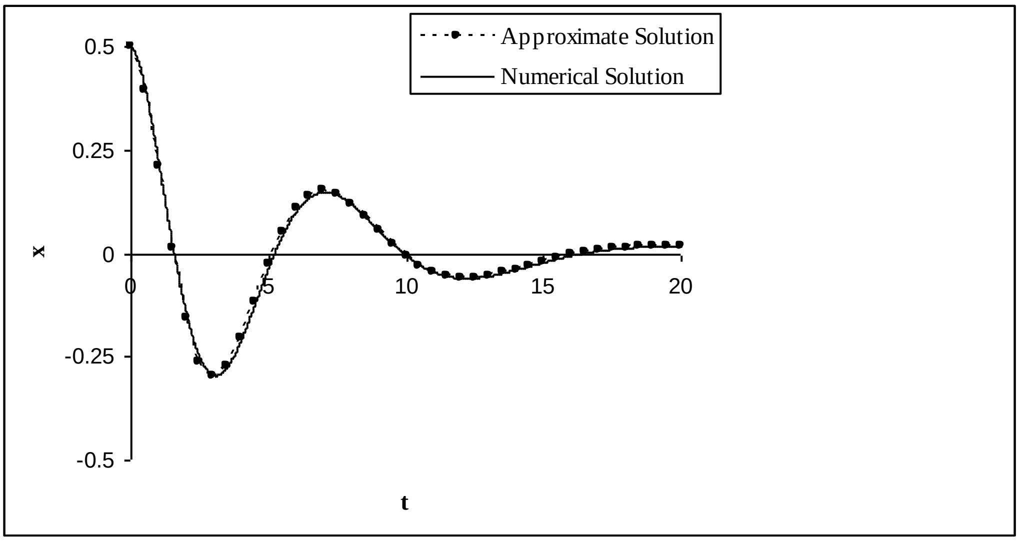

Fig. 4.2 (a) First approximate solution of equation (3.26) is denoted by $-\bullet-$ (dash - dots lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.05790]$ with $k = 0.2,\ \varepsilon = 0.1,\ \varepsilon_1 = 1.0,\ \alpha_3 = 1.0,\ \alpha_5 = 1.0, \alpha_7 = 1.0$ and $f_3 = x^3,\ f_5 = x^5,\ f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line).

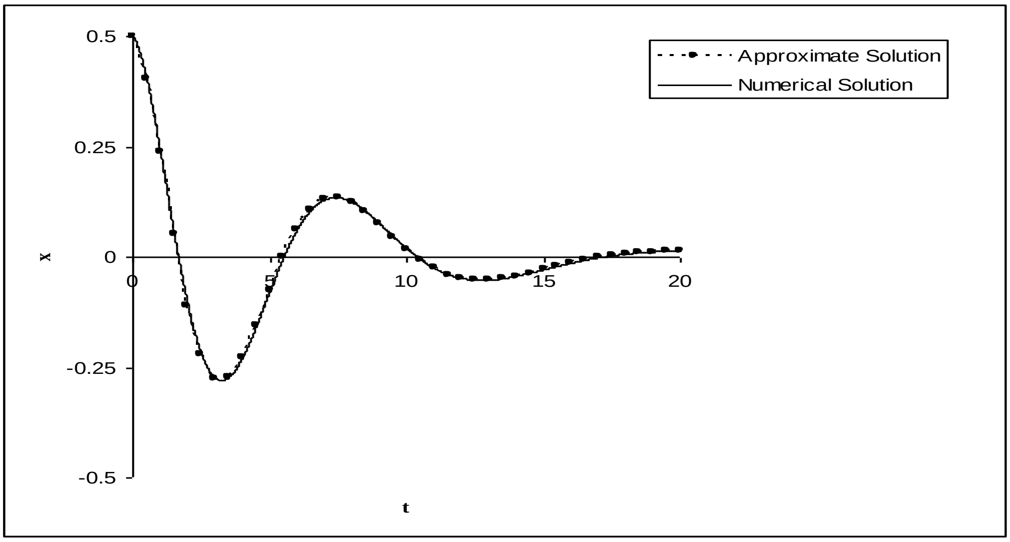

Fig. 4.2 (b) First approximate solution of equation (3.26) is denoted by $-\bullet-$ (dash-dots lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.08260]$ with $k = 0.2,\ \varepsilon = 0.1,\ \varepsilon_1 = 0.1,\ \alpha_3 = 1.0,\ \alpha_5 = 1.0, \alpha_7 = 1.0$ and $f_3 = x^3,\ f_5 = x^5,\ f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line).

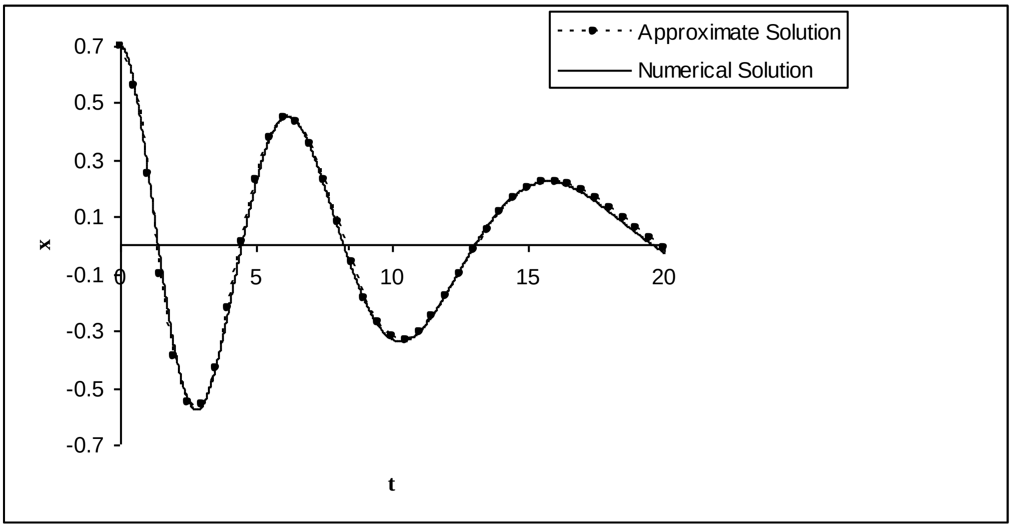

Fig.4.3 (a) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dash-dots lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.7,\ \dot{x}(0) = -0.00550]$ with $k = 0.1,\ \varepsilon = 0.1,\ \varepsilon_1 = 1.0, \alpha_3 = 1.0,\ \alpha_5 = 0.0,\ \alpha_7 = 0.0$ and $f_3 = x^3$. Corresponding numerical solution is denoted by - (solid line).

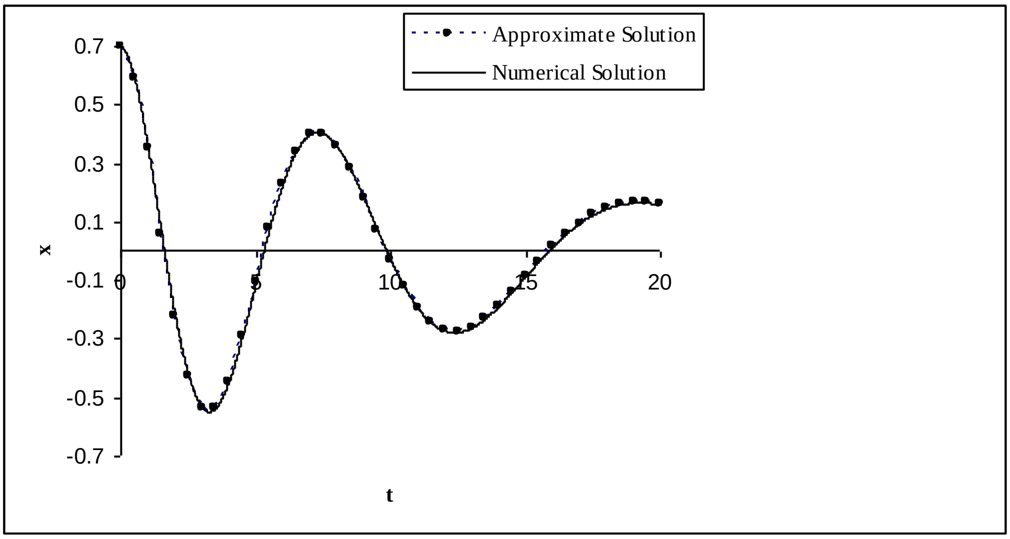

Fig. 4.3 (b) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dash-dots lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.7,\ \dot{x}(0) = -0.04156]$ with $k = 0.1,\ \varepsilon = 0.1,\ \varepsilon_1 = 0.1, \alpha_3 = 1.0,\ \alpha_5 = 0.0,\ \alpha_7 = 0.0$ and $f_3 = x^3$. Corresponding numerical solution is denoted by - (solid line).

CHAPTER V

Conclusions

It is well known that the classical perturbation methods are limited to weakly nonlinear systems. In our study, we have obtained approximate solutions of strongly as well as weakly generalized nonlinear Duffing equation with slow varying coefficients in presence of small damping. It is noticed that the first approximate solutions show a good agreement with the numerical (considered to be exact) solutions. The present method has been successfully implemented to illustrate the effectiveness and convenience of the suggested procedure. So the proposed method is more computationally efficient than the He's homopty perturbation and extended form of the KBM methods. The determination of position, amplitude and phase of nonlinear differential systems is very important in mechanics.

From the figures (**Figs. 4.1- 4.3**), it is clear that the first approximate solutions show good agreement with those solutions obtained by the fourth order Runge-Kutta method with the several small and significant damping in the whole solution domain. It is also noticed that He's HPM is incapable for solving nonlinear differential systems in presence of any damping and KBM method is fail to handle strongly nonlinear differential systems. Both limitations have been overcome by the proposed method.

This method is effective for solving second order strongly generalized nonlinear damped physical problems and converging rapidly to the exact solutions. So our proposed method can serve as a useful mathematical tool for dealing strongly as well as weakly generalized nonlinear damped systems with varying cofficients. The proposed method does not require a small parameter in the equation like the classical one.

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