Modified Solution of the Nonlinear Singular Oscillator by Iteration Procedure

By

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Dedicated to My

Beloved Parents

And

Affectionate Daughter & Son

Declaration

This is to certify that the thesis work entitled "Modified Solution of the Nonlinear Singular Oscillator by Iteration Procedure" has been carried out by M. M. Ayub Hossain in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

Signature of Supervisor	Signature of Student

Approval

This is to certify that the thesis work submitted by M. M. Ayub Hossain entitled “Modified Solution of the Nonlinear Singular Oscillator by Iteration Procedure” has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of Master of Philosophy in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh in September 2016.

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Abstract

An Iteration method for solving nonlinear oscillatory problems was introduced by R. E. Mickens. Mickens provided a general basis for this method as they are currently used in the calculation of approximations to the periodic solutions of various nonlinear oscillatory differential equations successfully. Latter the method of Iteration has been improved and justified by C. W. Lim, B. S. Wu and R. E. Mickens. It is a widely used technique for handling strong as well as weak nonlinear differential systems with periodic solutions. In this thesis, we have used the truncated Fourier series in each iterative step. The approximate frequencies obtained by the technique shows a good agreement with the exact frequency. Also the obtained solutions are much more accurate than other existing results.

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CHAPTER I

Introduction

Differential equation is one of the most attractive branch of mathematics and an essential tool for modeling many physical situations like mechanical vibration, nonlinear circuits, chemical oscillation, space dynamics and so on. These equations have showed their usefulness as well in the field of ecology, business cycle and biology. Therefore the solution of such problems lies essentially in solving the corresponding differential equations. The differential equations may be linear or nonlinear, autonomous or non-autonomous.

Pragmatically, a lot of differential equations that represent physical phenomena are nonlinear. Systems of nonlinear equations arise in many domains of practical importance such as engineering, mechanics, medicine, chemistry, and robotics. Nonlinear systems also have their importance in music, games etc.

In music, all sound comes through waves whose modeled form nonlinear equation. So, it helps to know how those waves interact with each other when recording music, or when at a live event and setting up the speakers. Thus we can say nonlinear equations are great in the range of importance. They help to predict a lot of things in our daily lives.

The ways of solving of linear differential equations are comparatively easy and highly developed. On the contrary, it is very little known of a general character about nonlinear equations. Ordinarily, the nonlinear problems are solved by converting them into linear equations attributing some terms; but such linearization is not feasible all time. The equation is generally confined to a variety of rather special cases, and one must resort to various methods of approximation.

Analytical solutions of nonlinear differential equations or linear differential equations with variables coefficients play an important role in the study of nonlinear dynamical systems, but sometimes it is difficult to find their solutions, especially for nonlinear problems with strong nonlinearities. Many methods such as Perturbation method, Harmonic balance (HB) method, Iteration method etc exist for constructing analytical approximations to the solution to the oscillatory system. Perturbation method is used only for small nonlinearities; Harmonic balance (HB) method is used for strong nonlinear problems. One the other hand Iteration method is used for small as well as large amplitude of oscillations.

In the Perturbation method, the expansion of a solution to a differential equation is represented in a series of a small parameter. It is used to construct uniformly valid periodic solution to second-order nonlinear differential equations.

Harmonic balance method is a procedure of determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation. An important advantage of the method is that it can be applied to nonlinear oscillatory problems for which the nonlinear terms are not “small” i.e., no perturbation parameter need to exist. A disadvantage of the method is that it is a priory difficult to predict for a given nonlinear differential equation whether a first order harmonic balance calculation will provide a sufficiently accurate approximation to periodic solution or not.

The Iteration method was proposed by R. E. Mickens [43]. The method introduces a reliable and efficient process for wide variety of scientific and engineering application for the case of nonlinear systems. There are two important advantages of Iteration method, one is “Only linear, inhomogeneous differential equations are required to be solved at each level of the calculation” and another is “ The coefficients of the higher harmonic, for a given value of the iteration index decrease rapidly with increasing harmonic number". The last point implies that higher order solutions may not be required.

It is noted that the majority of scientists have not been led to their discoveries by a process of deduction from general postulates, or general principles, but rather by a through examination of properly chosen particular cases. The generalizations have come later, because it is far easier to generalize an established result than to discover a new line of argument. Generalization is the temptation of a lot of researchers working now with nonlinear dynamical systems.

The important development of the theory of nonlinear dynamical systems, during these centuries, has essentially its origins in the studies if the “natural effects” encountered in these systems, and the rejection of non-essential generalizations, i.e. the study of concrete nonlinear systems have been possible due to the foundation of results from the theory or nonlinear dynamical system field.

In this thesis, a nonlinear dynamical system that can be described by a second order differential equation is considered as nonlinear singular oscillator and is discussed. The main object of this thesis is to determine the new approximate frequencies and corresponding analytical solutions of ‘Nonlinear Singular Oscillator’ by iteration procedure and compare them to some existing results. In this context a good agreement with existing results is found.

CHAPTER II

Some Basic Concepts

This chapter introduces the basic, but fundamental concepts relating to the thesis will be addressed. It may be noted that in the field of research on nonlinear system, nonlinear equation and nonlinear differential equation are interchangeably used.

Nonlinear Equation

If the degree of the dependent variable must in each term is not linear i.e. not of degree one, the equation is termed as nonlinear. Examples of nonlinear equations are Navier-Stokes equations in fluid dynamics, Lotka-Volterra equations in biology, Bellman equation for optimal policy, Richards equation for unsaturated water flow etc.

Nonlinear Ordinary Differential Equation

A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.

The following ordinary differential equations are all nonlinear:

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y^2=0$$

$$\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3+6y=0$$

$$\frac{d^2y}{dx^2}+5y\frac{dy}{dx}+6y=0$$

$$\frac{dy}{dx}+e^y=0$$

Truly Nonlinear Functions

If $f(x)$ has no linear approximation in any neighborhood of $x=a$, then $f(x)$ is a Truly Nonlinear function.

The following are several explicit examples of Truly Nonlinear functions

$$f_1(x)=\frac{1}{x-a}, \quad f_2(x)=\frac{1}{(x-a)^{\frac{3}{5}}}$$

Truly Nonlinear Oscillators

If $f(x)$ is a Truly Nonlinear function, then the second-order differential equations "$\ddot{x} + f(x) = 0$" is a Truly Nonlinear Oscillator.

The following are particular examples of Truly Nonlinear Oscillators

$\ddot{x} + x^3 = 0$

$\ddot{x} + x^{\frac{1}{3}} = 0$

$\ddot{x} + x + x^{\frac{1}{3}} = 0$

$\ddot{x} + \frac{1}{x} = 0$

Phase Plane

If a plane is such that, each point of that plane describe the position and velocity of a dynamical particle, then that plane is called phase plane.

The differential equation describing many nonlinear oscillators can be written in the form:

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right) = 0 \tag{2.1}$$

A convenient way to treat equation (2.1) is to rewrite it as a system of two first order ordinary differential equations

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -f(x, y) \tag{2.2}$$

Equilibrium point:

Equations (2.2) can be generalized in the form

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y) \tag{2.3}$$

where F and G have continuous first order partial derivatives for all (x,y).

A point which satisfies $F(x, y) = 0$ and $G(x, y) = 0$ is called an equilibrium point or critical point. The solution of (2.3) may be pictured as a curve in the x-y phase plane passing through the point of initial conditions (x_0, y_0). Each time a particle passes through a given point (x, y), its direction is always the same. This means a given particle takes some path which may not intersect itself. A periodic motion corresponds to a closed curve in the x-y plane. In the special case that the first equation of (2.3) is $\frac{dx}{dt} = y$, as in the case of equations

(2.2), the motion in the upper half-plane $y > 0$ must proceed to the right, that is, x must increase in time for $y > 0$, and vice versa for $y < 0$.

Trajectory

If a curve is such that each point of the curve represents the position and velocity of a dynamical particle, the curve is called the path or Trajectory of the particle.

For any number $t = t_0$ and any pair (x_0, y_0) of real number, there exists a unique solution of the equation (2.3), we obtain

$$\begin{aligned} x &= u(t) \\ y &= v(t) \end{aligned} \tag{2.4}$$

where $x_0 = u(t_0), \quad y_0 = v(t_0)$

If both f and g are not constant functions, then equation (2.4) defines a curve in the phase plane, which is called a path or orbit or trajectory of the system.

Limit Cycle

A closed trajectory in the phase plane such that other non-closed trajectories spiral toward it, either from the inside or the outside, as $t \to \infty$, is called a limit cycle . If all trajectories that start near a closed trajectory (both inside and outside) spiral toward the closed trajectory as $t \to \infty$, then the limit cycle is asymptotically stable. If the trajectories on both sides of the closed trajectory spiral away as $t \to \infty$, then the closed trajectory is unstable.

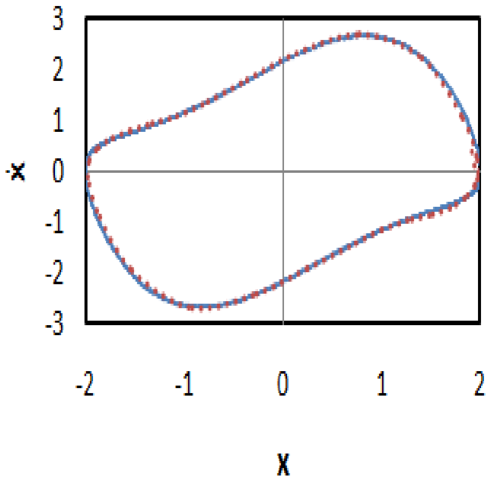

Fig: Limit cycle of the Van der Pol oscillator

The Autonomous System

Consider the systems of the form

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y) \tag{2.5}$$

where P and Q have continuous first partial derivatives for all (x, y). Such a system, in which the independent variable t is not explicitly appears in the function P and Q on the right, is called an autonomous system.

The following example is an autonomous system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x$$

The Non-autonomous system

Consider the systems of the form

$$\frac{dx}{dt} = P(x, y, t)$$
$$\frac{dy}{dt} = Q(x, y, t)$$

where P and Q have continuous first partial derivatives for all (x, y). Such a system, in which the independent variable t is explicitly appears in the function P and Q on the right, is called a non-autonomous system.

The following example is a non-autonomous system

$$\frac{dx}{dt} = \sin t + y$$
$$\frac{dy}{dt} = \cos t + x$$

Critical Point

Consider the autonomous system of the form of equation (2.5)

A point (x_0, y_0) at which both $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$ is called a critical point.

Isolated Critical Point

A critical point to a system of equation (2.5) is said to be isolated critical point if there is a neighborhood to the critical point that does not any other critical points.

Classifications of Critical Point:

(a) Centre

The isolated critical point (x_0, y_0) of the system of equation (2.5) is called a centre if there exists a neighborhood of (x_0, y_0) which contains accountably infinite numbers of closed path P_n, $(n = 1, 2, \cdots)$ each of which contains $(0, 0)$ as interior point and which are such that the diameters of the paths approaches to 0 as $n \to \infty$.

(b) Saddle Point

The isolated critical point (x_0, y_0) of the system of equation (2.5) is called a saddle point if there exists a neighborhood of (x_0, y_0) in which the following two conditions hold:

(i) There exists two paths which approaches and enter into $(0, 0)$ from a pair of opposite directions as $t \to +\infty$ and there exists two paths which approach and enter into $(0, 0)$ from a different pair of opposite directions as $t \to -\infty$.

(ii) In each of the four domains, between any two of the four directions in (i), there are infinitely many paths which are arbitrarily closed to $(0, 0)$ but which do not approach to $(0, 0)$ either as $t \to +\infty$ or as $t \to -\infty$.

(c) Spiral Point

The isolated critical point (x_0, y_0) of the system of equation (2.5) is called a spiral point if there exists a neighborhood of (x_0, y_0) such that every path $P(x, y)$ in this neighborhood has the following properties:

(i) P is defined for all $t > t_0$ or $t < t_0$, for some number t_0.

(ii) P approaches to $(0, 0)$ as $t \to +\infty$ or as $t \to -\infty$.

(iii) P approaches to $(0, 0)$ in a spiral like manner, winding around $(0, 0)$ an infinite number of times $t \to +\infty$ or as $t \to -\infty$.

(d) Node

The isolated critical point (x_0, y_0) of the system of equation (2.5) is called a node point if there exists a neighborhood of (x_0, y_0) such that every path $P(x, y)$ in this neighborhood has the following properties:

(i) P is defined for all $t > t_0$ or $t < t_0$, for some number t_0.

(ii) P approaches to $(0, 0)$ as $t \to +\infty$ or as $t \to -\infty$.

(iii) P enters into $(0, 0)$ as $t \to +\infty$ or as $t \to -\infty$.

There exists another way to classify the critical points, mainly isolated critical points.

(a) Stable critical points

Consider the system of equation (2.5). Suppose (x_0, y_0) is an isolated critical point of the above system. Let C be a path of the system (2.1) and $x = f(t),\ y = g(t)$ be a solution of (2.5), which define C parametrically. Let $(x, y) = (f(t),\ g(t))$ be a point on C. Define

$$D(t) = \sqrt{\left[f(t)\right]^2 + \left[g(t)\right]^2}$$

where $D(t)$ is the distance between the critical point (x_0, y_0) and $R\left(f(t),\ g(t)\right)$, then the critical point (x_0, y_0) is said to be stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$D(t_0) < \delta$, for some t_0

and $D(t_0) < \varepsilon$, for all $t_0 \le t < \infty$.

(b) Asymptotically Stable critical points

Consider the system of equation (2.5). Suppose (x_0, y_0) is an isolated critical point of the above system. Let C be a path of the system (2.4) and $x = f(t),\ y = g(t)$ be a solution of the system (2.5), which define C parametrically. Let $(x, y) = (f(t),\ g(t))$ be a point on C. Define

$$D(t) = \sqrt{\left[f(t)\right]^2 + \left[g(t)\right]^2}$$

where $D(t)$ is the distance between the critical point $(0,\ 0)$ and $R\left(f(t),\ g(t)\right)$, then the critical point (x_0, y_0) is said to be asymptotically stable if it is stable and

$$\lim_{t \to +\infty} f(t) = 0, \qquad \lim_{t \to +\infty} g(t) = 0$$

Characteristic Equation

Consider the linear system

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned} \tag{2.6}$$

where a, b, c, d are real constants.

Clearly the origin $(0,\ 0)$ is critical point of the above system. We assume that

$$\begin{vmatrix} a & d \\ c & b \end{vmatrix} \neq 0$$

and hence $(0, 0)$ is the only critical point of (2.6). By Euler method, the solution of (2.6) is found of the form

$$\begin{cases} x = Ae^{\lambda t} \\ y = Be^{\lambda t} \end{cases} \tag{2.7}$$

where A and B are arbitrary constants. If (2.7) is a solution of (2.6), then we have

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \tag{2.8}$$

Equation (2.8) is called the characteristic equation of (2.7) and its roots are called characteristic roots or Eigen values of equation (2.6). Characteristic roots or Eigen values have relations with the type and nature of the critical points and are presented in the following table-

Relations between Eigen values and Critical point:

Nature of the roots	**Nature of the critical point**	**Nature of the stability of critical point** (x_0, y_0)
Real, unequal and of same sign	Node	Asymptotically stable if the roots are negative; unstable if the roots are positive
Real, unequal and of opposite sign	Saddle point	Unstable
Real and equal	Node	Asymptotically stable if the roots are negative; unstable if the roots are positive
Complex conjugate but not purely imaginary	Spiral point	Asymptotically stable if the real part of the roots are is negative; unstable if the real part is positive
Purely imaginary	Centre	Stable

Free Oscillating System

A system on which no external forces are applies is called free oscillating system. For a free oscillating system, the applied force is proportional to the restoring force. If $f_1(x)$ is the restoring force and F is the applied force on the system, then

$$F \propto -f_1(x)$$

$$F = -k\, f_1(x)$$

where k is the constant of proportionality.

$$m\frac{d^2x}{dt^2} + k\, f_1(x) = 0 \ [\because F = ma]$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}\, f_1(x) = 0\,,$$

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{2.9}$$

Equation (2.9) is the governing equation for a free oscillating system.

Force Oscillating System

If some external forces are applied on system, then the system is called force oscillating system. A common example of a forced oscillator is a damped oscillator driven by an external force that varies periodically, such as $F = F_{ext}\cos wt$ where ω is the angular frequency of the periodic force and F_{ext} is a constant.

$$F_{ext}\cos wt - kx - b\dot{x} = m\ddot{x} \tag{2.10}$$

Natural Frequency

The frequency of the system in which no external force is acting is called the natural frequency.

CHAPTER III

The Survey and the Proposal

3.1 The Survey

A system of nonlinear equations is a set of simultaneous equations in which the unknowns appear as variables of a polynomial of degree other than unity or in the argument of a function which is not a polynomial of degree one. In other words, in a system of nonlinear equations, the equations to be solved cannot be written as a linear combination of the unknown variables or functions that appear in it or them. If nonlinear known functions appear in the equations, it does not matter. Specially, a differential equation is regarded as linear if it gets linear in terms of the dependent variable as well as its derivatives, even if nonlinear in terms of the other variables appearing in it.

As nonlinear equations are difficult to solve, nonlinear systems are commonly approximated by linear equations. This works well up to some accuracy and some range for the input values, but some interesting phenomena such as chaos and singularities are hidden by linearization. It follows that some aspects of the behavior of a nonlinear system (for example weather) appear commonly to be chaotic, unpredictable or counterintuitive. Although such chaotic behavior may resemble random behavior, it is absolutely not random. In this position there are several analytical approaches to find approximate solutions to nonlinear problems, such as: perturbation [28, 32, 52], harmonic balance (HB) [13, 18, 22, 35, 37, 42, 44, 48, 60], homotophy perturbation [14-16], cubication [17, 21], asymptotic [58, 59], iterative [23, 26, 29, 36, 39, 41, 43, 46, 49] methods, etc.

The perturbation method is the most widely utilized method in which the nonlinear term is small. HB originated by Mickens [41] is another latest method and farther work has been done by Lim, Lai and Wu [36- 39], Hu [29- 31] and so on for solving the strong nonlinear problems. Recently, some authors have used an iteration procedure [23, 25, 26, 29, 30, 31, 36, 39, 41, 43, 46, 49] to obtain the approximate frequency and the corresponding periodic solution of strong nonlinear problems. The main advantage of the iteration method is that it is straightforward. Some modified perturbation methods are being used for handling strong nonlinear problems. However, HB and iteration methods are widely used.

The solution of a differential equation is expanded in a power series of a small parameter in the perturbation method. The method of Lindstedt-Poincare(LP) [55-57], Krylov-Bogoliubov-Mtropolskii(KBM) [34, 51], Multiple Scale method [32, 33] and Homotopy perturbation [14- 16] are most important among all perturbation methods.

The method of Krylov and Bogoliubov is an asymptotic method in the sense $\varepsilon \to 0$. An asymptotic series may not be convergent itself, but for a fixed number of terms, the approximate solution tends to the exact solution as $\varepsilon \to 0$. It is to be noted that the term asymptotic is frequently used in the theory of oscillations also in the sense that $\varepsilon \to \infty$, But in this case the mathematical method is quite different.

Alam [1] has developed a new perturbation technique to find approximate analytical solution of both second order over-damped and critically damped nonlinear systems. Later he (Alam [6-8]) extended the method to n-th order nonlinear differential systems. Alam [3, 4, 9] has also extended the KBM method for certain non-oscillatory nonlinear systems when the Eigen values of the unperturbed equation are real and non-positive. Alam [2] has presented a new perturbation method based on the work of the Krylov-Bogoliubov-Mtropolskii method to find approximate solution of second order nonlinear systems with large damping. Alam et al. [37] investigated perturbation solution of a second order time-dependent nonlinear system based on the modified Krylov- Mtropolskii method.

Osiniskii [53], first extended the KBM method to a third order nonlinear differential equation. Osiniskii [54], has also extended the KBM method to a third order nonlinear differential equation with initial friction and relaxation. Mulholland [50] studied nonlinear oscillations governed by a third order nonlinear differential equation. Lardner and Bojadziev [35] investigated nonlinear damped oscillations governed by a third order nonlinear partial differential equation. Alam [5] has also presented a unified Krylov-Bogoliubov-Mtropolskii method, which is not the formal form of the original KBM method, for solving n-th order nonlinear systems. Alam [10] has also presented a modified and compact form of the Krylov-Bogoliubov-Mtropolskii unified method for solving n-th order nonlinear systems. Alarm [10] developed a general formula based on the extended Krylov-Bogoliubov-Mtropolskii method, for obtaining asymptotic solution of an n-th order time dependent quasi linear differential equation with damping. Bojadziev [19], Bojadziev and Hung [20] used at least two trial solutions to investigate time dependent differential systems; one is for resonant case and the other is for non-resonant case. But Alam [11] used only one set of variational equations, arbitrary for both resonant and non-resonant cases. Alam et al. [12] presented a general form of the KBM method for solving nonlinear partial differential equations.

Harmonic balance method is the most useful technique for finding the periodic solutions of nonlinear system. If a periodic solution does not exist of an oscillator, it may be sought in the form of Fourier series, whose coefficients are determined by requiring the series to satisfy the equation of motion. However, in order to avoid solving an infinite system of algebraic equations, it is better to approximate the solution by a suitable finite sum of trigonometric function. This is the main task of harmonic balance method. Thus approximate solutions of an oscillator are obtained by harmonic balance method using a suitable truncation of Fourier series.

The method is capable to determining analytic approximate solution to the nonlinear oscillator valid even for the case where the nonlinear terms are not small i.e., no particular parameter need to exist.

An enormous text on the method of harmonic balance is available. Selected lists of some of the articles that studied and applied this method to a variety of differential equations are given in the references [13, 18, 22, 35, 37, 42, 44, 48, 62]. The formulation of the method of harmonic balance focuses primarily by Mickens [42]. However, it should be noted that various generalizations of the method of harmonic balance have been made by several investigators: an intrinsic method of harmonic analysis by Huseyin and Lin [33]; the use of Jacobi elliptic functions by Garcia –Margallo, Bejarano and two time scale harmonic balance by Summers and Savage [60]. Lately, combing the method of averaging (Krylov and Bogoliubov [34]) and harmonic balance, Lim and Lai [39] presented analytic technique to obtain first approximate perturbation solution; their solutions gives desired results for some non-conservative systems when the damping force is very small. Another technique is developed by Yamgoue and Kofane [63] to determine approximate solutions of nonlinear problems with strong damping effect; more than two harmonic terms are involved in their solution. Further work has been done by various scientists. Some notable scientists are; Wu et al. [62], Beléndez et al. [18], Gottlieb [22], and so forth for handeling strong nonlinearities.

Mickens [49] has given the general procedure for calculating solutions by means of the method of direct harmonic balance is as follows:

He considered the equation for all Truly Nonlinear (TNL) Oscillators as

$$F(x,\dot{x},\ddot{x})=0 \tag{3.1}$$

where $F(x,\dot{x},\ddot{x})$ is of odd-parity, i.e.,

$$F(-x,-\dot{x},-\ddot{x})=-F(x,\dot{x},\ddot{x}) \ . \tag{3.2}$$

A major consequence of this property is that the corresponding Fourier expansions of the periodic solutions only contain odd harmonics [45], i.e.

$$x(t)=\sum_{k=1}^{\infty}\{A_k \cos(2k-1)\theta + B_k \sin(2k-1)\theta\} \ , \tag{3.3}$$

where $\theta=\Omega t$

The N -th order harmonic balance approximation to $x(t)$ is the expression

$$x_N(t)=\sum_{k=1}^{N}\{\overline{A}_k{}^N \cos(2k-1)\overline{\theta} + \overline{B}_k{}^N \sin(2k-1)\overline{\theta}\} \tag{3.4}$$

where $\overline{\theta}=\overline{\Omega}_N t$ and $(\overline{A}_k{}^N, \overline{B}_k{}^N, \overline{\Omega}_N)$ are approximations to (A_k, B_k, Ω_k) for

$k=1,2,3,\ldots\ldots,N$.

For the case of a conservative oscillator, Equation (3.1) generally takes the form

$$\ddot{x}+f(x,\lambda)=0 \ , \tag{3.5}$$

where λ denotes the various parameters appearing in $f(x,\lambda)$ and $f(-x,\lambda)$. The following initial conditions are selected

$$x(0)=A \ , \ \dot{x}(0)=0 \ , \tag{3.6}$$

And this has the consequence that only the cosine terms are needed in the Fourier expansions and therefore we have

$$x_N(t) = \sum_{k=1}^{N} \overline{A}_k^{\,N} \cos(2k-1)\overline{\theta} \tag{3.7}$$

where $\overline{\theta} = \overline{\Omega}_N t$

It is observed that $x_N(t)$ has (N+1) unknowns, the coefficients $\overline{A}_1^{\,N}$, $\overline{A}_2^{\,N}$, ………, $\overline{A}_N^{\,N}$ and Ω_N, the angular frequency. These quantities may be calculated by carrying out the following steps.

Step 1: Substitution of Equation (3.7) into Equation (3.5), which an expression will provide term in the following form

$$\sum_{k=1}^{N} H_k \cos\left[(2k-1)\Omega_N t\right] + HOH \cong 0\,, \tag{3.8}$$

where H_k are functions of the coefficients, the angular frequency and the parameters, i.e.,

$$H_k = H_k(\overline{A}_1^{\,N}, \overline{A}_2^{\,N}, \ldots, \overline{A}_N^{\,N}, \Omega_N, \lambda)\,.$$

and HOH is indicating the higher order terms of H.

Step 2: Setting of the value of the function H_k as zero, i.e.,

$$H_k = 0, \quad k = 1, 2, \ldots, N\,. \tag{3.9}$$

This assumption is acceptable because the cosine functions are linearly independent and, as a result, any linear sum of them that is equal to zero must have the property the coefficients are all zero.

Step 3: Solutions of N equations of equation (3.9) involving $\overline{A}_1^{\,N}$, $\overline{A}_2^{\,N}$, ………, $\overline{A}_N^{\,N}$ and Ω_N in terms of $\overline{A}_1^{\,N}$.

Using the initial conditions, Equation (3.6), we have for $\overline{A}_1^{\,N}$ the relation

$$x_N(0) = A = \overline{A_1}^N + \sum_{k=2}^{N} \overline{A_k}^N (\overline{A_1}^N \lambda). \tag{3.10}$$

For the case of non-conservative oscillator, where odd parity appears, the calculation of approximations to periodic solutions follows a procedure modified for the case of conservative oscillators presented above. Many of these equations take the form

$$\ddot{x} + f(x,\lambda_1) = g(x,\dot{x},\lambda_2)\dot{x}\,, \tag{3.11}$$

where

$$f(-x,\lambda_1) = -f(x,\lambda_1),\; g(-x,-\dot{x},\lambda_2) = g(x,\dot{x},\lambda_2)\,, \tag{3.12}$$

And (λ_1,λ_2) denote the parameters appearing in f and g. For this type of differential equation, a limit cycle may exist and the initial conditions, cannot in general, be a periodically specified.

Harmonic balancing, for systems where limit cycle may exist, uses the following procedures:

Step 1: The N-th order approximations to the periodic solution be taken as

$$x_N(t) = \overline{A_1}^N \cos(\overline{\Omega_N t}) + \sum_{k=2}^{N} \{\overline{A}_k{}^N \cos[(2k-1)\overline{\Omega_N t}] + \overline{B}_k{}^N \sin[(2k-1)\overline{\Omega_N t}]\}\,, \tag{3.13}$$

where the 2N unknowns $\overline{A_1}^N, \overline{A_2}^N, ..., \overline{A_N}^N; \overline{\Omega_N}, \overline{B_2}^N, ..., \overline{B_N}^N$ and $\overline{\Omega_N}$ are to be determined.

Step 2: Equation (3.13) be substituted into Equation (3.11) which gives the form

$$\sum_{k=1}^{N} \{H_k \cos\left[(2k-1)\Omega_N t\right] + L_k \sin[(2k-1)\overline{\Omega_N t}]\} + HOH \cong 0\,, \tag{3.14}$$

where the $\{H_k\}$ and $\{L_k\}$, $k = 1$ to N, are functions of the $2N$ unknowns which are mentioned above.

Step 3: The $2N$ functions $\{H_k\}$ and $\{L_k\}$ to be equated to zero and to be solved to provide $(2N-1)$ amplitudes and the angular frequencies. If a “valid” solution exists, then it corresponds to a limit-cycle. In general, the amplitudes and the angular frequencies are expressed in terms of the parameters λ_1 and λ_2.

Recently, some authors (Guo et al[23], Haque et al[24-427], Hu[30], Lim and Wu[36], Lim et al[39, 40], Mickens[45, 49]) have used a technique for calculating approximations to the periodic solutions and corresponding frequencies of TNL oscillators for small and as well as large amplitude of oscillations. The method was originated by R.E. Mickens in 1987. In the paper (Mickens [43]), he provided a general basis for iteration methods as they are currently used in the calculation of approximations to the periodic solutions of various nonlinear oscillatory differential equation successfully.

The general methodology of iteration procedure by Mickens [43] is as follows:

He considered the differential equation of the form

$$F(\dot{x}, x) = 0, \; x(0) = A, \; \dot{x}(0) = 0 \tag{3.15}$$

where over dots denote differentiation with respect to time, t.

Further he assumed that the equation of the nonlinear oscillator modeled by the equation

$$\ddot{x} + f(\dot{x}, x) = 0, \; x(0) = A, \; \dot{x}(0) = 0 \tag{3.16}$$

In general, the Equation (3.16) is of odd parity i.e. $f(-\dot{x}, -x) = -f(\dot{x}, x)$

Let the natural frequency of this system is Ω . Then $\Omega^2 x$ is added on both sides of Equation (3.16), to obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\dot{x}, x) \equiv G(x, \dot{x}). \tag{3.17}$$

He formulated the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \dot{x}_k); \; \mathrm{k} = 0, 1, 2, ... \tag{3.18}$$

Together with initial guess $x_0(t) = A\cos(\Omega_0 t)$ (3.19)

Hence x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{3.20}$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$ (Nayfeh [76]).

The above procedure gives the sequence of solutions which are labeled by $x_0(t),\ x_1(t), \cdots$, since all solutions are obtained from solving linear equations, they are, in particular, easy to calculate. The only difficulty might be the algebraic intensity required to complete the calculations. As the Equation (3.16) is of odd parity, the solution will only contain odd multiples of the angular frequency (Mickens [45]).

Further a generalization of this work was then given by Lim and Wu [36]. Their procedure is as follows:

They assumed the equation in the form

$$\ddot{x} + f(x) = 0,\ x(0) = A,\ \dot{x}(0) = 0 \tag{3.21}$$

where A is given positive constant and $f(x)$ satisfies the condition

$$f(-x) = -f(x) \tag{3.22}$$

Addition of $\omega^2 x$ on both sides of Equation (3.21), provides

$$\ddot{x} + \omega^2 \mathrm{x} = \omega^2 \mathrm{x} - f(x) \equiv g(x)\,. \tag{3.23}$$

where ω is priory unknown frequency of the periodic solution $x(t)$ being sought.

They proposed the iteration scheme of Equation (3.23) as

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = g(x_{k-1}) + (x_k - x_{k-1})g_x(x_{k-1});\ \mathrm{k} = 0,\ 1,\ 2\ \ldots \tag{3.24}$$

where $g_x = \dfrac{\partial g}{\partial x}$ and the inputs of starting functions are

$$x_{-1}(t) = x_0(t) = A\cos(\omega t) \tag{3.25}$$

with the initial conditions

$$x_k(0) = A,\quad \dot{x}_k(0) = 0,\quad k = 1, 2, 3, \ldots\ldots \tag{3.26}$$

Then substituting Equation (3.25) into Equation (3.24) and expanding the right hand side of Equation (3.24) into the Fourier series yields

$$g[x_{k-1}(t)] + g_x[x_{k-1}(t)][x_t(t) - x_{k-1}(t)] = a_1(A,\omega)\cos\omega t + \sum_{n=2}^{N} a_{2n-1}(A,\omega)\cos[(2n-1)\omega t] \tag{3.27}$$

where the coefficients $a_{2n-1}(A,\omega)$ are known functions of A and ω , and the integer N depends upon the function $g(x)$ of the right hand side of Equation (3.23) . On view of Equation (3.27), the solution of equation is taken to be

$$x_{k+1}(t) = B\cos\omega t - \sum_{n=2}^{N} \frac{a_{2n-1}(A,\omega)}{[(2n-1)^2-1]\omega^2}\cos[(2n-1)\omega t]\,, \tag{3.28}$$

where B is tentatively an arbitrary constant. In Equation (3.28), the particular solution is chosen such that it contains no secular terms (Nayfeh [52]), which requires that the coefficient $a_1(A,w)$ of right-side term $\cos\omega t$ in Equation (3.27) satisfy

$$a_1(A,w) = 0 \ . \tag{3.29}$$

Equation (3.29) allows the determination of the frequency as a function A. Next, the unknown constant B will be computed by imposing the initial conditions in Equation (3.26). Finally, putting these steps together gives the solution $x_{k+1}(t)$.

In 2005, this process was extended by Mickens. He considered the Equation (3.15) as

$$\ddot{x} + f(\ddot{x},\dot{x},x) = 0\,,\ x(0) = A\,,\ \dot{x}(0) = 0 \tag{3.30}$$

where over dots denote differentiation with respect to time, t.

Let Ω is the natural frequency of this system. Then adding $\Omega^2 x$ on both sides of Equation (3.15), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x},\dot{x},x) \equiv G(x,\dot{x},\ddot{x})\,. \tag{3.31}$$

Now, formulate the iteration scheme as

$$\begin{aligned}\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} =& G(x_{k-1},\dot{x}_{k-1},\ddot{x}_{k-1}); + (x_k - x_{k-1})G_x(x_{k-1},\dot{x}_{k-1},\ddot{x}_{k-1}) \\ &+ (\dot{x}_k - \dot{x}_{k-1})G_{\dot{x}}(x_{k-1},\dot{x}_{k-1},\ddot{x}_{k-1}) + (\ddot{x}_k - \ddot{x}_{k-1})G_{\ddot{x}}(x_{k-1},\dot{x}_{k-1},\ddot{x}_{k-1})\end{aligned} \tag{3.32}$$

$$\text{where}\ G_x = \frac{\partial G}{\partial x}\ ,\ G_{\dot{x}} = \frac{\partial G}{\partial \dot{x}},\ G_{\ddot{x}} = \frac{\partial G}{\partial \ddot{x}} \tag{3.33}$$

and x_{k+1} satisfies the initial conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{3.34}$$

The starting function are taken to be (Lim & Wu [38], Mickens [46])

$$x_{-1}(t) = x_0(t) = A\cos(\Omega_0 t) \tag{3.35}$$

The right hand side of Equation (3.32) essentially is the first term in a Taylor series expansion of the function $G(x_k, \dot{x}_k, \ddot{x}_k)$ at the point $(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})$ (Taylor and Mann [61]) To illustrate this point, note that

$$x_k = x_{k-1} + (x_k - x_{k-1}), \tag{3.36}$$

and for some function $G(x)$,

$$G(x_k) = G[x_{k-1} + (x_k - x_{k-1})] = G(x_{k-1}) + G_x(x_k - x_{k-1}) + ..., \tag{3.37}$$

An alternative, but very insightful modification of above scheme was proposed by Hu [30]. He used the following equation in place of Equation (3.36)

$$x_k = x_0 + (x_k - x_o) \tag{3.38}$$

Then Equation (3.37) is changed to

$$G(x_k) = G[x_0 + (x_k - x_0)] = G(x_0) + G_x(x_k - x_0) + ..., \tag{3.39}$$

And the corresponding modification to Equation (3.32) is

$$\begin{aligned} \ddot{x}_{k+1} + \Omega_k^2 x_{k+1} &= G(x_0, \dot{x}_0, \ddot{x}_0) + (x_k - x_0) G_x(x_0, \dot{x}_0, \ddot{x}_0) \\ &+ (\dot{x}_k - \dot{x}_0) G_{\dot{x}}(x_0, \dot{x}_0, \ddot{x}_0) + (\ddot{x}_k - \ddot{x}_0) G_{\ddot{x}}(x_0, \dot{x}_0, \ddot{x}_0) \end{aligned} \tag{3.40}$$

This scheme is computationally easier to execute, for $k \geq 2$, than the one given in Equation (3.32). The essential idea is that if $x_0(t)$ is a good approximation, then the expansion should take place at $x = x_0$. Also as pointed out by Hu [30], the $x_0(t)$ in $(x_k - x_o)$ is not the same for all k. In particular, $x_0(t)$ in $(x_1 - x_o)$ is the function $A\cos(\Omega_1 t)$, while the $x_0(t)$ in $(x_2 - x_o)$ is the function $A\cos(\Omega_2 t)$.

Mickens further used the iterative technique [47] to calculate a higher-order approximation to the periodic solutions of a conservative oscillator for which the elastic force term is proportional to $x^{\frac{1}{3}}$. Hu [29, 31] applied the modified iteration technique presented by Mickens [46] to find approximate solutions of nonlinear oscillators with fractional powers and quadratic nonlinear oscillator respectively. Recently, Zheng et al. [64] has applied Mickens extended iteration method and direct iteration method to determine approximate periodic solutions of a class of nonlinear jerk equations.

3.2 The Proposal

We propose to modify the direct iteration and extended iteration technique to determine the modified solution of Nonlinear Singular Oscillator that describes by the nonlinear dynamical systems governed by second order differential equation.

A general second order nonlinear ordinary differential equation is of the form

$$\ddot{x} + f(\dot{x}, x) = 0, \; x(0) = A, \; \dot{x}(0) = 0 \qquad (3.41)$$

where over dots denotes differentiation with respect to time, t. It is considered that the initial amplitude of Equation (3.41) is A and corresponding frequency is Ω which is to be determined and f is the given nonlinear function.

In Chapter IV, a modified direct iteration procedure is applied for finding the second and higher order approximate periodic solution of the Nonlinear Singular Oscillator $\ddot{x} + x^{-1} = 0$.

In Chapter V, a modified extended iteration procedure is applied for finding the second and higher order approximate periodic solution of the Nonlinear Singular Oscillator $\ddot{x} + x^{-1} = 0$.

CHAPTER IV

Modified Solution of the Nonlinear Singular Oscillator by Direct Iteration Procedure

4.1: Introduction

The attractive or repulsive singular oscillator exhibits an infinite number of solutions provided the parameter responsible for the singularity is greater than a certain critical value. The frequency-amplitude relations for the oscillator in which the restoring force is inversely proportional to the dependent variable are achieved analytically. There are different types of singular oscillator such as the singular harmonic oscillator, Quantum singular oscillator, Hamiltonian singular oscillator etc.

This chapter introduces the direct iteration method as a technique for calculating approximations to the periodic solutions of nonlinear singular oscillator. There are numerous methods to solve the problem like Mickens harmonic balance method [48], Mickens iteration method [49], Haques iteration method [25] etc.

In Mickens Iteration method the problem is solved by the following way-

The problem can be written as

$$x\ddot{x}+1=0$$

Or, $\ddot{x}=-(\ddot{x})^2 x$

$$\ddot{x}+\Omega^2 x=\Omega^2 x-(\ddot{x})^2 x$$

This last expression suggests the following iteration scheme

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=\Omega_k^2 x_k-\left(\ddot{x}_k\right)^2 x_k \tag{4.1}$$

For k=0 and $x_0(t)=A\cos\theta$, $\theta=\Omega_0 t$, we have

$$\ddot{x}_1+\Omega_0^2 x_1=\left(\Omega_0^2 A\cos\theta\right)-\left(-\Omega_0^2 A\cos\theta\right)^2\left(A\cos\theta\right)$$

$$= \Omega_0^2 \left[1 - \frac{3A^2\Omega_0^2}{4} \right] A\cos\theta - \left(\frac{A^3\Omega_0^4}{4} \right) \cos 3\theta \tag{4.2}$$

The elimination of secular terms gives

$$1 - \frac{3A^2\Omega_0^2}{4} = 0$$

$$\Omega_0^2(A) = \left(\frac{4}{3} \right) \frac{1}{A^2} \tag{4.3}$$

Therefore, $x_1(t)$ satisfies the equation

$$\ddot{x}_1 + \Omega_0^2 x_1 = \left(\Omega_0^2 A\cos\theta \right) - \left(-\Omega_0^2 A\cos\theta \right)^2 \left(A\cos\theta \right) \tag{4.4}$$

The particular solution, $x_1^{(p)}(t)$, is

$$x_1^{(p)}(t) = \left(\frac{A^3\Omega_0^2}{32} \right) \cos 3\theta = \left(\frac{A}{24} \right) \cos 3\theta$$

Therefore, the complete solution is

$$x_1(t) = C\cos\theta + \frac{A}{24}\cos 3\theta \tag{4.5}$$

Using $x_1(0) = A$, we have $C = \dfrac{23}{24}$ and

Therefore, $x_1(t) = A(\dfrac{23}{24}\cos\theta + \dfrac{1}{24}\cos 3\theta)$ (4.6)

Proceeding to the second level of iteration, $x_2(t)$ satisfies the equation

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - (\ddot{x}_1)^2 x_1 \tag{4.7}$$

where $x_1(t) = A[\dfrac{23}{24}\cos(\Omega_1 t) + \dfrac{1}{24}\cos(3\Omega_1 t)]$

Let $\theta = \Omega_1 t$ and substitute this $x_1(t)$ into the right-hand side of Equation (4.7); doing so gives

$$\ddot{x}_2 + \Omega_1^2 x_2 = {\Omega_1}^2 \left[\alpha - \left(\frac{3}{4} \right) A^2\Omega_1^2 g(\alpha, \beta) \right] + HOH\ , \tag{4.8}$$

where

$$g(\alpha,\beta)=\alpha^3+\left(\frac{19}{3}\right)\alpha^2\beta+66\alpha\beta^2+27\beta^3$$

and

$$\alpha=\frac{23}{24},\ \beta=\frac{1}{24}$$

The absence of secular terms gives

$$\Omega_1^2=\left[\left(\frac{4}{3}\right)\frac{1}{A^2}\right]\left[\frac{\alpha}{g(\alpha,\beta)}\right]$$

and

$$\Omega_1(A)=\frac{1.0175}{A}$$

In Haques Iteration [26] method the problem is solved by the following way-

Let us consider the Oscillator

$$\ddot{x}+x^{-1}=0. \tag{4.9}$$

Adding $\Omega^2 x$ on both sides of Equation (4.9), we get

$$\ddot{x}+\Omega^2\mathrm{x}=\Omega^2\mathrm{x}-x^{-1} \tag{4.10}$$

the iteration scheme of Equation (4.10) is

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=\Omega_k^2 x_k-x_k^{-1} \tag{4.11}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained by putting k=0

in Equation(4.23) we get

$$\ddot{x}_1+\Omega_0^2 x_1=\Omega_0^2 A\cos\theta-(A\cos\theta)^{-1} \tag{4.12}$$

Now expanding $(\cos\theta)^{-1}$ in a Fourier cosine series then Equation (4.12) reduces to

$$\ddot{x}_1+\Omega_0^2 x_1=\Omega_0^2 A\cos\theta-\frac{2}{A}\sum_{n=1}^{\infty}(-1)^{n-1}\cos(2n-1)\theta \tag{4.13}$$

Now secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

i.e. $\Omega_0{}^2 A-\dfrac{2}{A}=0$

$$\Omega_0{}^2 = \frac{2}{A^2}$$

$$\Omega_0 = \frac{\sqrt{2}}{A} = \frac{1.41421}{A}$$

This is the first approximate frequency of the oscillator.

Then solving Equation (4.13) and satisfying the initial condition, we obtain

$$x_1(t) = A\left(1 + \frac{1}{4}(-1 + 2\ln 2)\cos\theta\right) - A\sum_{n=2}^{\infty} \frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta \tag{4.14}$$

This is the second approximation and the related Ω_1 is to be determined.

Substituting $x_1(t)$ into the right-hand side of Equation (4.11), we obtain

$$\begin{aligned} \ddot{x}_2 + \Omega_1^2 x_2 &= A\Omega_1^2\left((1 + (-1 + 2\ln 2)/4)\cos\theta - \sum_{n=2}^{\infty} \frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta\right) \\ &\quad - \frac{1}{A}\sum_{n=1}^{\infty} (-1)^{n-1} a_{2n-1}\cos(2n-1)\theta \end{aligned} \tag{4.15}$$

where

$$a_1 = 1.599611, a_3 = 0.983636,\ a_5 = 1.102235,\ a_7 = 1.079400, \ldots\ldots$$

Now secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero

$$\Omega_1^2 = \frac{1.599611}{A^2(1 + (-1 + 2\ln 2)/4)}$$

Then Equation (3.27) becomes,

$$\begin{aligned} \ddot{x}_2 + \Omega_1^2 x_2 &= -A\Omega_1^2\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta\right) \\ &\quad - \frac{1}{A}\sum_{n=1}^{\infty} (-1)^{n-1} a_{2n-1}\cos(2n-1)\theta \end{aligned} \tag{4.16}$$

Then solving Equation (4.16) and satisfying the initial condition, we obtain the second approximation,

$$\begin{aligned} x_2(t) &= A\big((1 - (3 - 4\ln 2)/16 + 1.1(-1 + 2\ln 2)/(4z)\big)\cos\theta \\ &\quad + \sum_{n=2}^{\infty}\left(\frac{(-1)^n}{(4(n-1)n)^2} + \frac{1.1(-1)^{n-1}}{4(n-1)n}\right)\cos(2n-1)\theta \end{aligned} \tag{4.17}$$

where

$$z = \frac{8}{(1+(-1+2\ln 2)/4)\sqrt{(3+\ln 2)(4+\ln 16)}}$$

This is the third approximation and the related Ω_2 is to be determined.

Substituting $x_2(t)$ into the right-hand side of Equation (4.11), we obtain

$$\ddot{x}_3 + \Omega_2^2 x_3 = \sum_{n=2}^{\infty}\left(A\Omega_2^2\left(\frac{(-1)^n}{(4(n-1)n)^2} + \frac{1.1(-1)^{n-1}}{4\Omega_1^2(n-1)n}\right) + (-1)^n \frac{1.26}{A}\right)\cos(2n-1)\theta$$

where

$$\Omega_2^2 = 1.693744/(A^2(1-(3-4\ln 2)/16+1.1(-1+2\ln 2)/(4z)))$$

$$\Omega_2 = \frac{1.265}{A} \tag{4.18}$$

The main object of this thesis is to improve the accuracy of the approximate solution of 'Nonlinear Singular Oscillator' by iteration procedure so that it will help us to investigate the nature (amplitude, frequency *etc*) in the nonlinear dynamical systems.

4.2: The method

Let us suppose that a nonlinear oscillator modeled by

$$\ddot{x} + f(\dot{x}, x) = 0,\ x(0) = A,\ \dot{x}(0) = 0, \tag{4.19}$$

where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ on both sides of Equation (4.19), we obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(\dot{x}, x) \equiv G(x, \dot{x}). \tag{4.20}$$

Now, formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \dot{x}_k);\ k = 0, 1, 2, \ldots \tag{4.21}$$

Together with initial guess $x_0(t) = A\cos(\Omega_0 t)$ (4.22)

Hence x_{k+1} satisfies the conditions

$$x_{k+1}(0)=A, \quad \dot{x}_{k+1}(0)=0. \tag{4.23}$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$. The above procedure gives the sequence of solutions which are mentioned by $x_0(t), x_1(t), \cdots$. The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second.

4.3: Solution Procedure

Let us consider the Oscillator

$$\ddot{x} + x^{-1} = 0. \tag{4.24}$$

Adding $\Omega^2 x$ on both sides of Equation (4.24), we get

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - x^{-1} \tag{4.25}$$

According to Equation (4.21), the iteration scheme of Equation (4.25) is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - x_k^{-1} \tag{4.26}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained by putting $k=0$

in Equation(4.26) and using Equation(4.22), we get

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - (A\cos\theta)^{-1} \tag{4.27}$$

Now expanding $(\cos\theta)^{-1}$ in a truncated Fourier cosine series and substitution in Equation (4.27) provides

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - \frac{1}{A}(2\cos\theta - 2\cos 3\theta)$$

$$\ddot{x}_1 + \Omega_0^2 x_1 = (\Omega_0^2 A - \frac{2}{A})\cos\theta + \frac{2}{A}\cos 3\theta \tag{4.28}$$

Now secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

i.e. $\Omega_0{}^2 A - \dfrac{2}{A} = 0$

$$\Omega_0{}^2 = \frac{2}{A^2}$$

$$\Omega_0 = \frac{\sqrt{2}}{A} = \frac{1.41421}{A}$$

This is the first approximate frequency of the oscillator.

Note that $\Omega_{exact}(A) = \dfrac{1.2533141}{A}$ (4.29)

And $\left|\dfrac{\Omega_{exact} - \Omega_0}{\Omega_{exact}}\right|.100 = \left|\dfrac{1.2533141 - 1.4142136}{1.2533141}\right|.100 = 12.8379\,\%$ error.

After simplification the Equation (4.28) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \frac{2}{A}\cos 3\theta \qquad (4.30)$$

The particular solution, $x_1{}^{(p)}(t)$ is

$$x_1{}^{(p)}(t) = \frac{\frac{2}{A}}{-9\Omega_0{}^2 + \Omega_0{}^2}\cos 3\theta$$

$$= -\frac{1}{4\Omega_0{}^2 A}\cos 3\theta$$

$$= -\frac{1}{4.\frac{2}{A^2}.A}\cos 3\theta$$

$$= -\frac{A}{8}\cos 3\theta$$

Therefore, the complete solution is

$x_1(t) = c_1\cos\theta - \dfrac{A}{8}\cos 3\theta$ (C_1 is the arbitrary constant) (4.31)

Using $x_1(0) = A$, we have $C_1 = \dfrac{9}{8}A$

Therefore, $x_1(t) = A(\dfrac{9}{8}\cos\theta - \dfrac{1}{8}\cos 3\theta)$ (4.32)

This is the first approximate solution of the oscillator.

Proceeding to the second level of iteration, $x_2(t)$ satisfies the equation

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - (x_1)^{-1} \tag{4.33}$$

where $x_1(t) = A(\frac{9}{8}\cos\theta - \frac{1}{8}\cos 3\theta)$

Now expanding the second term on right hand side in a truncated Fourier cosine series and after substitution the Equation (4.33) reduces to

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1{}^2 A(\frac{9}{8}\cos\theta - \frac{1}{8}\cos 3\theta) - \{A(\frac{9}{8}\cos\theta - \frac{1}{8}\cos 3\theta)\}^{-1}$$

$$= \Omega_1{}^2 A(\frac{9}{8}\cos\theta - \frac{1}{8}\cos 3\theta) - 2\sqrt{\frac{2}{3}}.\frac{1}{A}\cos\theta + 1.30306\ \cos 3\theta$$

$$\ddot{x}_2 + \Omega_1^2 x_2 = (\frac{9\Omega_1^2 A}{8} - \frac{2\sqrt{2}}{\sqrt{3}A})\cos\theta - (\frac{\Omega_1^2 A}{8} - \frac{1.30306}{A})\cos 3\theta \tag{4.34}$$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

$$\frac{9\Omega_1^2 A}{8} - \frac{2\sqrt{2}}{\sqrt{3}A} = 0$$

or $\frac{9\Omega_1^2 A}{8} = \frac{2\sqrt{2}}{\sqrt{3}A}$

or $\Omega_1{}^2 = \frac{16\sqrt{2}}{9\sqrt{3}A}$

i.e. $\Omega_1 = \frac{1.2048}{A}$ (4.35)

This is the second approximate frequency of the oscillator.

And the percentage error is $\left|\frac{\Omega_{exact} - \Omega_0}{\Omega_{exact}}\right|.100 = \left|\frac{1.2533141 - 1.2048}{1.2533141}\right|.100 = 3.87087\,\%$.

After simplification the Equation (4.34) reduces to

$$\ddot{x}_2 + \Omega_1^2 x_2 = -(\frac{\Omega_1^2 A}{8} - \frac{1.30306}{A})\cos 3\theta \tag{4.36}$$

The particular solution, $x_2^{(p)}(t)$ is

$$x_2^{(p)}(t) = \frac{-(\frac{\Omega_1^{\,2} A}{8} - \frac{1.30306}{A})}{-9\Omega_0^{\,2} + \Omega_0^{\,2}} \cos 3\theta$$

$$= (\frac{A}{64} - \frac{1.30306}{8A\Omega_1^{\,2}}) \cos 3\theta$$

The complete solution of Equation (4.34) is

$$x_2(t) = C_2 \cos\theta + (\frac{A}{64} - \frac{1.30306}{8A\Omega_1^2}) \cos 3\theta \quad (C_2 \text{ is arbitrary constant}) \tag{4.37}$$

Using $x_2(0) = A$ then

$$A = C_2 + \frac{A}{64} - \frac{1.30306}{8A\Omega_1^{\,2}}$$

Thus we have $C_2 = (\frac{63}{64} A + \frac{1.30306}{8A\Omega_1^2})$

Therefore, $x_2 = (\frac{63}{64} A + \frac{1.30306}{8A\Omega_1^2}) \cos\theta + (\frac{A}{64} - \frac{1.30306}{8A\Omega_1^{\,2}}) \cos 3\theta$

$$x_2 = \{\frac{63}{64} A + \frac{1.30306}{8A(\frac{1.2048}{A})^2}\} \cos\theta + \{\frac{A}{64} - \frac{1.30306}{8(\frac{1.2048}{A})^2}\} \cos 3\theta$$

$$x_2(t) = 1.09659 A \cos\theta - 0.0965883 A \cos 3\theta \tag{4.38}$$

Proceeding to the third level of iteration $x_3(t)$ satisfies the equation

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 x_2 - (x_2)^{-1} \tag{4.39}$$

where $x_2(t) = 1.09659 A \cos\theta - 0.0965883 A \cos 3\theta$

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 A(1.09659 \cos\theta - 0.0965883 \cos 3\theta) - \{A(1.09659 \cos\theta - 0.0965883 \cos 3\theta)\}^{-1}$$

Now expanding the 2nd term on right hand side in a truncated Fourier cosine series and after substitution the Equation (4.37) reduces to

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 A(1.09659\cos\theta - 0.0965883\cos 3\theta)$$
$$-\frac{1}{A}(1.69861 + o.i)\cos\theta - (-1.42177 + o.i)\cos 3\theta$$

$$\ddot{x}_3 + \Omega_2^2 x_3 = (1.09659 A\Omega_2^2 - \frac{1.69861}{A})\cos\theta + (-0.0965883 A\Omega_2^2 + \frac{1.42177}{A})\cos 3\theta \qquad (4.40)$$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

$$1.09659 A\Omega_2^2 - \frac{1.69861}{A} = 0$$

$$\Omega_2^2 = \frac{1.69861}{A^2(1.09659)}$$

$$\Omega_2 = \frac{1.24459}{A} \qquad (4.41)$$

And the percentage error in this stage

$$\left|\frac{\Omega_{exact} - \Omega_0}{\Omega_{exact}}\right|.100 = \left|\frac{1.2533141 - 1.24459}{1.2533141}\right|.100 = 0.696082\,\% \,.$$

4.4: Results and discussions

An iterative approach is presented to obtain approximate solution of the 'nonlinear singular oscillator'. The present technique is very simple for solving algebraic equations analytically and the approach is different from the existing other approach for taking truncated Fourier series. Here we have calculated the first, second, third approximate frequencies $\Omega_0, \Omega_1, \Omega_2$. All the results are presented in the following Table-4.1. Frequencies obtained by Mickens iteration method [49] , Mickens HB method [48] and Haque's iteration method [25] are presented to compare the evaluated frequencies. To show the accuracy, we have calculated the percentage errors by the definitions $\left|\frac{\Omega_e - \Omega i}{\Omega_e}\right|.100$ where Ω_i; i=0, 1, 2…….represents the approximate frequencies obtained by the present method and Ω_e represents the corresponding exact frequency of the oscillator.

Table - 4.1:

Comparison of the approximate frequencies with exact frequency Ω_e [49] of $\ddot{x} + x^{-1} = 0$

Exact frequency Ω_e	$\frac{1.253}{A}$		
Amplitude A	First approximate frequencies, Ω_0 % Error	Second approximate frequencies, Ω_1 % Error	Third approximate frequencies, Ω_2 % Error
Adopted method	$\frac{1.414}{A}$ 12.84	$\frac{1.205}{A}$ 3.87	$\frac{1.245}{A}$ 0.696
Mickens iteration method [49]	$\frac{1.155}{A}$ 7.9	$\frac{1.018}{A}$ 18.1	-----
Mickens HB method [48]	$\frac{1.414}{A}$ 12.84	$\frac{1.273}{A}$ 1.6	$\frac{1.2731}{A}$ 1.58
Haque's iteration method [25]	$\frac{1.414}{A}$ 12.84	$\frac{1.208}{A}$ 3.63	$\frac{1.265}{A}$ 0.92

CHAPTER V

Modified Solution of the Nonlinear Singular Oscillator by Extended Iteration Procedure

5.1. Introduction

The extended iteration method generally is easier to apply, for a given equation, in comparison with similar direct iteration techniques. It requires fewer overall computations for the calculation of $x(t)$ and $\Omega(k)$ for a given value of k. In particular, for equations having cubic-type nonlinearities, the number of harmonics at the k-th level has approximately the following behaviors [49]

Direct iteration : $\dfrac{3^{k+1}}{2}$,

Extended iteration : $k+1$,

Since the coefficients of the harmonics have a rapid decrease in values, the extended iteration method is expected to be sufficient for most of the investigations. There are numerous extended methods to solve the problem, Mickens extended iteration method is the most useful technique.

In Mickens extended iteration method the problem

$$\ddot{x} + \frac{1}{x} = 0 \tag{5.1}$$

is solved by the following way-

To obtain the solution the following formula was used for the extended iteration scheme

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_0, \dot{x}_0, \Omega_k^2) + \left(x_k - x_0\right)G_x(x_0, \dot{x}_0, \Omega_k^2) + \left(\dot{x}_k - \dot{x}_0\right)G_{\dot{x}}(x_0, \dot{x}_0, \Omega_k^2)$$

For $k = 1$, we have

$$\ddot{x}_2 + \Omega_1^2 x_2 = 2x_0(\dot{x}_0)^2 + [\Omega_1^2 - (\dot{x}_0)^2]x_1 - 2x_0\dot{x}_0\dot{x}_1 \tag{5.2}$$

with

$$\begin{cases} x_0(t) = A\cos\theta, \\ x_1(t) = A[\alpha\cos\theta + \beta\cos 3\theta], \\ \theta = \Omega_1 t, \quad \alpha = \dfrac{23}{24}, \quad \beta = \dfrac{1}{24} \end{cases} \tag{5.3}$$

Substitution of the items in Equation (5.3) into the right-hand side of Equation (5.2) gives, after some algebraic and trigonometric simplification, the result

$$\ddot{x}_2 + \Omega_1^2 x_2 = (\Omega_1^2 A)\left[\alpha - (3-7\beta)\left(\frac{\Omega_1^2 A^4}{4}\right)\right]\cos\theta$$

$$-\left(\frac{A\Omega_1^2}{4}\right)\left[(1+35\beta)\Omega_1^2 A^2 - 4\beta\right]\cos 3\theta$$

$$-\left(\frac{19\beta}{4}\right)(\Omega_1^4 A^3)\cos 5\theta \tag{5.4}$$

Setting the coefficient of $\cos\theta$ to zero and solving for Ω_1^2 gives

$$\Omega_1^2(A) = \left[\left(\frac{4}{3}\right)\frac{1}{A^2}\right]\left(\frac{69}{65}\right) = \Omega_0^2(A)\left[\frac{69}{65}\right] \tag{5.5}$$

Or

$$\Omega_1(A) = \frac{1.189699}{A}$$

The main principle of this thesis is to improve the accuracy of the approximate solution of 'Nonlinear Singular Oscillator' by extended iteration procedure so that it will help us to investigate the nature (amplitude, frequency *etc*) in the nonlinear dynamical systems.

5.2: The method

Let us suppose that a nonlinear oscillator modeled by

$$\ddot{x} + f(\ddot{x}, \dot{x}, x) = 0, \; x(0) = A, \; \dot{x}(0) = 0 \tag{5.6}$$

where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ on both sides of Equation (5.1), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x}, \dot{x}, x) \equiv G(x, \dot{x}, \ddot{x}). \tag{5.7}$$

Now, formulate the iteration scheme as

$$\begin{aligned} \ddot{x}_{k+1} + \Omega_k^2 x_{k+1} &= G(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}) + \left(x_k - x_{k-1}\right) G_x(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}) \\ &+ \left(\dot{x}_k - \dot{x}_{k-1}\right) G_{\dot{x}}(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}) + \left(\ddot{x}_k - \ddot{x}_{k-1}\right) G_{\ddot{x}}(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}) \end{aligned} \tag{5.8}$$

where $G_x = \dfrac{\partial G}{\partial x}$, $G_{\dot{x}} = \dfrac{\partial G}{\partial \dot{x}}$, $G_{\ddot{x}} = \dfrac{\partial G}{\partial \ddot{x}}$ (5.9)

And x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{5.10}$$

The starting function are taken to be due to (Lim and Wu [38])

$$x_{-1}(t) = x_0(t) = A\cos(\Omega_0 t) \tag{5.11}$$

The right hand side of Equation (5.8) is essentially the first term in a Taylor series expansion of the function $G(x_k, \dot{x}_k, \ddot{x}_k)$ at the point $(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})$ (Taylor and Mann [61]). To illustrate this point, note that

$$x_k = x_{k-1} + (x_k - x_{k-1}), \tag{5.12}$$

And for some function $G(x)$,

$$G(x_k) = G[x_{k-1} + (x_k - x_{k-1})] = G(x_{k-1}) + G_x(x_k - x_{k-1}) + ..., \tag{5.13}$$

An alternative, but very insightful, modification of above scheme was proposed by Hu [30]. He used the following equation in place of Equation (5.12)

$$x_k = x_0 + (x_k - x_o) \tag{5.14}$$

Then Equation (5.13) is changed to

$$G(x_k) = G[x_0 + (x_k - x_0)] = G(x_0) + G_x(x_k - x_0) + ..., \tag{5.15}$$

And the corresponding modification to Equation (5.8) is

$$\begin{aligned} \ddot{x}_{k+1} + \Omega_k^2 x_{k+1} &= G(x_0, \dot{x}_0, \ddot{x}_0) + \left(x_k - x_0\right) G_x(x_0, \dot{x}_0, \ddot{x}_0) \\ &+ \left(\dot{x}_k - \dot{x}_0\right) G_{\dot{x}}(x_0, \dot{x}_0, \ddot{x}_0) + \left(\ddot{x}_k - \ddot{x}_0\right) G_{\ddot{x}}(x_0, \dot{x}_0, \ddot{x}_0) \end{aligned} \tag{5.16}$$

This scheme is computationally easier to work with for $k \geq 2$, than the one given in Equation (5.8). The essential idea is that if $x_0(t)$ is a good approximation, then the expansion should take place at $x = x_0$. Also as pointed out by Hu [32], the $x_0(t)$ in $(x_k - x_o)$ is not the same for all k. In particular, $x_0(t)$ in $(x_1 - x_o)$ is the function $A\cos(\Omega_1 t)$, while the $x_0(t)$ in $(x_2 - x_o)$ is the function $A\cos(\Omega_2 t)$.

5.3: Solution Procedure

Let us consider the Oscillator

$$\ddot{x} + x^{-1} = 0. \tag{5.17}$$

Adding $\Omega^2 x$ on both sides of Equation (5.17), we get

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - x^{-1} = G(x, \Omega^2) \tag{5.18}$$

where $G(x, \Omega^2) = \Omega^2 \mathrm{x} - x^{-1}$

Therefore $G_x = \Omega^2 + x^{-2}$

According to Equation (5.15), the extended iteration scheme of Equation (5.18) is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = (\Omega_k^2 x_0 - x_0^{-1}) + ({\Omega_k}^2 + {x_0}^{-2})(x_k - x_0) \tag{5.19}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained by putting $k = 0$ in Equation (5.13) and using Equation (5.11) we get

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - (A\cos\theta)^{-1} \tag{5.20}$$

Now expanding $(\cos\theta)^{-1}$ in a truncated Fourier cosine series and after substitution Equation (5.20) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = (\Omega_0^2 A - \frac{2}{A})\cos\theta + \frac{2}{A}\cos 3\theta \tag{5.21}$$

Now secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

i.e. ${\Omega_0}^2 A - \frac{2}{A} = 0$

$$\Omega_0{}^2 = \frac{2}{A^2}$$

$$\Omega_0 = \frac{\sqrt{2}}{A} = \frac{1.41421}{A}$$

This is the first approximate frequency of the oscillator.

Note that $\Omega_{exact}(A) = \dfrac{1.2533141}{A}$ (5.22)

And the percentage error is $\left|\dfrac{\Omega_{exact} - \Omega_0}{\Omega_{exact}}\right|.100 = \left|\dfrac{1.2533141 - 1.4142136}{1.2533141}\right|.100 = 12.8379\,\%$.

After simplification the Equation (5.21) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \frac{2}{A}\cos 3\theta \qquad (5.23)$$

The particular solution, $x_1{}^{(p)}(t)$ is

$$x_1{}^{(p)}(t) = \frac{\frac{2}{A}}{-9\Omega_0{}^2 + \Omega_0{}^2}\cos 3\theta$$

$$= -\frac{1}{4\Omega_0{}^2 A}\cos 3\theta$$

$$= -\frac{1}{4.\frac{2}{A^2}.A}\cos 3\theta$$

$$= -\frac{A}{8}\cos 3\theta$$

Therefore, the complete solution is

$$x_1(t) = c_1 \cos\theta - \frac{A}{8}\cos 3\theta \quad (C_1 \text{ is the arbitrary constant}) \qquad (5.24)$$

Using $x_1(0) = A$, we have $C_1 = \dfrac{9}{8}A$

Therefore, $x_1(t) = A(\dfrac{9}{8}\cos\theta - \dfrac{1}{8}\cos 3\theta)$ (5.25)

This is the first approximate solution of the oscillator.

Proceeding to the second level of iteration, $x_2(t)$ satisfies the equation

$$\ddot{x}_2 + \Omega_1^2 x_2 = (\Omega_1^2 x_0 - x_0^{-1}) + (\Omega_1^{\ 2} + x_0^{-2})(x_1 - x_0) \qquad (5.26)$$

where $x_0(t) = A\cos\theta$

and $x_1(t) = A(\alpha\cos\theta + \beta\cos 3\theta)$

$$\text{where } \theta = \Omega_1 t \quad \alpha = \frac{9}{8}, \beta = -\frac{1}{8} \qquad (5.27)$$

Equation (5.26) gives

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^{\ 2} x_1 + x_1 x_0^{-2} - 2x_0^{-1} \qquad (5.28)$$

Putting the values of Equation (5.26) in to Equation (5.27), we have

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^{\ 2} A(\alpha\cos\theta + \beta\cos 3\theta) + A(\alpha\cos\theta + \beta\cos\theta)(A\cos\theta)^{-2} - 2(A\cos\theta)^{-1}$$

$$= \Omega_1^{\ 2} A(\alpha\cos\theta + \beta\cos 3\theta) + \alpha(A\cos\theta)^{-1} + A\beta\cos 3\theta(A\cos\theta)^{-2} - 2(A\cos\theta)^{-1}$$

$$= \Omega_1^{\ 2} A(\alpha\cos\theta + \beta\cos 3\theta) + (\alpha - 2)(A\cos\theta)^{-1} + A\beta\cos 3\theta(A\cos\theta)^{-2} \qquad (5.29)$$

Now expanding second and third term on right hand side in a truncated Fourier cosine series and after substitution the Equation (5.29) reduces to

$$\ddot{x}_2 + \Omega_1^2 x_2 = (\Omega_1^{\ 2} A\alpha + \frac{2\alpha - 4 - 2\beta}{A})\cos\theta + (\Omega_1^{\ 2} A\beta - \frac{2\alpha - 4 - 6\beta}{A})\cos 3\theta \qquad (5.30)$$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

$$\Omega_1^{\ 2} A\alpha + \frac{2\alpha - 4 - 2\beta}{A} = 0$$

$$\Omega_1^{\ 2} A\alpha = -\frac{2\alpha - 4 - 2\beta}{A}$$

$$\Omega_1^{\ 2} = -\frac{2\alpha - 4 - 2\beta}{A^2\alpha}$$

$$= -\frac{2\frac{9}{8} - 4 - 2\frac{-1}{8}}{A^2\frac{9}{8}}$$

$$\Omega_1{}^2 = \frac{4}{3A^2}$$

$$\Omega_1 = \sqrt{(}\frac{4}{3A^2})$$

i.e. $\Omega_1 = \dfrac{1.1547}{A}$ (5.31)

This is the second approximate frequency of the oscillator.

And the percentage error in this stage is

$$\left|\frac{\Omega_{exact} - \Omega_1}{\Omega_{exact}}\right|.100 = \left|\frac{1.2533141 - 1.1547}{1.2533141}\right|.100 = 7.868\,\% \; .$$

After simplification the Equation (5.30) reduces to

$$\ddot{x}_2 + \Omega_1^2 x_2 = (\Omega_1{}^2 A\beta - \frac{2\alpha - 4 - 6\beta}{A})\cos 3\theta \qquad (5.32)$$

or $\ddot{x}_2 + \Omega_1^2 x_2 = (\dfrac{4}{3A^2} A(\dfrac{-1}{8}) - \dfrac{2\frac{9}{8} - 4 - 6(\frac{-1}{8})}{A})\cos 3\theta$

$$\ddot{x}_2 + \Omega_1^2 x_2 = \frac{\frac{10}{12}}{A}\cos 3\theta$$

$$\ddot{x}_2 + \Omega_1^2 x_2 = \frac{0.833}{A}\cos 3\theta \qquad (5.33)$$

The particular solution, $x_1{}^{(p)}(t)$ is

$$x_2{}^{(p)}(t) = \frac{\frac{0.833}{A}}{-9\Omega_1{}^2 + \Omega_1{}^2}\cos 3\theta$$

$$= -\frac{\frac{0.833}{A}}{-8\Omega_0{}^2}\cos 3\theta$$

$$= -\frac{0.833}{8.\frac{4}{3A^2}.A}\cos 3\theta$$

$$= -0.078125\,A\cos 3\theta$$

The complete solution of Equation (5.30) is

$x_2(t) = C\cos\theta - 0.078125A\cos 3\theta$ (C_2 is the arbitrary constant) (5.34

Using $x_2(0) = A$, we have $C_2 = 1.078125A$

Therefore, $x_2(t) = 1.078125A\cos\theta - 0.078125A\cos 3\theta$ (5.35)

$$x_2(t) = \alpha_1 A\cos\theta + \beta_1 A\cos 3\theta \tag{5.36}$$

where $\alpha_1 = 1.078125$, $\beta_1 = -0.078125$ and $\theta = \Omega_2 t$

Proceeding to the third level of iteration $x_3(t)$ satisfies the equation

$$\ddot{x}_3 + \Omega_2^2 x_3 = ({\Omega_2}^2 x_2 - {x_0}^{-1}) + (x_2 - x_0)({\Omega_2}^2 + {x_0}^{-2}) \tag{5.37}$$

Putting the values of Equation (5.36) in to Equation (5.37), we have

$$\ddot{x}_3 + \Omega_2^2 x_3 = {\Omega_2}^2 A(\alpha_1\cos\theta + \beta_1\cos 3\theta) + A(\alpha_1\cos\theta + \beta_1\cos 3\theta)(A\cos\theta)^{-2} - 2(A\cos\theta)^{-1}$$

$$= {\Omega_2}^2 A(\alpha_1\cos\theta + \beta_1\cos 3\theta) + \alpha_1(A\cos\theta)^{-1} + A\beta_1\cos 3\theta(A\cos\theta)^{-2} - 2(A\cos\theta)^{-1}$$

$$= {\Omega_2}^2 A(\alpha_1\cos\theta + \beta_1\cos 3\theta) + (\alpha_1 - 2)(A\cos\theta)^{-1} + A\beta_1\cos 3\theta(A\cos\theta)^{-2} \tag{5.38}$$

Now expanding second and third term on right hand side in a truncated Fourier cosine series and after substitution the Equation (4.38) reduces to

$$\ddot{x}_3 + \Omega_2^2 x_3 = ({\Omega_2}^2 A\alpha_1 + \frac{2\alpha_1 - 4 - 2\beta_1}{A})\cos\theta + ({\Omega_2}^2 A\beta_1 - \frac{2\alpha_1 - 4 - 6\beta_1}{A})\cos 3\theta \tag{5.39}$$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to zero.

$${\Omega_2}^2 A\alpha_1 + \frac{2\alpha_1 - 4 - 2\beta_1}{A} = 0$$

$${\Omega_2}^2 A\alpha_1 = \frac{2\beta_1 + 4 - 2\alpha_1}{A}$$

$${\Omega_2}^2 = \frac{2\beta_1 + 4 - 2\alpha_1}{A^2\alpha_1}$$

$$\Omega_2^{\ 2} = \frac{2(-0.078125) + 4 - 2(1.078125)}{A^2(1.078125)}$$

$$\Omega_2^{\ 2} = \frac{1.56522}{A^2}$$

$$\Omega_2 = \frac{1.2511}{A} \tag{5.40}$$

And the percentage error in this stage is

$$\left|\frac{\Omega_{exact} - \Omega_2}{\Omega_{exact}}\right|.100 = \left|\frac{1.2533141 - 1.2511}{1.2533141}\right|.100 = 0.176\,\%\ .$$

5.4: Results and discussions

An extended iterative approach is presented to obtain approximate solution of the 'nonlinear singular oscillator'. The present technique is very simple for solving algebraic equations analytically and the approach is different from the existing other approach for taking truncated Fourier series. Here we have calculated the first, second, third approximate frequencies $\Omega_0, \Omega_1,\ \Omega_2$. All the results are given in the following Table -5.1.Frequencies obtained by Mickens iteration method [49], Mickens HB method [38] and Haque's iteration method [25] are presented to compare the evalutaed frequencies. To show the accuracy, we have calculated the percentage errors by the definitions $\left|\frac{\Omega_e - \Omega i}{\Omega_e}\right|.100$ where Ω_i ;i=0, 1, 2…….represents the approximate frequencies obtained by the present method and Ω_e represents the corresponding exact frequency of the oscillator.

Table - 5.1:

Comparison of the approximate frequencies with exact frequency Ω_e [49] of $\ddot{x} + x^{-1} = 0$

Exact frequency Ω_e	$\frac{1.253}{A}$		
Amplitude A	First approximate frequencies, Ω_0 % Error	Second approximate frequencies, Ω_1 % Error	Third approximate frequencies, Ω_2 % Error
Adopted method	$\frac{1.414}{A}$ 12.84	$\frac{1.1547}{A}$ 7.87	$\frac{1.2511}{A}$ 0.17
Mickens direct iteration method [49]	$\frac{1.155}{A}$ 7.9	$\frac{1.018}{A}$ 18.1	-----
Mickens extended iteration method [49]	$\frac{1.155}{A}$ 7.9	$\frac{1.189699}{A}$ 5.1	-----
Mickens HB method [48]	$\frac{1.414}{A}$ 12.84	$\frac{1.273}{A}$ 1.6	$\frac{1.2731}{A}$ 1.58
Haque’s iteration method [25]	$\frac{1.414}{A}$ 12.84	$\frac{1.208}{A}$ 3.63	$\frac{1.265}{A}$ 0.92

CHAPTER VI

Conclusion

In chapter 4, the result has been improved only by rearranging the governing equation. Although most of the articles, the results have been improved by modifying the method. So not only modification of model is important but also rearranging is important in the case of iteration procedure. Rearranging the equation by a direct iteration technique from the first to the third approximate frequencies is better than corresponding frequencies which have been obtained by other techniques. It is observed that the third approximation provided excellent result.

In chapter 5, we used a simple but effective modification of the iteration method to attach a nonlinear singular oscillator. From the table, it is seen that the third-order approximate frequency obtained by adopted method provide an excellent result which is significantly better than the other existing results. The obtained results show that the modification of the extended iteration method is more accurate than other method.

In this thesis, a Direct Iteration method and an Extended Iteration method have been applied to obtain analytic approximate solutions for a nonlinear singular oscillator. All the results show a good agreement with exact results. The approximate frequency derived by the extended Iteration method is more accurate and closer to the exact solution and the relative error in the frequency is reduced and maximum error is less than 0.17%. The both method provide a very good accuracy and is a promising technique to a lot of practical engineering and physical problems.

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