Analytical Solutions of Second Order Strongly Nonlinear Differential Systems with Slowly Varying Coefficients

by

Chumki Rani Dey

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Dedicated to My

Beloved Parents

Declaration

This is to certify that the thesis work entitled "**Analytical Solutions of Second Order Strongly Nonlinear Differential Systems with Slowly Varying Coefficients**" has been carried out by **Chumki Rani Dey,** Roll No. **1451562** in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna 9203, Bangladesh. The above thesis work or any part of the thesis work has not been submitted anywhere for the award of any degree or diploma.

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Abstract

Considerable attention has been directed toward the study of strongly nonlinear differential systems. Nonlinear differential systems have been widely used in many areas of applied mathematics, physics, plasma and laser physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion.

The aim of the present study is to develop an analytical technique for obtaining the approximate solutions of second order strongly nonlinear differential systems with slowly varying coefficients and higher order nonlinearity in presence of small damping based on the He's homotopy perturbation method (HPM) and the extended form of the Krylov-Bogoliubov- Mitropolskii (KBM) method. Graphical representation of any physical system is important for its locations, amplitudes and phases. So the results obtained by the presented method are compared with those solutions obtained by the fourth order Runge-Kutta method in graphically.

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The following paper has been published from this thesis:

1. **Chumki Rani Dey, M. Saiful Islam, Deepa Rani Ghosh and M. Alhaz Uddin**, Approximate solutions of second order strongly and high order nonlinear Duffing equation with slowly varying coefficients in presence of small damping, **Progress in Nonlinear Dynamics and Chaos , Vol.4, No.1 (2016) 7-15**

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CHAPTER I

Introduction

In the last three decades, with the rapid development of nonlinear science, there has been ever-increasing interest from applied mathematicians, researchers, scientists and engineers in the approximate analytical techniques for addressing nonlinear differential systems. Though, it is very easy for us now to find the solutions of linear systems by means of computers, it is still very difficult to solve nonlinear differential systems either numerically or analytically.

The key to physical understanding of real-life physics is the use of approximation. Most of the physical problems that are faced by applied mathematicians, researchers, engineers, and physicists today exhibit certain essential features which preclude exact analytical solutions (except in the case of solitary solutions of nonlinear differential systems). Even if the exact solutions, with complicated and unfamiliar functions of the variable, can be found explicitly, they may be useless for mathematical and physical interpretation or numerical evaluation. By contrast, approximate solutions can strip away the overlying detail to show the essential relationships between the physical variables that are familiar to all applied mathematicians, researchers, scientists and engineers. In addition, most important information, such as the natural circular frequency of a nonlinear oscillation, depends on the initial condition (i.e. amplitude of oscillation), and will be lost during the procedure of numerical simulation.

Most of the real-life physical systems are modeled by nonlinear differential systems. Obtaining exact solutions for these nonlinear physical problems are very difficult and time consuming for researchers. Thus, applied mathematicians, researchers, engineers and scientists are tried to find new approaches to overcome this difficulties. The subject of differential equations constitutes a large and very important branch of modern mathematics. Numerous physical, electrical, mechanical, chemical, biological, mechanics in which we want to describe the motion of the body (plasma and laser physics, automobile, electron, or satellite) under the action of a given force, and many other relations appear mathematically in the form of differential equations that are linear or nonlinear, autonomous or non-autonomous. Also, in ecology and economics the differential equations are vastly used. Basically, many differential equations involving physical phenomena are nonlinear such as spring-mass systems, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, the motion of a pendulum, the motion of the rotating mass around another body, population dynamics etc. In mathematics and physics, linear generally means "simple" and nonlinear means "complicated". The theory for solving linear equations is very well developed because linear equations are simple enough to be solvable. Nonlinear differential equations can not be usually solved exactly and are the subject of much on-going research. In such situations, applied mathematicians, physicists and engineers convert the nonlinear differential equations into linear equations i.e., they linearize them by imposing some special conditions. Method of small oscillations is well-known example of the linearization for the physical problems. But, such a linearization is not always possible and when it is not, then the original nonlinear differential equation itself must be used.

At first van der Pol [1] paid attention to the new (self-excited) oscillations and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. Thus, this nonlinearity appears as the very essence of these phenomena and by linearizing the differential equations in the sense of small oscillations, one simply eliminates the possibility of investigating such problems. Thus, it is necessary to deal with the nonlinear differential equations directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential equations, there exist some methods such as perturbation method [2-39], homotopy perturbation method [40-47], variational iterative method [48], harmonic balance method [49] etc. Among the methods, the method of perturbations, i.e., asymptotic expansions in terms of a small parameter are first and foremost.

A perturbation method known as “the asymptotic averaging method” in the theory of nonlinear oscillations was first introduced by Russian famous scientists Krylov and Bogoliubov (KB) [2] in 1947. Primarily, the method was developed only for obtaining the periodic solutions of second order weakly conservative nonlinear differential systems. Later, the method of KB has been improved and justified by Bogoliubov and Mitropolskii [3] in 1961. In literature, this method is known as the Krylov-Bogoliubov-Mitropolskii (KBM) [2, 3] method.

A perturbation method is based on the following aspects: the equations to be solved are sufficiently "smooth" or sufficiently differentiable a number of times in the required regions of variables and parameters. The KBM [2, 3] method was developed for obtaining only the periodic solutions of second order weakly nonlinear differential equations without damping. Now a days, this method is used for obtaining the solutions of second, third and fourth order weakly nonlinear differential systems for oscillatory, damped oscillatory, over damped, and critically damped cases by imposing some special restrictions with quadratic and cubic nonlinearities. Several authors [4-39] have investigated and developed many significant results concerning the solutions of the weakly nonlinear differential systems based on the **KBM** method.

The method of KB [2] is an asymptotic method in the sense that $£ \to 0$. An asymptotic series itself may not be convergent, but for a fixed number of terms, the approximate solution approaches toward the exact solution. Two widely spread methods in this theory are mainly used in the literature; one is averaging asymptotic KBM method and the other is multiple-time scale method [36-38]. The KBM method is particularly convenient and extensively used technique for determining the approximate solutions among the methods used for studying the weakly nonlinear differential systems with small nonlinearity specially cubic nonlinearity. The KBM method starts with the solution of linear equation (sometimes called the generating solution of the linear equation) assuming that in the nonlinear case, the amplitude and phase variables in the solution of the linear differential equation are time dependent functions instead of constants. This method introduces an additional condition on the first derivative of the assumed solution for determining the solution of second order nonlinear differential systems. The KBM method demands that the asymptotic solutions are free from secular terms. These assumptions are mainly valid for second and third order equations. But, for the fourth order differential equations, the correction terms sometimes contain secular terms, although the solution is generated by the classical KBM asymptotic method. For this reason, the traditional solutions fail to explain the proper situation of the systems. To remove the presence of secular terms for obtaining the desired results, one needs to impose some special conditions.

Ji-Huan He [40-43] has developed a homotopy perturbation technique for solving second order strongly nonlinear differential systems without damping effects. In this method the solution is considered as the summation of an infinite series which converges rapidly to the exact solutions. This technique has been employed to solve a large variety of nonlinear differential equations. Uddin *et al.* [44], Uddin and Sattar [45], Ghosh *et al.* [46] have been extended the homotopy perturbation method to damped nonlinear differential systems.

In this thesis, He's homotopy perturbation method (HPM) has been extended for obtaining the approximate analytical solutions of second order strongly and weakly nonlinear differential systems with slowly varying coefficients and high order nonlinearity for small amplitude in presence of small damping based on the extended form of the KBM method.

In Chapter II, the review of literature is presented. In **Chapter III**, an approximate analytical technique has been extended for solving second order strongly and weakly nonlinear differential systems with slowly varying coefficients and high order nonlinearity for small amplitude in presence of small damping**.** Finally, **in Chapter IV**, the concluding remarks are given.

CHAPTER II

Literature Review

The nonlinear differential equations are generally difficult to solve and their exact solutions are difficult to obtain. But, mathematical formulations of many physical problems often results in differential equations that are linear or nonlinear. In many situations, linear differential equation is substituted for a nonlinear differential equation, which approximates the original equation closely enough to give expected results. In many cases such a linearization is not possible and when it is not, the original nonlinear differential equation must be considered directly. During last several decades in the $20th$ century, some famous Russian scientists like Krylov and Bogoliubov [2], Bogoliubov and Mitropolskii [3], Mitropolskii [4], have investigated the nonlinear dynamics. For solving nonlinear differential equations, there exist some methods. Among the methods, the method of perturbations, i.e., an asymptotic expansion in terms of small parameter is foremost. In 1947, Krylov and Bogoliubov (KB) [2] considered the equation of the form

$$\ddot{x} + \mu^2 x = \pounds\, f(x, \dot{x}, t, \pounds), \tag{2.1}$$

where $\ddot{x}$ denotes the second order derivative with respect to *t*, $\pounds$ is a small positive parameter and *f* is a power series in $\pounds$, whose coefficients are polynomials in $x, \dot{x}, \sin t$ and $\cos t$ and the procedure proposed by Krylov and Bogoliubov [2]. In general, f does not contain either $\pounds$ or t explicitly. In KBM [2, 3] method, it is assumed that the amplitude and phase variables in the solution of the linear equations are time dependent functions instead of constants in nonlinear differential systems. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the desired results. To describe the behavior of nonlinear oscillations by the solutions obtained by the perturbation method, Poincare [5] discussed only periodic solutions. Duffing [6] has investigated many significant results for the periodic solutions of the following damped nonlinear differential systems

$$\ddot{x} + 2k\,\dot{x} + \mu^2 x = -\pounds\, x^3. \tag{2.2}$$

Sometimes different types of nonlinear phenomena occur, when the amplitude of a dynamic system is less than or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation having these phenomena is

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0. \tag{2.3}$$

In literature, this equation is known as van der Pol [1] equation and is used in electrical circuit theory. Kruskal [7] has extended the KB [2] method to solve the fully nonlinear differential equation of the following form

$$\ddot{x} = F(x, \dot{x}, \varepsilon). \tag{2.4a}$$

Cap [8] has studied nonlinear differential system of the form

$$\ddot{x} + \omega^2 x = \varepsilon\, F(x, \dot{x}). \tag{2.4b}$$

Generally, F does not contain ε or t explicitly, so the equation (2.1) leads to

$$\ddot{x} + \omega^2 x = \varepsilon\, f(x, \dot{x}). \tag{2.5}$$

In the treatment of nonlinear oscillations by the perturbation methods, only periodic solutions are discussed, transients are not considered by different investigators, where as KB [2] have discussed transient response.

When $\varepsilon = 0,$ the equation (2.5) reduces to linear equation and its solution can be obtained as

$$x = a\cos(\omega\, t + \varphi). \tag{2.6}$$

where a and φ are arbitrary constants and the values of these arbitrary constants are determined by using the given initial conditions.

When $\varepsilon \neq 0,$ but is sufficiently small, then in KB [2] it is assumed that the solution of equation (2.5) is still given by equation (2.6) together with the derivative of the form

$$\dot{x} = -a\omega\sin(\omega\, t + \varphi). \tag{2.7}$$

where a and φ are functions of t, rather than being constants. In this case, the solution of equation (2.5) is

$$x = a(t)\cos(\omega\, t + \varphi(t)) \tag{2.8}$$

and the derivative of the solution is

$$\dot{x} = -a(t)\omega\sin(\omega\, t + \varphi(t)). \tag{2.9}$$

Differentiating the assumed solution equation (2.8) with respect to time t, we obtain

$$\dot{x} = \dot{a}\cos\psi - a\omega\sin\psi - a\dot{\varphi}\sin\psi\,, \quad \psi = \omega\, t + \varphi(t). \tag{2.10}$$

Using the equations (2.7) and (2.10), we get

$$\dot{a}\cos\psi = a\dot{\varphi}\sin\psi. \tag{2.11}$$

Again, differentiating equation (2.9) with respect to *t*, we have

$$\ddot{x} = -\dot{a}\omega\sin\psi - a\omega^2\cos\psi - a\omega\dot{\varphi}\cos\psi. \tag{2.12}$$

Putting the value of $\ddot{x}$ from equation (2.12) into the equation (2.5) and using equations (2.8) and (2.9), we obtain

$$\dot{a}\omega\sin\psi + a\omega\dot{\varphi}\cos\psi = -\varepsilon f(a\cos\psi, -a\omega\sin\psi). \tag{2.13}$$

Solving equations (2.11) and (2.13), we have

$$\dot{a} = -\frac{\varepsilon}{\omega}\sin\psi\, f(a\cos\psi, -a\omega\sin\psi), \tag{2.14}$$

$$\dot{\varphi} = -\frac{\varepsilon}{a\omega}\cos\psi\, f(a\cos\psi, -a\omega\sin\psi). \tag{2.15}$$

It is observed that, a basic differential equation (2.5) of the second order in the unknown x, leads to two first order differential equations (2.14) and (2.15) in the unknowns a and φ to get its solutions.

Moreover, $\dot{a}$ and $\dot{\varphi}$ are proportional to ε and they are slowly varying functions of the time t with period $T = \frac{2\pi}{\omega}$. It is noted that these first order differential equations are now written in terms of the amplitude a and phase φ as dependent variables. Therefore, the right sides of equations (2.14) and (2.15) show that both $\dot{a}$ and $\dot{\varphi}$ are periodic functions of period T. In this case, the right-hand terms of these equations contain a small parameter ε and also contain both a and φ, which are slowly varying functions of the time t with period $T = \frac{2\pi}{\omega}$. We can transform the equations (2.14) and (2.15) into more convenient form. Now, expanding $\sin\psi\, f(a\cos\psi, -a\omega\sin\psi)$ and $\cos\psi\, f(a\cos\psi, -a\omega\sin\psi)$ in a Fourier series with phase ψ, the first approximate solution of equation (2.5) by averaging equations (2.14) and (2.15) with period $T = \frac{2\pi}{\omega}$, is obtained as

$$\begin{aligned} \langle\dot{a}\rangle &= -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi}\sin\psi\, f(a\cos\psi, -a\omega\sin\psi)\,d\psi, \\ \langle\dot{\varphi}\rangle &= -\frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi}\cos\psi\, f(a\cos\psi, -a\omega\sin\psi)\,d\psi, \end{aligned} \tag{2.16}$$

where a and φ are independent of time t under the integral signs. Later, KB method has been extended mathematically by Bogoliubov and Mitropolskii [3], and has been extended to non-stationary vibrations by Mitropolskii [4]. They have assumed the solution of equation (2.5) in the following form

$$x = a\cos\psi + \varepsilon\, u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \ldots\ldots\ldots\ldots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \qquad (2.17)$$

where u_k, $(k = 1, 2, \ldots\ldots., n)$ are periodic functions of ψ with period 2π, and the terms a and ψ are functions of time t and is obtained by solving the following first order ordinary differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots\ldots\ldots. + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \\ \dot{\psi} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots\ldots. + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}). \end{aligned} \qquad (2.18\text{ a, b})$$

The functions u_k, A_k and B_k, $(k = 1, 2, \ldots\ldots., n)$ are to be chosen in such a way that the equation (2.17), after replacing a and ψ by the functions defined in equation (2.18), is a solution of equation (2.5). Since there are no restrictions in choosing functions A_k and B_k, it generates the arbitrariness in the definitions of the functions u_k (Bogoliubov and Mitropolskii [3]). To remove this arbitrariness, the following additional conditions are imposed

$$\begin{aligned} &\int_0^{2\pi} u_k(a,\psi)\cos\psi \, d\psi = 0, \\ &\int_0^{2\pi} u_k(a,\psi)\sin\psi \, d\psi = 0. \end{aligned} \qquad (2.19\text{a, b})$$

Secular terms are removed by using these conditions in all successive approximations. Differentiating equation (2.17) two times with respect to t, and then substituting the values of $\ddot{x}, \dot{x}$ and x into equation (2.5), and using the relations equation (2.18) and equating the coefficients of ε^k, $(k = 1, 2, \ldots\ldots., n)$, we obtain

$$\omega^2((u_k)_{\psi\psi} + u_k) = f^{(k-1)}(a,\psi) + 2\omega\,(a B_k \cos\psi + A_k \sin\psi), \qquad (2.20)$$

where $(u_k)_\psi$ denotes partial derivative with respect to ψ.

$$\begin{aligned} f^{(0)}(a,\psi) &= f(a\cos\psi, -a\omega\sin\psi), \\ f^{(1)}(a,\psi) &= u_1 f_x(a\cos\psi, -a\omega\sin\psi) + (A_1\cos\psi - aB_1\sin\psi + \omega(u_1)_\psi) \\ &\quad \times f_{\dot{x}}(\cos\psi, -a\omega\sin\psi) + (aB_1^2 - A_1\frac{dA_1}{da})\cos\psi \\ &\quad + (2A_1B_1 - aA_1\frac{dB_1}{da})\sin\psi - 2\omega(A_1(u_1)_{a\psi} + B_1(u_1)_{\psi\psi}). \end{aligned} \qquad (2.21\text{a, b})$$

Here $f^{(k-1)}$ is a periodic function of φ with period 2π which depends also on the amplitude a. Therefore, $f^{(k-1)}$ and u_k can be expanded in a Fourier series in the following form

$$f^{(k-1)}(a,\varphi) = g_0^{(k-1)}(a) + \sum_{n=1}^{\infty}(g_n^{(k-1)}(a)\cos n\varphi + h_n^{(k-1)}(a)\sin n\varphi),$$
$$u_k(a,\varphi) = v_0^{(k-1)}(a) + \sum_{n=1}^{\infty}(v_n^{(k-1)}(a)\cos n\varphi + \omega_n^{(k-1)}(a)\sin n\varphi), \tag{2.22a, b}$$

where

$$g_0^{(k-1)}(a) = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\varphi, -a\omega\sin\varphi)\,d\varphi. \tag{2.23}$$

Here, $v_1^{(k-1)} = \omega_1^{(k-1)} = 0$ for all values of k, since both integrals of equation (2.19) are vanished. Substituting these values into the equation (2.20), we obtain

$$\omega^2 v_0^{(k-1)}(a) + \sum_{n=2}^{\infty}\omega^2(1-n^2)[v_n^{(k-1)}(a)\cos n\varphi + \omega_n^{(k-1)}(a)\sin n\varphi]$$
$$= g_0^{(k-1)}(a) + (g_1^{(k-1)}(a) + 2\omega a B_k)\cos n\varphi + (h_1^{(k-1)}(a) + 2\omega A_k)\sin\varphi \tag{2.24}$$
$$+\sum_{n=2}^{\infty}[g_n^{(k-1)}(a)\cos n\varphi + h_n^{(k-1)}(a)\sin n\varphi].$$

Now, equating the coefficients of the harmonics of the same order, we get

$$g_1^{(k-1)}(a) + 2\omega a B_k = 0, \quad h_1^{(k-1)}(a) + 2\omega A_k = 0, \quad v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2},$$
$$v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad \omega_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad n \geq 1. \tag{2.25}$$

These are the sufficient conditions to obtain the desired order of approximations. For the first approximation, we have

$$A_1 = -\frac{h_1^{(0)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos t\varphi, -a\omega\sin\varphi)\sin\varphi\,d\varphi,$$
$$B_1 = -\frac{g_1^{(0)}(a)}{2a\omega} = -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos t\varphi, -a\omega\sin\varphi)\cos\varphi\,d\varphi. \tag{2.26a, b}$$

Thus, the variational equations in equation (2.18) become

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\varphi, -a\omega\sin\varphi)\sin\varphi\,d\varphi,$$
$$\dot{\varphi} = \omega - \frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi} f(a\cos\varphi, -a\omega\sin\varphi)\cos\varphi\,d\varphi. \tag{2.27a, b}$$

It is seen that the equation (2.27) are similar to the equation (2.16). Thus, the first approximate solution obtained by Bogoliubov and Mitropolskii [3] is identical to the original solution obtained by KB [2]. The correction term u_1 is obtained from equation (2.22) by using equation (2.25) as

$$u_1 = \frac{g_0^{(0)}(a)}{\mu^2} + \sum_{n=2}^{\tilde{}} \frac{g_n^{(0)}(a)\cos n\cdot + h_n^{(0)}(a)\sin n\cdot}{\mu^2(1-n^2)} \tag{2.28}$$

The solution equation (2.17) together with u_1 is known as the first order improved solution in which a and $\cdot$ are obtained from equation (2.27). If the values of the functions A_1 and B_1 are substituted from equation (2.26) into the second relation of equation (2.21b), the function $f^{(1)}$ is determined. In the similar way, the functions A_2, B_2 and u_2 can be found. Therefore, the determination of the second order approximation is completed. The KB [2] method is very similar to that of van der Pol [1] method and related to it. van der Pol [1] has applied the method of variation of constants to the basic solution $x = a\cos\mu t + b\sin\mu t$ of $\ddot{x} + \mu^2 x = 0,$ on the other hand KB [2] has applied the same method to the basic solution $x = a\cos(\mu t + \ddot{\ })$ of the same equation. Thus, in the KB [2] method the varied constants are a and $\ddot{\ }$, while in the van der Pol's method the constants are a and b. The KB [2] method seems more interesting from the point of view of physical applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillations. The solution of the equation (2.4a) is based on recurrent relations and is given as the power series of the small parameter. Cap [8] has solved the equation (2.4b) by using elliptical functions in the sense of KB [2]. The KB [2] method has been extended by Popov [9] to damped nonlinear differential systems represented by the following equation

$$\ddot{x} + 2k\dot{x} + \mu^2 x = \pounds\, f(\dot{x}, x), \tag{2.29}$$

where $2k\dot{x}$ is the linear damping force and $0 < k < \mu$. It is noteworthy that, because of the importance of the Popov's method in the physical nonlinear differential systems, involving damping force, Mendelson [10] and Bojadziev [11] have retrieved Popov's [9] results. In case of damped nonlinear differential systems, the first equation of equation (2.18a) has been replaced by

$$\dot{a} = -k\,a + \pounds\, A_1(a) + \pounds^2 A_2(a) + \dots\dots\dots\dots + \pounds^n A_n(a) + O(\pounds^{n+1}). \tag{2.18a}$$

Murty and Deekshatulu [12] have developed a simple analytical method to obtain the time response of second order over damped nonlinear differential systems with small nonlinearity represented by the equation (2.29), based on the KB [2] method. In accordance to the KBM [2, 3] method, Murty *et al.* [13] have found a hyperbolic type asymptotic solution of an over damped system represented by the nonlinear differential equation (2.29), i.e., in the case $k > \mu$. They have used hyperbolic functions, $\cosh\varphi$ and $\sinh\varphi$ instead of their circular counterpart, which are used by KBM [2, 3], Popov [9] and Mendelson [10]. Murty [14] has presented a unified KBM method for solving the nonlinear differential systems represented by the equation (2.29), which cover the undamped, damped and over-damped cases. Bojadziev and Edwards [15] have investigated solutions of oscillatory and non-oscillatory systems represented by equation (2.29) when k and μ are slowly varying functions of time *t*. Initial conditions may be used arbitrarily for the case of oscillatory or damped oscillatory process. But, in case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions (Murty *et al.* [13], Murty [14], Bojadziev and Edwards [15]). Arya and Bojadziev [16, 17] have examined damped oscillatory systems and time dependent oscillating systems with slowly varying parameters and delay. Sattar [18] has developed an asymptotic method to solve a second order critically damped nonlinear differential system represented by equation (2.29). He has found the asymptotic solution of the equation (2.29) in the following form

$$x = a(1+\psi) + \varepsilon u_1(a,\psi) + \dots\dots\dots\dots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \qquad (2.30)$$

where a is defined by the equation (2.18a) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \varepsilon^2 C_2(a) + \dots\dots\dots\dots + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}) \qquad (2.18b)$$

Also Sattar [19] has extended the KBM asymptotic method for three dimensional over damped nonlinear systems. Osiniskii [20] has extended the KBM method to the following third order nonlinear differential equation

$$\dddot{x} + c_1 \ddot{x} + c_2 \dot{x} + c_3 x = \varepsilon f(\ddot{x}, \dot{x}, x), \qquad (2.31)$$

where ε is a small positive parameter and *f* is a given nonlinear function. He has assumed the asymptotic solution of equation (2.31) in the form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \dots\dots\dots\dots + \varepsilon^n u_n(a,b,\psi) + O(\varepsilon^{n+1}), \qquad (2.32)$$

where each u_k $(k = 1, 2, ..., n)$ is a periodic function of φ with period 2π and a, b and φ are functions of time t, and they are given by

$$\begin{aligned}
\dot{a} &= -\lambda\, a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + ... + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \\
\dot{b} &= -\mu\, b + \varepsilon B_1(b) + \varepsilon^2 B_2(b) + ... + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}), \qquad (2.33a, b,c) \\
\dot{\varphi} &= \omega + \varepsilon C_1(b) + \varepsilon^2 C_2(b) + ... + \varepsilon^n C_n(b) + O(\varepsilon^{n+1}),
\end{aligned}$$

where $-\lambda, -\mu \pm i\omega$ are the eigen values of the equation (2.31) when $\varepsilon = 0$.

Lin and Khan [21] have also used the KBM method for solving biological problems. Proskurjakov [22] has investigated periodic solutions of nonlinear systems by using the Poincare and KBM methods, and compared the two solutions. Mulholland [23] has studied nonlinear oscillations governed by a third order differential equation. Lardner and Bojadziev [24] have investigated the solutions of nonlinear damped oscillations governed by a third order partial differential equation. They have introduced the concept of “couple amplitude” where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b.

Alam and Sattar [25] have extended the KBM method for solving third order critically damped autonomous nonlinear differential systems. Alam and Sattar [26] have presented a unified KBM method for solving third order nonlinear differential systems. Alam [27] has extended the KBM method to over damped nonlinear differential systems. Also, Alam *et al*. [28] have extended the KBM method to certain non-oscillatory nonlinear differential systems with slowly varying coefficients. Alam [29] has also presented a unified KBM method, which is not the formal form of the original KBM method for solving *nth*, $(n \geq 2, 3)$ order nonlinear differential systems. The solution contains some unusual variables, yet this solution is very important. Alam [30] has extended the KBM method presented in [25] to find the approximate solutions of critically damped nonlinear differential systems in presence of different damping forces by considering the different sets of variational equations. Alam [31] has also extended the KBM method for solving third order over damped nonlinear differential system when two of the eigen values are almost equal (i.e., the system is near to the critically damped) and the rest is small. Alam [32] has presented an asymptotic method for certain type of third order non-oscillatory nonlinear differential systems, which gives desired results when the damping force is near to the critically damping force. Alam [33] has investigated a unified KBM method for solving *nth* $(n = 2, 3)$ order nonlinear differential equation with varying coefficients. Roy and Alam [34] have discussed the effect of higher approximation of Krylov- Bogoliubov-

Mitropolskii solution and matched asymptotic differential systems with slowly varying coefficients and damping near to a turning point. Alam and Sattar [35] have developed an asymptotic method for third order nonlinear systems with slowly varying coefficients. Nayfeh [36, 37] and Murdock [38] have developed perturbation methods and theory for obtaining the solutions of weakly nonlinear differential systems. Sachs *et al.* [39] have developed a simple ODE model of tumor growth and anti-angiogenic or radiation treatment.

The HPM was first proposed by the Chinese mathematician Ji Huan He [40]. The essential idea of this method is to introduce a homotopy parameter, say p, which varies from 0 to 1. At $p = 0$, the system of equations usually has been reduced to a simplified form which normally admits a rather simple solution. As p gradually increases continuously toward 1, the system goes through a sequence of deformations, and the solution at each stage is closed to that at the previous stage of the deformation. Eventually at $p = 1$ the system takes the original form of the equation and the final stage of the deformation give the desired solution.

He [40] has investigated a novel homotopy perturbation technique for finding a periodic solution of a general nonlinear oscillator for conservative systems. He [40] has considered the following nonlinear differential equation in the form

$$A(u) - f(r) = 0, \; r \in \Omega, \tag{2.34}$$

with the boundary conditions

$$B(u, \frac{\partial u}{\partial t}) = 0, \quad r \in \Gamma, \tag{2.35}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of the domain Ω. Then He [40] has written Eq. (2.34) in the following form

$$L(u) + N(u) - f(r) = 0, \tag{2.36}$$

where L is linear part, while N is nonlinear part. He [40] has constructed a homotopy $v(r, p) : \Omega \times [0,1] \rightarrow \Re$ which satisfies

$$\mathrm{H}(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(u) - f(r)] = 0, \; p \in [0,1], \; r \in \Omega \tag{2.37a}$$

or

$$\mathrm{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{2.37b}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of equation (2.34), which satisfies the boundary conditions. Obviously, from equation (2.37), it becomes

$$\mathrm{H}(v,0) = L(v) - L(u_0) = 0, \tag{2.38}$$

$$\mathrm{H}(v,1) = A(v) - f(r) = 0. \tag{2.39}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. He [40] has assumed the solution of Eq. (2.37) as a power series of p in the following form

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{2.40}$$

The approximate solution of Eq. (2.34) is given by setting $p = 1$ in the form

$$u = v_0 + v_1 + v_2 + \cdots. \tag{2.41}$$

The series (2.41) is convergent for most of the cases, and also the rate of convergence depends on how one chooses $A(u)$.

He [41] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities. He [42] has developed some new approaches to solve Duffing equation with strongly and high order nonlinearity without damping. Also, He [43] has presented a new interpretation of homotopy perturbation method. Uddin *et al.* [44] and Uddin and Sattar [45] have presented an approximate technique for solving second order strongly nonlinear differential systems with damping by combining the He's [40-43] homotopy perturbation and the extended form of the KBM [2-4] methods. Recently, Ghosh *et al.* [46] have also developed an approximate technique for solving second order strongly nonlinear differential systems with high order nonlinearity in presence of small damping based on He's [40-43] homotopy perturbation and the extended form of the KBM [2-4] methods with constant coefficients. Belendez *et al.* [47] have applied He's homotopy perturbation method to Duffing harmonic oscillator. He [48] has presented a variational iteration method for solving strongly nonlinear differential systems. Ghadimi and Kaliji [49] have presented an application of the harmonic balance method on nonlinear equation. Ganji *et al.* [50] have presented an approximate solutions to van der Pol damped nonlinear oscillators by means of He's energy balance method.

CHAPTER III

Approximate Solutions of Second Order Strongly and High Order Nonlinear Differential Systems with Slowly Varying Coefficients in Presence of Small Damping

3.1 Introduction

The study of nonlinear differential systems is of great importance not only in all areas of physics but also in engineering and other disciplines, since most of the phenomena in our real-life problems are nonlinear and are described by nonlinear differential equations. Most of these nonlinear differential systems occur in nature with slowly varying coefficients in presence of small damping. The common perturbation methods for constructing the approximate analytical solutions to the nonlinear differential equations are the Krylov-Bogoliubov- Mitropolskii (KBM) [2, 3] method, the Lindstedt-Poincare (LP) method [36-38] and the method of multiple time scales [36]. Almost all perturbation methods are based on an assumption that small parameter must exist in the equations. Arya and Bojadziev [17] have presented the analytical technique for time depending oscillating systems with slowly varying parameters, damping, and delay. Roy and Alam [34] have presented the effects of higher approximation of Krylov-Bogoliubov-Mitropolskii solution and matched asymptotic differential systems with slowly varying coefficients and damping near to a turning point for weakly nonlinear differential systems. He [40] has investigated the homotopy perturbation technique. In another paper, He [41] has developed a coupling method of a homotopy perturbation technique and a perturbation technique for strongly nonlinear differential systems. He [42] has also presented a new interpretation of homotopy perturbation method for solving strongly nonlinear differential systems. Uddin *et al.* [44] have presented an approximate technique for solving strongly nonlinear differential systems with cubic nonlinearity in presence of damping effects. Uddin and Sattar [45] have presented an approximate technique to Duffing` equation with small damping and slowly varying coefficients for cubic nonlinearity. Recently Ghosh *et al*. [46] have presented an approximate technique for solving second order strongly and high order nonlinear differential systems in presence of small damping without slowing varying coefficients. Belendez *et al.* [47] have presented the application of He's homotopy perturbation method to the Duffing harmonic oscillators. The authors [40-44] have studied the differential systems with cubic nonlinearity in absence of damping effects. But most of the physical and oscillating systems encounter in presence of small damping in nature and it plays an important role to the nonlinear physical systems. In this thesis, we have developed an approximate analytical technique for solving second order differential systems having strong and high order nonlinearities with slowly varying coefficients in presence of small damping based on the He's homotopy perturbation [40-44] and the extended form of the KBM [2-4] methods. Figures are provided to compare between the solutions obtained by the presented method with the corresponding numerical (considered to be exact) solutions obtained by fourth order Runge-Kutta method.

3.2 The method

We are interested to consider a second order differential system having strong and high order nonlinearity [34] with slowly varying coefficients in presence of small damping in the following form

$$\ddot{x} + 2k(²)\dot{x} + e^{-²} x = -£_1 f(x), \quad k << 1, \tag{3.1}$$

subject to the initial conditions

$$x(0) = b_0, \ \dot{x}(0) = 0, \tag{3.2}$$

where $k > 0$ and $2k$ is the linear damping coefficient which varies slowly with time t, $² = £t$ is the slowly varying time, $£$ is a small parameter, $£_1$ is parameter not necessarily small, b_0 is positive constant and known as the initial amplitude of the systems and $f(x)$ is a given high order nonlinear function which satisfies the following condition

$$f(-x) = -f(x)\,. \tag{3.3}$$

For simplicity, we are going to use the following transformation [44]

$$x = y(t)e^{-kt}. \tag{3.4}$$

Differentiating equation (3.4) twice with respect to time t and substituting $\ddot{x}, \dot{x}$ together with x into equation (3.1) and then simplifying them, we obtain the following equation

$$\ddot{y} + (e^{-²} - k^2)y = -£_1 e^{kt} f(ye^{-kt}). \tag{3.5}$$

According to the homotopy perturbation method [40-46], equation (3.5) can be re-written as the following form

$$\ddot{y} + \omega^2 y = \lambda\, y - \varepsilon_1 e^{kt} f(y\, e^{-kt}), \tag{3.6}$$

where

$$\omega^2 = e^{-\tau} - k^2 + \lambda. \tag{3.7}$$

Herein λ is an unknown constant which can be determined by eliminating the secular terms. However, for a damped nonlinear differential system ω is a time dependent function and it varies slowly with time *t*. To handle this situation, we need to use the extended form of the KBM [2-4] method. According to this technique, we are going to choose the first approximate analytical solution of equation (3.6) in the following form

$$y = b\cos\varphi, \tag{3.8}$$

where b and φ represent the amplitude and phase variable respectively and they vary slowly with time t. According to the KBM [2, 3] method b and φ satisfy the following first order differential equations

$$\begin{aligned} \dot{b} &= \varepsilon A_1(b,\tau) + \varepsilon^2 A_2(b,\tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + \varepsilon B_1(b,\tau) + \varepsilon^2 B_2(b,\tau) + \cdots, \end{aligned} \tag{3.9 a, b}$$

where ε is a small positive parameter and A_j and B_j are unknown functions. Now differentiating equation (3.8) twice with respect to time *t*, utilizing the relations equation (3.9) and substituting $\ddot{y}$ and y into equation (3.6) and then equating the coefficients of $\sin\varphi$ and $\cos\varphi$, we obtain the value of the unknown functions A_1 and B_1 as the form

$$A_1 = -\omega' b/(2\omega), \; B_1 = 0, \tag{3.10}$$

where prime denotes differentiation with respect to slowly varying time τ. Now putting equation (3.8) into equation (3.4) and equation (3.10) into equation (3.9) we obtain the following equations

$$x = b e^{-kt} \cos\varphi, \tag{3.11}$$

and

$$\begin{aligned} \dot{b} &= -\varepsilon\, \omega' b/(2\omega), \\ \dot{\varphi} &= \omega(\tau). \end{aligned} \tag{3.12 a, b}$$

Thus, the first approximate solution of equation (3.1) is obtained by equation (3.11) with the assist of equations (3.7) and (3.12). Usually, the integration of equation (3.12 a, b) is performed by well-known techniques of calculus [36-38], but sometimes they are solved by a numerical procedure [13-35, 44-46].

3.3 Example

For implementing and justifying the above procedure, we are going to assume the following Duffing equation with slowly varying coefficients in presence of small damping as the form

$$\ddot{x} + 2k(^2)\dot{x} + e^{-^2} x = -\pounds_1 x^5, \qquad (3.13)$$

where $f(x) = x^5$. Now using the transformation [44] equation (3.4) into equation (3.13) and then simplifying them, we obtain

$$\ddot{y} + (e^{-^2} - k^2) y = -\pounds_1 e^{-4k\,t} y^5. \qquad (3.14)$$

According to the homotopy perturbation [40-46] method, equation (3.14) can be re-written as

$$\ddot{y} + \mu^2 y = \text{ª}\, y - \pounds_1 e^{-4k\,t} y^5, \qquad (3.15)$$

where

$$\mu^2 = e^{-^2} - k^2 + \text{ª}. \qquad (3.16)$$

According to the extended form of the KBM [2-4] method, the solution of equation (3.15) is given by equation (3.8). In presence of secular terms, the solution will be non-uniform and break down. So, researchers must be needed to remove the secular terms from their obtained solutions for finding the uniform solutions. For avoiding the secular terms in particular solution of equation (3.15), setting the coefficients of the $\cos\cdot$ terms is zero, we obtain

$$\text{ª}\, b - \frac{5\pounds_1 b^4 e^{-4k\,t}}{8} = 0, \qquad (3.17)$$

For the nontrivial solution *i.e.*, $b \neq 0,$ equation (3.17) leads to

$$\text{ª} = \frac{5\pounds_1 b^3 e^{-4k\,t}}{8}. \qquad (3.18)$$

Substituting the value of ª from equation (3.18) into equation (3.16), it yields

$$\mu^2 = e^{-2} - k^2 + \frac{5£_1 b^3 e^{-4kt}}{8}. \quad (3.19)$$

This is a time dependent frequency equation of the given nonlinear differential systems. As $t \to 0$, equation (3.19) leads to

$$\mu_0 = \mu(0) = \sqrt{1 - k^2 + \frac{5£_1 b_0^3}{8}}, \quad (3.20)$$

which is known as the constant frequency equation of the given nonlinear differential systems. Now integrating the equation (3.12 a), we get

$$b = b_0 \sqrt{\frac{\mu_0}{\mu}}, \quad (3.21)$$

where b_0 is a constant of integration which represents the initial amplitude of the nonlinear differential systems. Now putting equation (3.21) into equation (3.19), we obtain the following bi-quadratic equation

$$\mu^4 - q\mu^2 - r = 0, \quad (3.22)$$

where

$$q = e^{-2} - k^2, \quad r = \frac{5£_1 b_0^4 \mu_0^2 e^{-4kt}}{4}. \quad (3.23)$$

Solving equation (3.22) for the real angular frequency μ, we obtain

$$\mu = \sqrt{\frac{q + \sqrt{q^2 + 4r}}{2}}, \quad (3.24)$$

The solution of equation (3.12 b) becomes

$$\cdot = \cdot_0 + \int_0^t \mu(t)\,dt, \quad (3.25)$$

where $\cdot_0$ is the initial phase and μ is given by equation (3.24). Therefore, the first approximate solution of equation (3.13) is obtained by equation (3.11) and the amplitude b and the phase $\cdot$ are calculated from equation (3.21) and equation (3.25) respectively with the help of equations (3.23)-(3.24). Thus, the determination of the first order analytical approximate solution of equation (3.13) is completed by the presented approximate analytical technique.

3.4 Initial conditions

The initial conditions of $\ddot{x} + 2k(²)\dot{x} + e^{-²} x = -£_1 x^5$ are obtained as

$$x(0) = b_0 \cos \cdot_0 ,$$
$$\dot{x}(0) = \left(\frac{b_0(2£ + 5£_1 k b_0^4)}{4£ \mu_0 (4\mu_0^2 + 2)} - k b_0 \right) \cos \cdot_0 - b_0 \mu_0 \sin \cdot_0 . \tag{3.26}$$

In general, the initial conditions $[x(0), \dot{x}(0)]$ are specified. Then one has to solve nonlinear algebraic equation in order to determine the initial amplitude b_0 and the initial phase $\cdot_0$ that appear in the solutions from the initial conditions equation (3.26).

3.5 Results and discussion

For specified values of the parameters $k, £$, and $£_1$, the initial conditions are calculated. For those initial conditions, the first approximate solutions of the considered Duffing equation are plotted along with the corresponding numerical solution of that equation in Figs. 3.1-3.2. In this thesis He's homotopy perturbation technique has been extended for solving second order strong and high order nonlinear differential systems with slowly varying coefficients in presence of small damping based on the extended form of the KBM method [2-4]. From the Figs. 3.1-3.2, it is seen that the first approximate solutions show a good agreement with the corresponding numerical solutions for small several damping. The presented method is very simple in its principle, and is very easy to implement for both strong $(£_1 = 1.0)$ and weak $(£_1 = 0.1)$ nonlinear differential systems with slowly varying coefficients in presence of small damping. The variational equations of the amplitude and phase variables appear in a set of first order nonlinear ordinary differential equations. The integrations of these variational equations are obtained by well-known techniques of calculus [36-38]. In a lack of analytical solutions, they are solved by numerical procedure [13-35, 44-46]. The amplitude and phase variables change slowly with time t. The behavior of amplitude and phase variables characterizes the oscillating processes. Moreover, the variational equations of amplitude and phase variables are used to investigate the stability of the nonlinear differential equations. Ji-Huan He [40-42] has developed homotopy perturbation for conservative nonlinear differential systems. But the presented method is valid for non-conservative nonlinear differential systems. The presented method can also overcome some limitations of the classical perturbation techniques; it does not require a small parameter $(i.e, £_1 = 1.0)$ in the equations. The advantage of the presented method is that the first order approximate analytical solutions show a good agreement with the corresponding numerical solutions in presence of small damping for small amplitude. The method has been successfully implemented for solving the second order slowly varying differential systems with high order nonlinearity in presence of small damping for both strongly and weakly nonlinear cases.

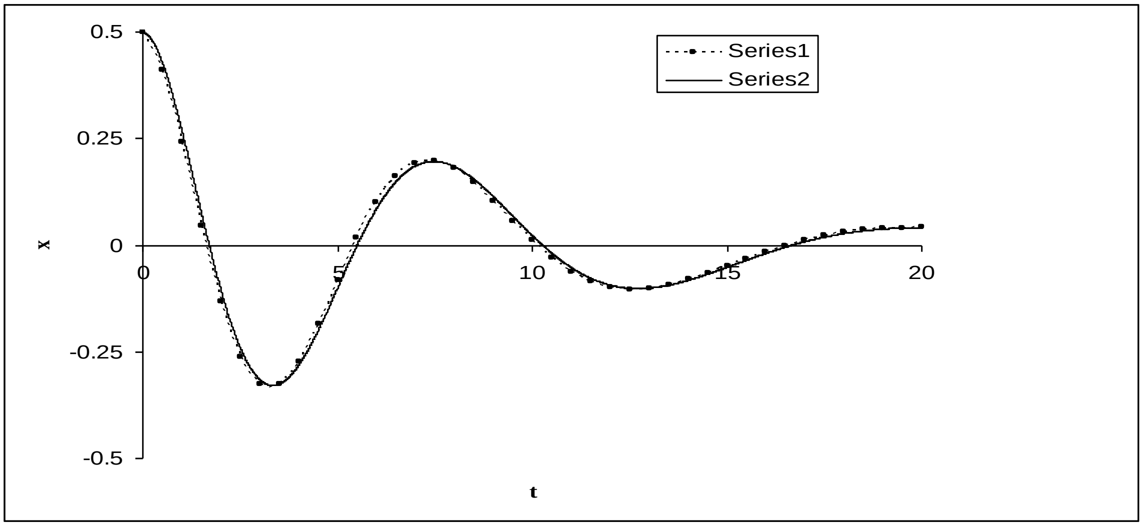

Fig. 3.1 (a) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dotted lines) by the presented analytical technique with the initial conditions $b_0 = 0.5, \ \cdot_{\,0} = 0$ or $[x(0) = 0.5, \ \dot{x}(0) = -0.02339]$ with $k = 0.15, \ £ = 0.1, \ £_1 = 1.0$ and $f = x^5$. Corresponding numerical solution is denoted by - (solid line).

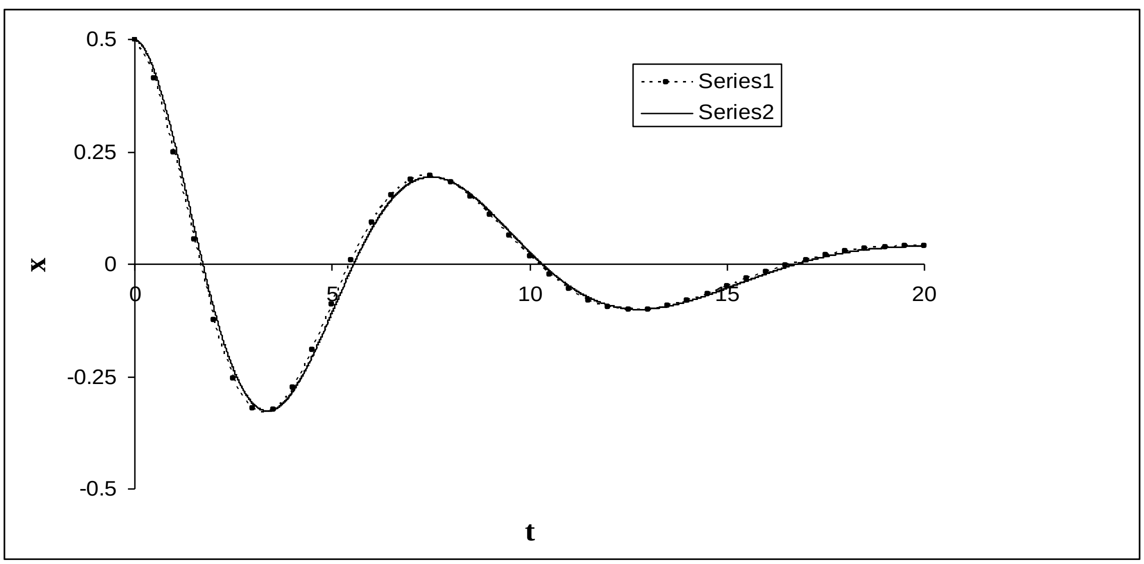

Fig. 3.1 (b) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dotted lines) by the presented analytical technique with the initial conditions $b_0 = 0.5, \ \cdot_{\,0} = 0$ or $[x(0) = 0.5, \ \dot{x}(0) = -0.03131]$ with $k = 0.15, \ £ = 0.1, \ £_1 = 0.1$ and $f = x^5$. Corresponding numerical solution is denoted by - (solid line).

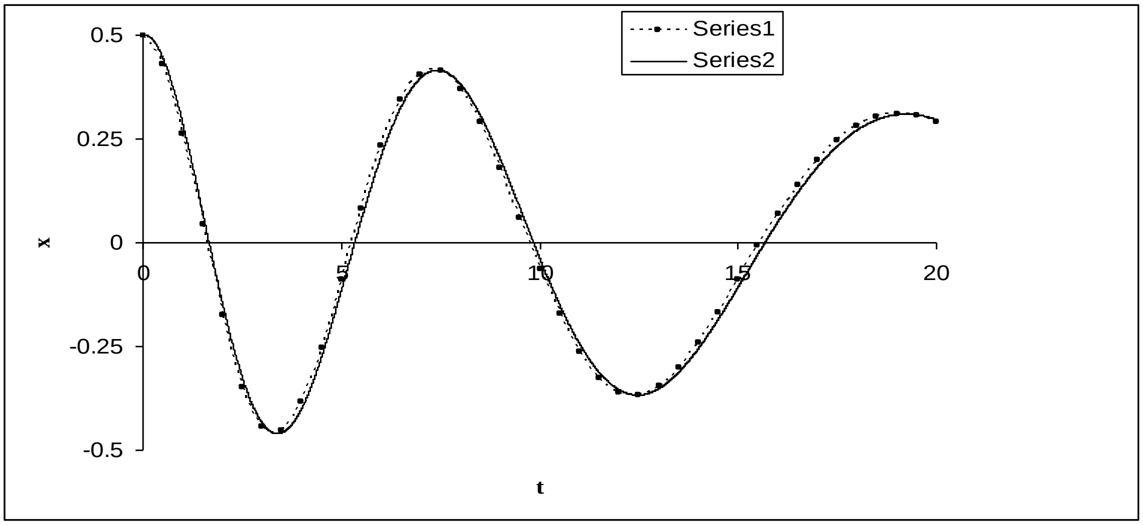

Fig. 3.2 (a) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dotted lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \cdot_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = 0.01904]$ with $k = 0.05,\ \pounds = 0.1,\ \pounds_1 = 1.0$ and $f = x^5$. Corresponding numerical solution is denoted by - (solid line).

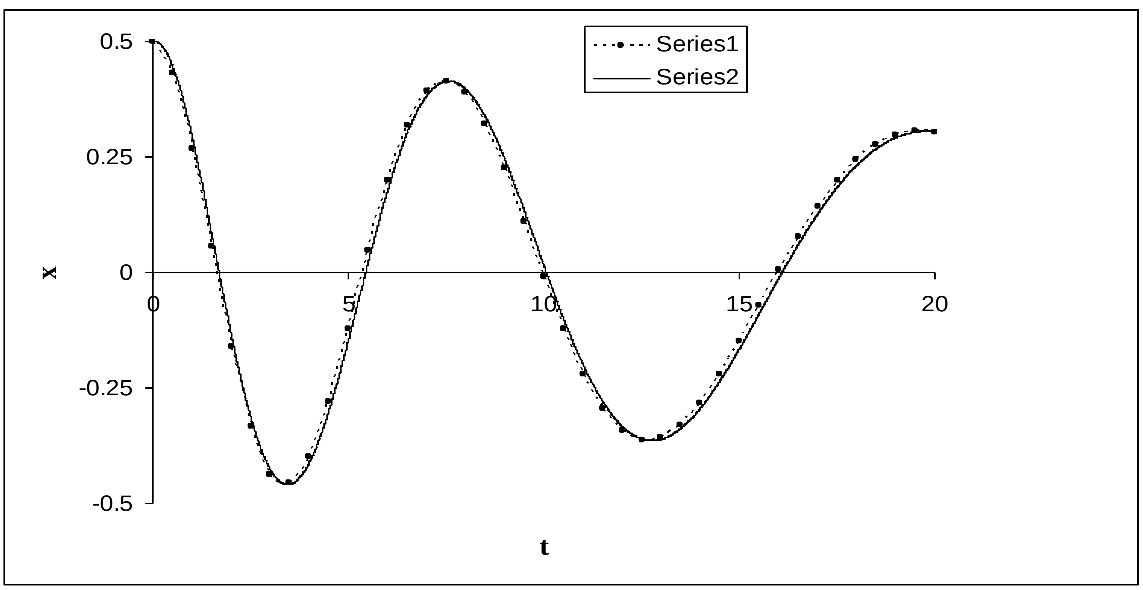

Fig.3.2 (b) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dotted lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \cdot_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = 0.01702]$ with $k = 0.05,\ \pounds = 0.1,\ \pounds_1 = 0.1$ and $f = x^5$. Corresponding numerical solution is denoted by - (solid line)

CHAPTER IV

Conclusions

The great achievement of this thesis is that the presented approximate analytical technique is suitable for solving the second order nonlinear differential systems with slowly varying coefficients and high order nonlinearity in presence of small damping for strong $(£_1 = 1.0)$ as well as weak $(£_1 = 0.1)$ nonlinear cases but the classical perturbation and He's homotopy methods are not capable to handle for these situations.

The determination of amplitude and phase variable is important for both strong and weak nonlinear differential systems in presence of small damping and they play very important role for physical problems. The advantage of the present technique is that it is able to give the position of the physical objects at any time as well as amplitude and phase. The amplitude and phase variable characterize the oscillatory processes. In presence of damping, the amplitude $a \to 0$ as $t \to \infty$ (i.e., for large time t).

It is also mentioned that, the classical KBM method is failed to tackle the second order strong nonlinear differential systems with high order nonlinearity in presence of damping and He's homotopy perturbation method is failed to handle nonlinear differential systems in presence of damping. Some limitations of He's homotopy perturbation (without damping) technique and the KBM method (strong and high order nonlinearity) have been overcome by the present method.

The present method does not require a small parameter in the equation like the classical one. The method has been successfully implemented to illustrate the effectiveness and convenience of the suggested procedure and shown that the first approximate solutions show a good agreement with those solutions obtained by the numerical procedure with high order nonlinearity in presence of several small damping for both strong $(£_1 = 1.0)$ and weak $(£_1 = 0.1)$ nonlinear physical systems with slowly varying coefficients. The graphical representations show good agreement (**Figs. 3.1- 3.2**) between the first approximate analytical solutions and the corresponding numerical solutions for second order nonlinear differential systems with slowly varying coefficients and high order nonlinearity in presence of small damping for both strong and weak nonlinearities cases.

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